Math 213a: Complex analysis
Problem Set #8 (12 November 2003):
Harmonic functions and their uses, cont'd

First, an observation on the coefficients of the linear equations used to determine the logarithm of our conformal map of a finitely connected region (Ahlfors, V.3.1, Ex.2 (p.204)):

1. Let $\Omega \subset \mathbf{C}$ be a region whose boundary consists of the simple closed curves $C_1, \ldots, C_n$. As in Ahlfors V.3.1, for $j, k \in \{1, \ldots, n\}$ let $\alpha_{jk} = \oint_{C_j} *d\omega_k$, where ω_k is the harmonic measure of C_k with respect to Ω, and C_j is traversed in the positive direction relative to Ω. Prove that $\alpha_{jk} = \alpha_{kj}$.

Next, some problems on the nice special case of a doubly-connected region. We begin with an elementary problem (that is, a problem not requiring the machinery of (sub)harmonic functions from Chapter V) that picks up a thread from the fourth problem set, where we studied the action of $\mathrm{PGL}_2(\mathbf{C})$ on pairs of disjoint circles.

2. If Ω is the doubly connected region bounded by two disjoint circles, show that the conformal bijections of Ω to an annulus are fractional linear transformations. Deduce that the subgroup of $\mathrm{PGL}_2(\mathbf{C})$ that preserves two disjoint circles is conjugate in $\mathrm{PGL}_2(\mathbf{C})$ to the group of fractional linear transformations of the form $z \mapsto cz$ for some $c \in \mathbf{C}^*$ with $|c| = 1$. Obtain a similar description of the stabilizer in $\mathrm{PGL}_2(\mathbf{C})$ of two circles that are tangent or meet in two points.

3. Let Ω be the annulus $\{z \in \mathbf{C} : r < |z| < R\}$. Prove that if u is a bounded harmonic function on Ω then, for any $\rho \in (r, R)$,

$$\left(\log \frac{R}{r}\right) \oint_{|z|=\rho} *du \;\leq\; 2\pi\bigl(\sup_z u(z) - \inf_z u(z)\bigr),$$

   with equality if and only if there exist $C_0, C_1 \in \mathbf{R}$, with $C_1 \geq 0$, such that $u(z) = C_0 + C_1 \log |z|$ for all $z \in \Omega$. Conclude that the only conformal bijections of Ω are $z \mapsto cz$ and $z \mapsto cRr/z$ ($|c| = 1$).

4. Suppose Ω and $\Omega' \in \mathbf{C}$ are doubly connected regions conformally equivalent to the annuli $r < |z| < R$ and $r' < |z| < R'$. Let their complements in $\mathbf{P}^1(\mathbf{C})$ have components E_1, E_2 and E_1', E_2', with E_1, E_1' containing ∞. Prove that if $E_j \supseteq E_j'$ for $j = 1, 2$ (whence $\Omega \subseteq \Omega'$) then $R/r \leq R'/r'$, with equality only when $\Omega = \Omega'$.

5. Suppose Ω is a simply-connected bounded subset of $\mathbf{C}$, and fix $z_0 \in \Omega$. For each $r > 0$ such that Ω contains the closed disc $\{z : |z - z_0| \leq r\}$, the region $\Omega_r := \{z \in \Omega : |z - z_0| > r\}$ is doubly connected, and thus conformal with an annulus A_r with outer radius 1. Let $\rho(r)$ be the inner radius of A_r. By

the previous problem, ρ is a decreasing function of r. Prove that $\rho(r) \to 0$ as $r \to 0$. What happens to the conformal bijections from Ω_r to A_r as $r \to 0$?

Finally:

6. Let $\Omega \in \mathbf{C}$ be a region of connectivity n.
   i) If $n \leq 3$, prove that Ω is conformally equivalent to its image under complex conjugation. Do this even in the (easier) case that the complement of Ω in $\mathbf{P}^1(\mathbf{C})$ has one or more components that reduce to a point.
   ii*) When $n = 4$, construct or prove the existence of an Ω not conformally equivalent to its complex conjugate, with none of the components of $\mathbf{P}^1(\mathbf{C}) - \Omega$ reducing to a point.

This problem set is due Wednesday, November 19, at the beginning of class.