Math 213a: Complex analysis
Problem Set #9 (19 November 2003):
Hypergeometric functions

Recall that the hypergeometric function $F(a,b;c;z)$ (a.k.a. ${}_2F_1(a,b;c;z)$) is defined by the power series

$$1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\cdots$$

($|z|<1$, $c\neq 0,-1,-2,\ldots$), and satisfies the hypergeometric differential equation

$$(z-z^2)w''+\bigl(c-(a+b+1)z\bigr)w'-abw=0.$$

1. Prove that if $\mathrm{Re}(c)>\mathrm{Re}(b)>0$ then

$$F(a,b;c;z)=\frac{1}{\mathrm{B}(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,$$

and deduce the value of $F(a,b;c;1)=\lim_{z\to 1-}F(a,b;c;z)$.

2. Prove the identity $F(a,b;c;z)=(1-z)^{-a}F(a,c-b;c;z/(z-1))$. What is the relation between $F(a,b;c;z)$ and $F(c-a,c-b;c;z)$? [Note that (a,b,c) and $(c-a,c-b,c)$ yield the same angles $\pi|1-c|,\pi|c-a-b|,\pi|a-b|$ at $z=0,1,\infty$.]

3. Suppose $2c=a+b+1$.
i) Prove that $F(a,b;c;z)=F(a/2,b/2;c;4z(1-z))$, and use this to evaluate $F(a,b;c;1/2)$.
ii) Show that (unless $a,b,c\in\{0,-1,-2,\ldots\}$) $F(a,b;c;z)$ and $F(a,b;c;1-z)$ constitute a basis of the solutions of the same hypergeometric differential equation.

4. Assume further that $a+b=0$, so $c=1/2$. Evaluate $F(a,-a;1/2;z)$ in closed form, and verify directly that if $0<a<1$ then

$$F(a,-a;1/2;z)\big/F(a,-a;1/2;1-z)$$

conformally maps the upper half-plane to a spherical triangle on the Riemann sphere.

5.* In the first problem set we encountered the finite subgroup $G \subset \mathrm{PGL}_2(\mathbf{C})$ generated by the fractional linear transformations $w \mapsto (1+w)/(1-w)$ and $w \mapsto iw$. Find an explicit rational function $z(w)$ such that $z(w) = z(w')$ if and only if w' is in the orbit $G(w)$, normalized so that if $w = g(w)$ for some nontrivial $g \in G$ then $z(w) \in \{0, 1, \infty\}$. Express (a branch of) the inverse function $w(z)$ explicitly in terms of hypergeometric functions. Be sure to check that the first few coefficients of the power-series expansion of your formula do agree with the desired inverse function.

It is known that $G \cong S_4$, and that there are two further exceptional discrete subgroups of $\mathrm{PGL}_2(\mathbf{C})$: the "tetrahedral" and "icosahedral" groups, isomorphic with A_4 and A_5. Be thankful that I didn't ask you to do the icosahedral version of this exercise!

6. Fix distinct $z_1, \ldots, z_n \in \mathbf{P}^1(\mathbf{C})$ and $\alpha_j, \beta_j \in \mathbf{C}$ $(j = 1, \ldots, n)$ such that $\alpha_j - \beta_j \notin \mathbf{Z}$. We seek differential equations $w'' + pw' + qw = 0$ whose only singularities are regular singular points at the z_j with exponents α_j, β_j. When $n = 3$ we saw, following Riemann, that such an equation exists if and only if $\sum_{j=1}^{3} \alpha_j + \beta_j = 1$, and then the equation is uniquely determined. Find the analogous condition on the α_j, β_j when $n > 3$, and show that under this condition the equation is determined up to $n - 3$ parameters.

This problem set is due Wednesday, November 26, at the beginning of class.