Math 213a: Complex analysis

Problem Set #5 (22 October 2003):

Series of analytic functions; partial fractions and product formula

1. i) [Ahlfors IV.3.1, Ex.2 (p.153)] Express $\sum_{n=-\infty}^{\infty} 1/(z^3 - n^3)$ in closed form.
   ii) [Seen in B. Amend's *Foxtrot* cartoon, 6.ii.1996] Evaluate in closed form: $\sum_{k=1}^{\infty}(-1)^{k+1}k^2/(k^3+1)$.

2. We showed that if f_n are analytic functions with $f_n \to f$ uniformly on compact subsets of Ω then f is analytic and $f' = \lim_{n\to\infty} f_n'$ uniformly on compacta. We noted that as it stands this result requires that $\Omega \subseteq \mathbf{C}$ and that f_n, f take values in $\mathbf{C}$.
   i) Give a definition for "uniform convergence on compacta" for sequences of maps between any two Riemann surfaces S, S', and show that your definition is reasonable in that it agrees with the usual definition for subsets of $\mathbf{C}$ and does not depend on any choices made in the definition (such as local coordiante patches).
   ii) Do the same for "$f' = \lim_{n\to\infty} f_n'$ uniformly on compacta".
   iii) Now extend "our" theorem to sequences of analytic maps from S to S'. [This should be the easy part.]

3. [Suggested by the remark in Ahlfors p.155] For $a_n \in \mathbf{C}$ such that $|a_n| < 1$ for each n, let P and S be respectively the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ and the infinite series $\sum_{n=1}^{\infty} a_n$.
   i) Assume $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Prove then P converges if and only if S converges.
   ii) Without the hypothesis on $\sum_{n=1}^{\infty} |a_n|^2$, show that there exist a_n for which P converges but S does not, and a_n for which P does not converge but S does. Can all a_n be real?

4. [Prelude to doubly periodic functions] Let A_0 be the ring of meromorphic functions f on $\mathbf{C}$ such that $f(z+1) = f(z)$ for all $z \in \mathbf{C}$ and there exists a limit $L = L(f) \in \mathbf{C}$ such that $f(z) \to L$ as $|\operatorname{Im}(z)| \to \infty$.
   i) Prove that there exists $M < \infty$ and poles b_ν ($\nu = 1, 2, \ldots, M$) of f such that every pole of f is $b_\nu + k$ for some unique $\nu \in \{1, 2, \ldots, M\}$ and $k \in \mathbf{Z}$. (Of course if f is analytic then we take $M = 0$ and interpret "$\{1, 2, \ldots, M\}$" as the empty set.)
   ii) Prove that $\sum_{\nu=1}^{M} \operatorname{Res}_{z=b_\nu} f(z)\,dz = 0$.
   iii) Prove that

$$f(z) = L(f) + \sum_{k\in\mathbf{Z}} \left[\sum_{\nu=1}^{M} P_\nu \left(\frac{1}{z - (b_\nu + k)} \right) \right],$$

   where $P_\nu(1/(z-b_\nu))$ is the principal part of f at b_ν, and the series converges uniformly in compact subsets of $\mathbf{C}$ (with the usual interpretation at $z = b_\nu$).

iv) Conclude that A_0 is the ring of rational functions of $e^{2\pi iz}$, say $g(e^{2\pi iz})$, that are regular at 0 and ∞ with $g(0) = g(\infty)$.

v) Let $A \supset A_0$ be the ring of meromorphic functions f on $\mathbf{C}$ such that $f(z+1) = f(z)$ for all $z \in \mathbf{C}$ and there exist limits $L_\pm = L_\pm(f) \in \mathbf{C}$ such that $f(z) \to L_+$ as $\mathrm{Im}(z) \to +\infty$ and $f(z) \to L_-$ as $\mathrm{Im}(z) \to -\infty$. Adapt the argument for (i–iv) to identify A with a suitable ring of rational functions of $e^{2\pi iz}$.

A Jensen-style inequality for analytic functions on a half-plane, and an application:

5. i) Let $f(z)$ be a bounded analytic function on the right half-plane $\mathrm{Re}(z) > 0$, with real zeros at x_k of multiplicity m_k. Show that either f vanishes identically or $\sum_{x_k>1} m_k/x_k$ converges.

ii) Let $\alpha(t)$ be a continuous complex-valued function on the interval $[0,1]$. Show that if $\int_0^1 t^{p_k}\alpha(t)dt = 0$ for distinct $p_k > 0$ with $\sum_k p_k^{-1} = \infty$ then $\alpha = 0$ identically.

In the special case $p_k = 1, 2, 3, \ldots$, this is an easy consequence of Weierstrass' Approximation Theorem, stating that any continuous function on a closed interval is a uniform limit of polynomials. [Consider $0 \le \int_0^1 (t-t^2)|\alpha(t)|^2\,dt = \int_0^1 \big((t-t^2)\bar\alpha(t)\big)\alpha(t)\,dt$, and approximate the $\bar\alpha(t)$ uniformly by a polynomial.] That Weierstrass result is generalized by a memorable theorem of Müntz: fix $0 = p_0 < p_1 < p_2 < p_3 < \ldots$; then *every continuous function on* $[0,1]$ *is a uniform limit of linear combinations of the powers* t^{p_k} *if and only if* $\sum_{k=1}^\infty p_k^{-1}$ *diverges.* Indeed, one approach to this theorem uses part (ii) above and related ideas from complex analysis.

This problem set is due Wednesday, October 29, at the beginning of class.