

**MATH 21b Final Exam**  
S 2005  
**Regular sections**

Name: \_\_\_\_\_

Circle the time of your section:

MWF10      MWF11      MWF12      TH10

**Instructions:**

- This exam booklet is only for students in the Regular sections.
- Print your name in the line above and circle the time of your section.
- ∑ Answer each of the questions below in the space provided. If more space is needed, use the back of the facing page or the extra blank pages at the end of this booklet. Please direct the grader to the extra pages used.
- Please give justification for answers if you are not told otherwise.
- Please write neatly. Answers deemed illegible by the grader will not receive credit.
- ∑ No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students. Use only your brain and a pencil.
- Each of the problems counts for the same total number of points, so budget your time for each problem.
- ∑ Do not detach pages from this exam booklet.
- ∑ There are 8 problems on this exam.

*In agreeing to take this exam, you are implicitly agreeing to act with fairness and honesty.*

**Problem 1:**

Let A denote the matrix  $\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$

- a) Find the eigenvalues and eigenvectors of A.
- b) Solve the dynamical system  $\bar{x}(m+1) = A\bar{x}(m)$  given that  $\bar{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ . Thus, give  $\bar{x}(m)$  for  $m = 1, 2, 3, \dots$ .
- c) Give the vector valued function  $t \rightarrow \bar{x}(t)$  that obeys  $\frac{d}{dt} \bar{x} = A\bar{x}$  and  $\bar{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ .
- d) Describe the set of real numbers  $r$  with the following property: Every vector valued function  $t \rightarrow \bar{x}(t)$  that obeys  $\frac{d}{dt} \bar{x} = A\bar{x} + r\bar{x}$  also obeys  $\lim_{t \rightarrow \infty} |\bar{x}(t)| = 0$ .

**Problem 2:**

Which of the following is the equation for the best fit as determined by the least squares method for a line through the four points  $\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$  in the x-y plane? Please justify your work.

- a)  $y = \frac{14}{11}x + \frac{27}{22}$
- b)  $y = \frac{11}{13}x + \frac{27}{11}$
- c)  $y = \frac{13}{11}x + \frac{21}{11}$
- d)  $y = \frac{11}{14}x + \frac{21}{22}$
- e)  $y = \frac{13}{11}x + \frac{27}{11}$
- f)  $y = \frac{14}{11}x + \frac{21}{22}$

**Problem 3:**

The vector  $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$  is the eigenvector with eigenvalue 3 of a symmetric  $2 \times 2$  matrix with trace equal to 1. Write down the matrix.

**Problem 4:**

Circle **T** if the accompanying statement is true, and circle **F** if it is false. You need not justify your answers.

- T F** a) Suppose that  $A$  is a symmetric  $n \times n$  matrix,  $\vec{v} \in \ker(A)$  and  $\vec{w}$  is in the image of  $A$ . Then  $\vec{v}$  and  $\vec{w}$  must be orthogonal.
- T F** b) There are infinitely many  $2 \times 2$  matrices with determinant equal to 1 and trace equal to 2.
- T F** c) All invertible matrices are diagonalizable.
- T F** d) A non-zero matrix with 2 columns and 4 rows must have 2-dimensional image.

- T F** e) Suppose that  $\vec{e} = \begin{pmatrix} 1+i \\ 1-i \\ -1+i \\ -1-i \end{pmatrix}$  is an eigenvector of a  $4 \times 4$  matrix with real number

entries. Then  $\begin{pmatrix} 1-i \\ 1+i \\ -1-i \\ -1+i \end{pmatrix}$ ,  $\begin{pmatrix} -1-i \\ -1+i \\ 1-i \\ 1+i \end{pmatrix}$ , and  $\begin{pmatrix} -1+i \\ -1-i \\ 1+i \\ 1-i \end{pmatrix}$  are eigenvectors of the same matrix.

- T F** f) The point  $(0, 0)$  is a stable equilibrium point for the non-linear system of equations  $\frac{d}{dt}x = \sin(x) \cos(y)$  and  $\frac{d}{dt}y = \cos(x) \sin(y)$ . Here,  $x(t)$  and  $y(t)$  are functions of  $t$ .

- T F** g) The complex valued function  $t \rightarrow z(t) = e^t \cos(2t) - i e^t \sin(2t)$  obeys the equation  $\frac{d}{dt}z = (1-2i)z$ .

- T F** h) A diagonalizable  $4 \times 4$  matrix with just one negative eigenvalue and negative determinant cannot have  $i$  for a complex eigenvalue.

- T F** i)  $SAS^{-1}$  is diagonal if  $S = \frac{1}{4} \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$  and if  $A$  is a  $2 \times 2$  matrix with eigenvectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

**T F** j) If  $z$  is any complex number, then  $z$  is an eigenvalue of  $\begin{pmatrix} \operatorname{re}(z) & -\operatorname{im}(z) \\ \operatorname{im}(z) & \operatorname{re}(z) \end{pmatrix}$ .

