

MATH 21b Final Exam
S 2004

Name: _____

Circle the time of your section:

MWF10 MWF11 MWF12 TH10

Instructions:

- This exam booklet is only for students in the Regular sections.
- Print your name in the line above and circle the time of your section.
- ∑ Answer each of the questions below in the space provided. If more space is needed, use the back of the facing page or the extra blank pages at the end of this booklet. Please direct the grader to the extra pages used.
- Please give justification for answers if you are not told otherwise.
- Please write neatly. Answers deemed illegible by the grader will not receive credit.
- ∑ No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students. Use only your brain and a pencil.
- Each of the problems counts for the same total number of points, so budget your time for each problem.
- ∑ Do not detach pages from this exam booklet.
- ∑ There are 8 problems on this exam.

In agreeing to take this exam, you are implicitly agreeing to act with fairness and honesty.

Problem 1:

Let A denote the matrix $\begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$

- Find the eigenvalues and eigenvectors of A.
- Solve the dynamical system $\bar{x}(m+1) = A\bar{x}(m)$ given that $\bar{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Thus, give $\bar{x}(m)$ for $m = 1, 2, 3, \dots$.
- Give the vector valued function $t \rightarrow \bar{x}(t)$ that obeys $\frac{d}{dt} \bar{x} = A\bar{x}$ and $\bar{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Problem 2:

Which of the following is the equation for the best fit for a line through the four points $\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ as determined by the least squares method? Please justify your work.

- $y = -3x + 10$
- $y = -\frac{8}{3}x + 9$
- $y = -\frac{20}{9}x + 12$
- $y = -12x + 3$
- $y = -\frac{16}{9}x + \frac{35}{3}$
- $y = -\frac{5}{3}x + 5$

Problem 3:

Let c denote the space of all continuous functions on $[-\pi, \pi]$ with the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$. Use V to denote the subspace of c that is spanned by the vectors $\{1, \cos(x), \sin(x+1)\}$.

- Find an orthonormal basis for V .
- Find the orthogonal projection of x onto V .
- Define a linear transformation, T , from V to itself by the formula $Tf = f'$. Find the eigenvalues of T .

Problem 4:

Find a 2×2 matrix that has all of the following properties: Both $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors, their respective eigenvalues are not zero, but the sum of their eigenvalues is zero.

Problem 5:

Let $x \rightarrow f(x)$ denote the function on $[-\pi, \pi]$ that obeys $f(x) = x + \pi$ where $x \leq 0$ and $f(x) = x$ where $x > 0$. Which of the following is the Fourier series for f ? Please justify your answer.

- a) $\frac{\pi}{2} - 2 \sum_{k=1}^{\infty} (-1)^k \frac{1}{k} \sin(kx)$.
- b) $\frac{\pi}{2} + 2 \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2} \cos(kx)$.
- c) $\frac{\pi}{2} + 2 \sum_{k=1}^{\infty} (-1)^k \left[\frac{1}{1+k^2} \cos(kx) - \frac{k}{1+k^2} \sin(kx) \right]$.
- d) $\frac{\pi}{2} - 2 \sum_{k=2,4,\dots}^{\infty} \frac{1}{k} \sin(kx)$.
- e) $\frac{\pi}{2} + 2 \sum_{k=2,4,\dots}^{\infty} \frac{1}{k} \cos(kx)$.
- f) $\frac{\pi}{2} - \sum_{k=1,3,\dots}^{\infty} \frac{1}{k} \sin(kx) + \frac{2}{\pi} \sum_{k=2,4,\dots}^{\infty} \frac{1}{k^2} \cos(kx)$.

Problem 6:

This problem concerns the differential equation $\frac{d^4 f}{dx^4} = 16f$.

- a) Find the general solution.
- b) Find three different solutions that obey $f(0) = 6$ and $f'(0) = 2$.
- c) Let V denote the space of solutions. Define a map, $T: V \rightarrow \mathbb{R}^2$ by $T(f) = \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}$.

Compute the dimension of the kernel of T .

