Math 213a: Complex analysis
Problem Set #7 (5 November 2003):
Harmonic functions and the Dirichlet problem

1. In Ahlfors' first exercise for V.1.3 (p.183) he gives a geometric description of the harmonic function $P_U(z)$ on the open unit disc Δ obtained from Poisson's integral formula when U is the characteristic function of an arc on the unit circle. Explain this result using a suitable conformal map from Δ to an infinite strip.

Some more problems from Ahlfors (V.1.4, p.184; V.2.1, p.196)

2. If E is a compact subset of a region Ω, prove that there exists a constant M, depending only on E and Ω, such that every positive harmonic function $u : \Omega \to (0, \infty)$ satisfies the inequality $u(z') \leq M u(z)$ for all $z, z' \in E$.

3. Show that the functions $|x| = |\operatorname{Re}(z)|$, $|z|^\alpha$ (all $\alpha \geq 0$), and $\log(1 + |z|^2)$ are subharmonic.

4. If v is a $\mathcal{C}^2$ function on some region, prove that v is subharmonic if and only if $\Delta v \geq 0$. [See Ahlfors for a hint.]

5. Extend Harnack's principle [Ahlfors V.1.4, Thm.6, 183–4] to subharmonic functions. (Warning: the conclusion must be weaker, as evidenced by such counterexamples as $\Omega_n = \Omega = \mathbf{C}$, $u_n = n|z|$.)

The final two problems concern a discrete analogue of the Laplacian. We work on the cubic lattice $L = \mathbf{Z}^n \in \mathbf{R}^n$, regarded as an infinite graph of degree $2n$ (so $z, z' \in L$ are adjacent iff $|z' - z| = 1$). An "interior point" of a subset $S \in L$ is a point all of whose neighbors are in S; these points constitute the "interior" of S, whose complement in S is the "boundary" ∂S of S. A function $u : S \to \mathbf{R}$ is *harmonic* if its value at each interior point $z \in S$ is the average of its values at the neighbors of z, and *subharmonic* if $u(z) \leq (2n)^{-1} \sum_{|z'-z|=1} u(z')$ for all interior $z \in S$. We similarly define (sub)harmonic functions on cL for any $c > 0$.

6. Suppose S is finite. Prove that any function $U : \partial S \to \mathbf{R}$ extends to a unique harmonic function $u : S \to \mathbf{R}$, which is nonnegative if U is.

7.* Suppose $K \in \mathbf{R}^n$ is a convex compact set and v is a harmonic function on some neighborhood of K. For $c > 0$ let $S_c = cL \cap K$, and let u_c be the harmonic function on S_c such that $u_c(z) = v(z)$ for all $z \in \partial S_c$ (this is well-defined, by the previous problem). Prove that there exists a constant A, depending only on K and u, such that $|u_c(z) - v(z)| \leq Ac^2$ for all $z \in S_c$. You may assume the theorem that v is automatically real-analytic; we didn't prove this in class for $n \geq 3$, but it follows easily from the Poisson integral representation of a harmonic function on a closed sphere in $\mathbf{R}^n$.

[This is the beginning of one approach to justifying the numerical approximation of solutions of the Dirichlet problem on $\mathbf{R}^n$ by discrete harmonic functions.]

This problem set is due Wednesday, November 12, at the beginning of class.