Math 21b: CHAPTER 10. DIFFERENTIAL EQUATIONS

Section 1. Vector spaces whose elements are functions

1. (a) This is not a subspace of C^∞ because it is not a subset: There are continuous functions $f: \mathbb{R} \to \mathbb{R}$ that are not infinitely differentiable.
   (b) Given $f, g \in C^\infty$ satisfying this condition, for any $\alpha, \beta \in \mathbb{R}$, $(\alpha f+\beta g)(0)+(\alpha f+\beta g)'(0) = \alpha(f(0)+f'(0)) + \beta(g(0)+g'(0)) = \alpha \cdot 0 + \beta \cdot 0 = 0$, so any linear combination of f and g also satisfies the condition. Therefore this is a subspace.
   (c) $(\alpha f + \beta g) + (\alpha f + \beta g)' = \alpha(f+f') + \beta(g+g') = \alpha \cdot 0 + \beta \cdot 0 = 0$, so this is a subspace.
   (d) The zero function is not contained, so this is not a subspace.

2. (a) Linearly indepedent.
   (b) Linearly dependent, since the fourth function is the sum of the first and third.
   (c) Linearly independent.
   (d) Linearly dependent, since $\sin(t + \frac{\pi}{3}) = \cos(\frac{\pi}{3})\sin t + \sin(\frac{\pi}{3})\cos t$, which is a linear combination of the first two functions.

3. (a) $T(\alpha f + \beta g) = (\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha T(f) + \beta T(g)$, so T is linear.
   (b) Not linear. $T(\alpha f) = \alpha^2 f^2 + \alpha f' \neq \alpha T(f)$.
   (c) $T(\alpha f + \beta g) = ((\alpha f + \beta g)(0), (\alpha f + \beta g)(1)) = \alpha(f(0), f(1)) + \beta(g(0), g(1)) = \alpha T(f) + \beta T(g)$, so T is linear.
   (d) $T(\alpha f + \beta g) = \int_0^1 (\alpha f + \beta g)(t)dt = \alpha \int_0^1 f(t)dt + \beta \int_0^1 g(t)dt = \alpha T(f) + \beta T(g)$, so T is linear.

4. $T(f)(t) = f(0)(1+t^2) + f'(0)(t+t^2)$, so a basis for the image is $\{1+t^2, t+t^2\}$.

5. Evaluate at $t = 0$ to see $0 \cdot f(0) = 0 = 1$, which is a contradiction.

6. (a) If $g(t) = t^2 f(t)$, then $g'(t) = t^2 f'(t) + 2tf(t) = T(f)(t)$. If $T(f)$ is the zero function, then $g'(t) = T(f)(t)$ is the zero function evaluated at some t, therefore $g'(t) = 0$ as well.
   (b) A function $f \in \ker T$ satisfies $T(f)(t) = g'(t) = 0$ for all t, so the function g is constant. Therefore for all t we have $g(t) = g(0) = 0^2 f(0) = 0$. If $t \neq 0$ then $f(t) = g(t)/t^2 = 0$, and $f(0) = 0$ as well by continuity of f.
   (c) $T(f)(0) = 0^2 f'(0) + 2 \cdot 0 f(0) = 0$, so $T(f)$ cannot be the constant function 1 because it evaluates to 0 at $t = 0$.

Section 2. Constant coefficient linear differential operators

1. $\lambda^2 + \lambda - 12 = (\lambda - 3)(\lambda + 4) = 0$. A basis for the kernel is therefore $\{e^{3t}, e^{-4t}\}$. Our desired function is in $\ker T$ so it is a linear combination of the basis elements $f(t) = a_1 e^{3t} + a_2 e^{-4t}$. $f(0) = 0$ so $a_1 + a_2 = 0$, and $f'(0) = 1$ so $3a_1 - 4a_2 = 7a_1 = 1$. Therefore $a_1 = \frac{1}{7}$, $a_2 = -\frac{1}{7}$, and the function is $f(t) = \frac{1}{7}e^{3t} - \frac{1}{7}e^{-4t}$.

2. $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm 1i$. A basis for the kernel is $\{e^{-t}\cos t, e^{-t}\sin t\}$. The desired function $f(t) = a_1 e^{-t}\cos t + a_2 e^{-t}\sin t$ obeys $f(0) = 1 \Rightarrow a_1 = 1$, and $f(1) = 1 \Rightarrow e^{-1}\cos 1 + a_2 e^{-1}\sin 1 = 1 \Rightarrow a_2 = \frac{e - \cos 1}{\sin 1}$. The function is $f(t) = e^{-t}\cos t + \frac{e-\cos 1}{\sin 1}e^{-t}\sin t$.

3. $\lambda^2+6\lambda+9=(\lambda+3)^2=0$. A basis for the kernel is $\{e^{-3t}, te^{-3t}\}$. The function $f(t)=a_1e^{-3t}+a_2te^{-3t}$ satisfies $f'(0)=1 \Rightarrow -3a_1+a_2=1$, and $f(1)=0 \Rightarrow (a_1+a_2)e^{-3}=0$. Therefore $a_1=-\frac{1}{4}$, $a_2=\frac{1}{4}$, and $f(t)=-\frac{1}{4}e^{-3t}+\frac{1}{4}te^{-3t}$.

4. $\lambda^2+a^2=0 \Rightarrow \lambda=\pm ai$. A basis for the kernel is $\{\cos at, \sin at\}$. If a is an integer, then $\sin at=0$ at $t=0$ and $t=\pi$.