Problem 7:

Circle **T** if the accompanying statement is true, and circle **F** if it is false.

- T F** a) The vector $\vec{0}$ is never a stable equilibrium point for the dynamical system $\vec{x}(m+1) = A\vec{x}(m)$ if $\det(A) = 2$.
- T F** b) A function $t \rightarrow f(t)$ solves the equation $f'' + f' - 3f = 0$ if and only if the vector valued function of time $\vec{x}(t) = \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix}$ solves the equation $\frac{d}{dt} \vec{x} = A\vec{x}$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$.
- T F** c) If f is any function on $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} f(x)^2 dx < 2\pi$ then $|f(x)| < 1$ for all x .
- T F** d) If functions $u(t,x)$ and $v(t,x)$ solve the wave equation and if they are equal at each x when $t = 0$, then it is necessary that $u(t,x) = v(t,x)$ for all t and x .
- T F** e) The map, T , from the space of continuous functions on $[-\pi, \pi]$ to \mathbb{R} that sends f to its largest value is not a linear map.

Problem 8:

Circle **T** if the accompanying statement is true, and circle **F** if it is false. You need not justify your answers.

- T F** a) If $n \geq 3$ and if \vec{v} and \vec{u} are eigenvectors of an $n \times n$ matrix, then $\vec{v} + \vec{u}$ must also be an eigenvector.
- T F** b) There are invertible 3×3 matrices A and S with the property that $SAS^{-1} = -A$.
- T F** c) If $n \geq 3$ and A and B are two diagonalizable $n \times n$ matrices with the same eigenbasis, there must exist an invertible $n \times n$ matrix S such that $A = SBS^{-1}$.
- T F** d) If A is an $n \times n$ matrix with $\det(A) = 0$, then at least one column of A must be a scalar multiple of some other column of A .

- T F** e) If A has 5 rows and 3 columns, then the dimension of its kernel must equal 2.
- T F** f) Any 4×4 matrix with negative determinant has either two or zero complex eigenvalues with non-zero imaginary part.
- T F** g) Given any matrix A , then $A^T A$ is symmetric.
- T F** h) Given any square matrix A , then $A^T A$ and AA^T have the same eigenvalues.
- T F** i) If A is an $n \times n$ matrix with n distinct, real eigenvalues, then all eigenvalues of A^2 are non-negative.
- T F** j) If A and B are any two matrices whose product is orthogonal, then both A and B are orthogonal.

Answers to the Math 21b 2004 exam

1. a) Eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -2$. Corresponding eigenvectors are:

$$\bar{e}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \bar{e}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

b) $\bar{x}(m) = 5^m \frac{10}{7} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - (-2)^m \frac{9}{7} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

c) $\bar{x}(t) = e^{5t} \frac{10}{7} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - e^{-2t} \frac{9}{7} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

2. The answer is b). Here is why: The generic line has the form $y = ax + b$. To find a

and b , introduce the $\bar{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ given by $\bar{x} = (A^T A)^{-1} A^T \bar{y}$ where $\bar{y} = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 6 \end{pmatrix}$ and

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{pmatrix}. \text{ Now, } A^T \bar{y} = \begin{pmatrix} 25 \\ 12 \end{pmatrix} \text{ and } A^T A = \begin{pmatrix} 21 & 9 \\ 9 & 4 \end{pmatrix} \text{ so } (A^T A)^{-1} = \frac{1}{3} \begin{pmatrix} 4 & -9 \\ -9 & 21 \end{pmatrix}. \text{ Thus,}$$

$\bar{x} = \frac{1}{3} \begin{pmatrix} -8 \\ 27 \end{pmatrix}$. This gives line (b).

3. a) Since $\sin(x+1) = \sin(1) \cos(x) + \cos(1) \sin(x)$, the function $\sin(x)$ is in V , and the collection $(\frac{1}{\sqrt{2}}, \cos(x), \sin(x))$ span V and are orthonormal.
 b) The orthogonal projection is $2\sin(x)$.

c) The matrix of T with respect to the basis given in a) is $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Its eigenvalues are $0, i, -i$.

