Math 213a: Complex analysis
Problem Set #4 (15 October 2003):
Contour integrals II

1. [Qualifying Exam, Spring 1997] For $b > 0$, compute $\int_0^\infty \log x \, dx/(x^2 + b^2)$.

2. Evaluate $\int_0^\pi \log|a + \cos(\theta)| \, d\theta$ as a function of $a \geq 0$. [As you might expect, the formula will depend on whether $a \leq 1$ or $a \geq 1$. Can you obtain this formula without using complex analysis?]

3. Prove that
$$\int_0^\infty \frac{\cos(mx)}{\cosh(\pi x)} \, e^{-mx^2} dx = \frac{1}{2} \, e^{-m/4}$$
for all $m > 0$. [From the Fall 1998 Qualifying Exam.] Can you evaluate any other such integrals this way (other than those obtained trivially from this formula by linear change of variable etc.)?

4. Let f be a nonconstant analytic function on a neighborhood of z_0, and let n be the multiplicity of the zero at z_0 of the function $f(z) - f(z_0)$. We have seen that, for any a sufficiently close to $f(z_0)$, the equation $f(z) = a$ has n solutions (counted with multiplicity) near z_0, call them $z_1, \ldots, z_n$ in some order. Prove that the coefficients of the polynomial $\prod_{j=1}^n (X - z_j)$ are analytic functions of a in that neighborhood of z_0. [These coefficients are (up to sign) the elementary symmetric functions in the z_j. For $n = 1$ the claim is equivalent to the existence of an analytic inverse function. To prove it in general, show that $\sum_{j=1}^n z_j^k$ is an analytic function of a for each $k = 1, 2, 3, \ldots$.]

5. For any two disjoint circles C_1, C_2 in the Riemann sphere $\mathbf{P}^1(\mathbf{C})$, define $I(C_1, C_2)$ as the value of the double path integral with respect to arc length:
$$I(C_1, C_2) := \oint_{C_1} \oint_{C_2} \frac{|dz_1| \, |dz_2|}{|z_1 - z_2|^2} \, .$$
This may be an improper integral if either C_1 or C_2 passes thru ∞ (i.e., is a straight line in $\mathbf{C}$), but even in that case the integral clearly converges.
i) Show that $I(C_1, C_2)$ is $\mathrm{PGL}_2(\mathbf{C})$-invariant; that is, that $I(\phi C_1, \phi C_2) = I(C_1, C_2)$ for any fractional linear transformation $\phi : \mathbf{P}^1(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$.
ii) Determine $I(C_1, C_2)$ as a function of the radii R_1, R_2 of the circles and the distance d between their centers. [Check: when $d = 0$, so C_1, C_2 are concentric, you should obtain
$$I(C_1, C_2) = 4\pi^2 \frac{R_1 R_2}{|R_1^2 - R_2^2|} \, ;$$
more generally, the formula should diverge when $d = |R_1 - R_2|$.] What is $I(C_1, C_2)$ if $C_1 = \mathbf{R}$?
To be continued...

This problem set is due Wednesday, October 22, at the beginning of class.