5. $f(0)$ is the function f evaluated at $t=0$, so it is a constant. If $f''+f(0)$ is the zero function, then f'' must be a constant function as well. Therefore f is a polynomial of degree at most 2, $f(t)=a_2t^2+a_1t+a_0$. $f''(t)+f(0)=2a_2+a_0=0$, so one basis element is t^2-2. Note that we can choose a_1 arbitrarily, so a linearly independent basis element is t. The kernel therefore has basis $\{t, t^2-2\}$.

Section 3. Fourier series

1. We find an orthonormal basis for the span of $\{e^t, e^{-t}\}$. Note that $\cosh t=\frac{e^t+e^{-t}}{2}$ and $\sinh t=\frac{e^t-e^{-t}}{2}$ are orthogonal to each other. We then normalize them to find the basis $\{\frac{2\pi}{\sinh 2\pi+2\pi}\cosh t, \frac{2\pi}{\sinh 2\pi-2\pi}\sinh t\}$.

2. Constant term $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}|t|dt=\frac{1}{\pi}\int_0^{\pi}tdt=\frac{\pi}{2}$.

   Coefficients on cosine terms $b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(kt)|t|dt=\frac{2}{\pi}\int_0^{\pi}t\cos(kt)dt=\frac{2}{\pi k^2}(\cos k\pi-1)=\frac{2}{\pi k^2}((-1)^k-1)$. (Use integration by parts.) This is 0 for k even and $\frac{-4}{\pi k^2}$ for k odd.

   Coefficients on sines c_k are zero because $\sin(kt)|t|$ is an odd function.

   Therefore $|t|=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k\text{ odd}}\frac{1}{k^2}\cos(kt)$.

3. Constant term $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{at}+e^{-at}}{2}dt=\frac{\sinh a\pi}{a\pi}$.

   Coefficients on cosine terms $b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(kt)\frac{e^{at}+e^{-at}}{2}dt=\frac{2a(-1)^k}{\pi(a^2+k^2)}\sinh a\pi$. (Again, we can integrate by parts to find this.)

   Coefficients on sine terms c_k are zero because $\sin(kt)\cosh(at)$ is an odd function.

   Therefore $\cosh at=\frac{\sinh a\pi}{a\pi}\left(1+\sum_{k=1}^{\infty}\frac{2a^2(-1)^k}{a^2+k^2}\cos kt\right)$.

   Set $t=\pi$ to find $\frac{1}{2a^2}(a\pi\coth a\pi-1)=\sum_{k=1}^{\infty}\frac{1}{a^2+k^2}$.

4. When evaluating the integrals, make the substitution $t\mapsto t+r$.

5. When evaluating the integrals, make the substitution $t\mapsto\frac{b-a}{2\pi}t+r$.

Section 4. Partial differential equations I: The heat/diffusion equation

1. $T(0,x)=\sin^2x-\cos^4x=\frac{1-\cos 2x}{2}-\left(\frac{1+\cos 2x}{2}\right)^2=\frac{1}{2}-\frac{1}{2}\cos 2x-\left(\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x\right)=\frac{1}{4}-\cos 2x-\frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)=\frac{1}{8}-\cos 2x-\frac{1}{8}\cos 4x$, which is a Fourier series. Then $T(t,x)=\frac{1}{8}-e^{-4\mu t}\cos 2x-e^{-16\mu t}\frac{1}{8}\cos 4x$.

2. $T(0,x)=\sinh x=\frac{e^x-e^{-x}}{2}$. Also, $e^x=\frac{\sinh\pi}{\pi}\left(1+2\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2}(\cos kx-k\sin kx)\right)$, which then implies $e^{-x}=\frac{\sinh\pi}{\pi}\left(1+2\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2}(\cos kx+k\sin kx)\right)$. Subtract the series to obtain $T(0,x)=\frac{\sinh\pi}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{1+k^2}k\sin kx$, so $T(t,x)=\frac{\sinh\pi}{\pi}\sum_{k=1}^{\infty}e^{-\mu k^2t}\frac{(-1)^{k+1}}{1+k^2}k\sin kx$.

3. $\frac{\partial T}{\partial t} = \mu c^2 e^{\mu c^2 t} e^{cx}$, and $\mu \frac{\partial^2 T}{\partial x^2} = \mu c^2 e^{\mu c^2 t} e^{cx}$ also, so this solves the heat equation.

4. $T(t,x) = e^{\mu t} e^x = e^{\mu t} \frac{\sinh \pi}{\pi} \left(1 + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{1+k^2} (\cos kx - k \sin kx)\right)$, which grows as $t \to \infty$. The text gives the solution $T(t,x) = \frac{\sinh \pi}{\pi} \left(1 + 2\sum_{k=1}^{\infty} e^{-\mu k^2 t} \frac{(-1)^k}{1+k^2} (\cos kx - k \sin kx)\right)$, which decreases as $t \to \infty$.

5. From text, $T(0,x) = x = 2\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \sin kt$, so $T(t,x) = 2\sum_{k=1}^{\infty} e^{-\mu k^2 t} (-1)^{k+1} \frac{1}{k} \sin kt$.

6. $\frac{\partial T}{\partial t} = 0 = \mu \frac{\partial^2 T}{\partial x^2}$. But $T(t,x) = x$ is different from the solution we found using the Fourier series because it is time-independent.