### Problem 5

This problem concerns the function,  $f$ , that is defined for  $-\pi \leq x \leq \pi$  by the Fourier series

$$f(x) = \sum_{n=1,2,\dots} 2^{-n} \cos(nx).$$

- Which of the numbers that follow is  $f(0)$ :  $-4, -2, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 2, 4$ ?
- Is 0 a local maximum of  $f$ , a local minimum of  $f$ , or neither?
- Which of the numbers that follow is  $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$ :  $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1, 2$ ?
- View the function  $\sin^2(x)$  as a function of  $x$  for  $-\pi \leq x \leq \pi$ . Which of the following is the inner product of  $f$  with  $\sin^2(x)$ :  $-2, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, 0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 2$ ?

Here, take  $\frac{1}{\pi} \int_{-\pi}^{\pi} g(x)h(x) dx$  to be the inner product between any two functions,  $f$  and  $g$ , that are defined on  $[-\pi, \pi]$ .

(Remember that  $1 + a + a^2 + a^3 + \dots = \frac{1}{1-a}$  if  $|a| < 1$ .)

### Problem 6

This problem concerns the linear transformation on the space of smooth functions of  $x \in \mathbb{R}$

that maps a function  $f(x)$  to  $Tf = \frac{d^4}{dx^4} f + 5 \frac{d^2}{dx^2} f + 4f$ .

- Write down a basis for the kernel of  $T$ .
- Let  $M: \text{kernel}(T) \rightarrow \mathbb{R}$  denote the linear map that sends  $f$  to  $f(\pi/2)$ . Write down a basis for  $\text{kernel}(M)$ .
- Use the basis from your answer to Part a) to write the matrix for the linear

transformation on kernel(T) that sends  $f$  to  $\frac{d}{dx}f$ ?

### Problem 7

Write down the function,  $u(x, y)$ , that is defined for  $-\pi \leq x \leq \pi$  and  $-\pi \leq y \leq \pi$  and satisfies the Laplace equation  $\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = 0$  with the following boundary conditions:

$$u(-\pi, y) = u(\pi, y) = 2 \sin(y) + \sin(3y) \quad \text{and} \quad u(x, -\pi) = u(x, \pi) = 0.$$

Here,  $\cosh(x)$  is short hand for  $\frac{1}{2}(e^x + e^{-x})$ .

### Problem 8

Circle **T** if the accompanying statement is true, and circle **F** if it is false. There is no need to justify your answers.

**T F** a) If both  $u_1(t)$  and  $u_2(t)$  are solutions to the equation  $2 \frac{d^2}{dt^2}u + 3 \frac{d}{dt}u - u = t$ , then

$$u_1 + u_2 \text{ solves the equation } 4 \frac{d^2}{dt^2}u + 6 \frac{d}{dt}u - 2u = 2t$$

**T F** b) No differentiable function defined for  $-\pi \leq x \leq \pi$  can simultaneously satisfy

$$\int_{-\pi}^{\pi} |f(x)|^2 dx < 2\pi, f(0) = 0 \text{ and } f(\pi) = 3\pi.$$

**T F** c) There is a function  $u(t, x)$  defined for  $t \geq 0$  and  $x \in [-\pi, \pi]$  that

simultaneously solves the wave equation  $\frac{\partial^2}{\partial t^2}u - 4 \frac{\partial^2}{\partial x^2}u = 0$ , the initial conditions  $u(0, x) = \sin(x)$  and  $(\frac{\partial}{\partial t}u)(0, x) = \sin(2x)$ , and the boundary conditions  $u(t, \pi) = u(t, -\pi) = 0$ .

**T F** d) If a continuous function,  $f$ , defined for  $-\pi \leq x \leq \pi$  has Fourier series

$$f(x) = \sum_{n=1,2,\dots} a_n \sin(nx), \text{ then } f(x)^2 \text{ has Fourier series } \sum_{n=1,2,\dots} a_n^2 \sin(2nx).$$

**T F** e) Suppose that  $u(t, x)$  solves the heat equation  $\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u$  for  $t \geq 0$  and  $-\pi \leq x \leq \pi$  with the boundary condition  $u(t, \pi) = u(t, -\pi) = 0$ . If, in addition,  $u(0, x) = \pi^2 - x^2$ , then the time derivative of  $u$  is negative at  $(0, 0)$ .

**Answers:**

1. a) 2, -3 with eigenvectors  $\vec{e}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$  and  $\vec{e}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$   
b)  $2^m \vec{e}_1 + 2(-3)^m \vec{e}_2$ .  
c)  $e^{2t} \vec{e}_1 + 2e^{-3t} \vec{e}_2$ .  
d)  $r < -2$ .

2. e)

3.  $\begin{pmatrix} -1 & -2 \\ -2 & 2 \end{pmatrix}$

4. T, T, F, F, T, F, T, F, T, T.

5. 1, local maximum,  $\frac{1}{3}$ ,  $-\frac{1}{8}$ .

6. a)  $\{\cos(2t), \sin(2t), \cos(t), \sin(t)\}$

b)  $\{\cos(t), \sin(2t), \cos(t)+\sin(t)\}$

c)  $\begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$

7.  $u(x,y) = 2 \sin(y) \cosh(x)/\cosh(\pi) + \sin(3y) \cosh(3x)/\cosh(3\pi)$ .

8. F, F, T, F, T.