Math 213a: Complex analysis

Problem Set #6 (29 October 2003):

The Gamma function; univalent functions and normal families

1. [Gauss multiplication formula] Let n be a positive integer, and define

$$F(z) = \prod_{k=0}^{n-1} \Gamma\Big(\frac{z+k}{n}\Big).$$

   i) Show that $F(z)$ has the same poles as $\Gamma(z)$, and satisfies the functional equation $F(z+1) = zF(z)/n$.

   ii) This suggests that $F(z)$ should be proportional to $n^{-z}\Gamma(z)$. Prove that this is in fact the case, and determine the constant of proportionality.

2. Determine for each $n = 0, 1, 2, \ldots$ the residue of $\Gamma(z)$ at the pole $z = -n$. Use this to compute $\int_\gamma \Gamma(s)x^s\, ds$ for all $x > 0$, where γ is the contour $\{1 + it \mid t \in \mathbf{R}\}$.

3. Prove the integral formula

$$\int_{-\infty}^{\infty} |\Gamma(\sigma + it)|^2 a^{it}\, dt = 2\pi\Gamma(2\sigma)\frac{a^\sigma}{(1+a)^{2\sigma}}$$

   for all real and positive a, σ. For which complex a does this formula remain valid?

This result, together with the inversion formula for Fourier transforms, yields a closed form for the Fourier transform of $(\operatorname{sech} x)^{2\sigma}$; In particular, for $\sigma = 1/2$ we recover the formula for the Fourier transform of $\operatorname{sech}(x)$. For general $\sigma > 0$ we can also obtain the orthogonal polynomials for the weight function $|\Gamma(\sigma+it)|^2$ — that is, polynomials $P_n(t)$ of degree n in t such that

$$I(m, n) := \int_{-\infty}^{\infty} |\Gamma(\sigma + it)|^2 P_m(t)P_n(t)dt$$

vanishes unless $m = n$. One nice way is to use the coefficients in the generating function

$$(1 + ix)^{-\sigma-it}(1 - ix)^{-\sigma+it} = \sum_{n=0}^{\infty} P_n(t)x^n.$$

Can you determine $I(n, n)$, or more generally

$$\int_{-\infty}^{\infty} |\Gamma(\sigma + it)|^2 P_l(t)P_m(t)P_n(t)dt$$

for nonnegative integers l, m, n? These $P_n(t)$ are the symmetric Meixner-Pollaczek polynomials ("symmetric" because one can more generally describe the orthogonal polynomials for the weight function $|\Gamma(\sigma+it)|^2 e^{ct}$ for any constant c with $|c| < \pi/2$).

Back to complex analysis: the remaining problems concern normal families and univalent functions.

4. Let
$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$
be a univalent meromorphic function on the open unit disc $\Delta = \{|z| < 1\}$. Show that for each positive $r < 1$ the complement in $\mathbf{C}$ of $\{f(z) \colon |z| < r\}$ has area
$$\pi\left(\frac{1}{r^2} - \sum_{n=0}^{\infty} n|a_n|^2 r^{2n}\right),$$
and thus that the complement of $f(\Delta)$ has area
$$\pi\left(1 - \sum_{n=0}^{\infty} n|a_n|^2\right).$$
[Recall from the zeroth problem set that the area enclosed by a simple closed analytic arc γ is $(1/2i)\oint_\gamma \bar{z}\,dz$.] Conclude that the univalent functions $1/z + O(z)$ on Δ constitute a normal family.

5. Let $\mathcal{F}$ be the family of analytic functions $f(\cdot)$ on Δ such that f is univalent and normalized by $f(0) = 0$ and $f'(0) = 1$.
   i) Show that if $f \in \mathcal{F}$ then $\mathcal{F}$ also contains a function $g(\cdot)$ such that $(g(z))^2 = f(z^2)$ for all $z \in \Delta$.
   ii) Apply the result of Problem 4 to conclude that $\mathcal{F}$ is a normal family.

In particular it follows that for each n there is an upper bound B_n on the absolute value of the z^n coefficient in the Taylor expansion of any $f \in \mathcal{F}$. The Bieberbach conjecture, finally proved in the mid-1980's by L. de Branges, asserts that the least bound is n, and specifies all functions attaining this bound. The next problem gives an easy first step in this direction, and some more information about $\mathcal{F}$.

6. i) Show that $|f''(0)| \leq 4$ for all $f \in \mathcal{F}$, with equality if and only if $f(z) = z/(1 - cz)^2$ for some $c \in \mathbf{C}$ with $|c| = 1$. [Such f turn out to be the only functions attaining the Bieberbach bound for any n.]
   ii) Prove that for each $f \in \mathcal{F}$ there exists $w \in \mathbf{C}$ such that $w \notin f(\Delta)$ and $|w| \leq 1$.

This problem set is due Wednesday, November 5, at the beginning of class.