4. Write the desired matrix as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since the eigenvalues are λ and $-\lambda$, this matrix must have zero trace. Thus, $d = -a$. Also, using what we know about the eigenvectors, we find that $3a+b = 3\lambda$, $3c - a = \lambda$, $a - b = -\lambda$, $c+a = \lambda$. This implies that $a = c$, $b = 3c$, and $\lambda = 2c$. Thus, any matrix of the form $c \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ with $c \neq 0$ fits the bill.

5. The answer is d). Here is why: The constant term in the Fourier series for f is $\frac{\pi}{2}$. Meanwhile, $\frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(kx) dx = 0$ and $\frac{1}{\pi} \int_{-\pi}^0 \cos(kx) dx = 0$, so there are no cosine terms in the Fourier series for f . As for the sine terms, $\frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) dx = \frac{2}{k}$ if k is odd and it is $-\frac{2}{k}$ if k is even. On the other hand, $\frac{1}{\pi} \int_{-\pi}^0 \sin(kx) dx = 0$ if k is even and $-\frac{2}{k}$ if k is odd. Thus, there are no $\sin(kx)$ terms when k is odd, and when k is even, the coefficient of $\sin(kx)$ is $-\frac{2}{k}$.

6. a) The general solution has the form $a e^{2x} + b e^{-2x} + c \cos(2x) + d \sin(2x)$ where a, b, c, d are constants.
 b) If $f(0) = 6$, then $a + b + c = 6$. If $f'(0) = 2$, then $2a - 2b + 2d = 2$. Thus, the

- general solution to the equation plus these constraints has the form
 $a e^{2x} + b e^{-2x} + (6-a-b) \cos(x) + (1-a+b) \sin(x)$. Take any three values for a and b .
- c) The kernel of V is two dimensional; it is spanned by the two functions
 $e^{2x} - \cos(x) - \sin(x)$ and $e^{-2x} - \cos(x) + \sin(x)$.
7. a) **T.** Here is why: The absolute values of the eigenvalues must be less than 1 if $\bar{0}$ is stable, yet their product is the determinant, this greater than 1.
- b) **T.** To see why, write out the two components of the equation for $\frac{d}{dt} \bar{x} = A\bar{x}$.
- c) **F.** For example, the function that is 10 where $0 \leq x \leq \frac{2\pi}{101}$ and 0 elsewhere has the property that the integral of its square is less than 2π .
- d) **F.** For example, the functions $u(t,x) = t$ and $v(t,x) = 0$ solve the wave equation and are equal for all x at $t = 0$.
- e) **T.** For example, the largest values of both $\frac{1}{2}(\pi-x)$ and x equal π , but this is also the largest value of their sum, the function $\frac{1}{2}(\pi+x)$.
8. a) **F.** Consider, for example, the 2×2 diagonal matrix with different entries.
- b) **F.** Here is why: Since $\det(A) = \det(S^{-1}AS)$ and $\det(-A) = -\det(A)$, the desired condition would require $\det(A)$ to vanish. Thus, A would not be invertible.
- c) **F.** The condition $A = SBS^{-1}$ requires that A and B also have the same eigenvalues.
- d) **F.** All that is required is that one column of A be a linear combination of the remaining columns.
- e) **F.** Consider the matrix that gives the linear map from \mathbb{R}^3 to \mathbb{R}^5 that sends the domain to the 3-dimensional subspace where the coordinates x_4 and x_5 are zero.
- f) **T.** To see this, note that it has an even number of complex eigenvalues, and if it had four, they would come in complex conjugate pairs. Thus, their product would be a product of two positive numbers and so positive.
- g) **T.** This is because the transpose of the product AB is $B^T A^T$ for any square matrices A and B .
- h) **F.** To see this, consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.
- i) **T.** The eigenvalues of A^2 are the squares of those of A .
- j) **F.** Consider, for example, when A is a non-orthogonal, but invertible matrix and B is A^{-1} .