Math 213a: Complex analysis
Problem Set #3 (8 October 2003):
Differentials, residues, Laurent series

1. Let $A, B \in \mathbf{C}[z]$ be polynomials such that B has distinct roots $z_1, \ldots, z_n$. Let ω be the differential $(A(z)/B(z))\,dz$ on $\mathbf{C} - \{z_1, \ldots, z_n\}$. Show that the residue of ω at each z_j is $A(z_j)/B'(z_j)$. Conclude that if $\deg(A) \leq \deg(B) - 2$ then $\sum_{j=1}^{n} A(z_j)/B'(z_j) = 0$. What happens if $\deg(A) = \deg(B) - 1$?

Since these identities are purely algebraic results, they must hold for polynomials over any algebraically closed field; but — as with invariance of the residue under coordinate change — a direct algebraic proof, though possible, is harder and less revealing.

2. Determine for any entire function f the residue at $z = 0$ of the differential $f(\cot(z))\,dz$. In particular, what is the residue at the origin of $\sin(\cot(z))\,dz$?

As far as I know, the other odd-order coefficients of the Laurent expansion of $\sin(\cot(z))$ about $z = 0$ are not known in closed form.

3. Fix a complex number t. Define an analytic function $z(\cdot)$ on the open unit disc $|w| < 1$ by $z(w) = w(1 - w)^t$. Here "$(1 - w)^t$" is $\exp(t \log(1 - w))$, using the branch of $\log(1 - w)$ that vanishes at $w = 0$. Since $z'(0) = 1 \neq 0$, this function has a local inverse: an analytic function $w(\cdot)$ on a neighborhood of the origin such that $z = w(z)\big(1 - w(z)\big)^t$.

   i) Determine the coefficients a_n in the power-series expansion $w(z) = \sum_{n=0}^{\infty} a_n z^n$.

   ii) For $t = \pm 1$ the function $w(z)$ may be evaluated in closed form and expanded in a power series directly. Check that in these two cases your answer agrees with your formula from (i).

   iii) For any integer $\beta > 0$ or $\beta < 0$, determine the Taylor or Laurent expansion of w^{β} in powers of z. Can you make sense of this for arbitrary complex β?

   iv) From your answer to (i), recover Abel's formula for the coefficients of the solution of $ye^{-y} = z$. What are the coefficients of y^{β}?

4. [Cf. Ahlfors, p.148, Ex.1] For any s_1, s_2 such that $0 < s_1 < s_2$, let Ω_1 be the complement of the closed square $\{x + iy : |x|, |y| \leq s_1\}$, and let Ω_2 be the open square $\{x + iy : |x|, |y| < s_2\}$. Show that any analytic function on the region $\Omega_1 \cap \Omega_2$ can be written as $f_1 + f_2$ where each f_j is an analytic function on the corresponding Ω_j and $f_1(z) \to 0$ as $|z| \to \infty$.

This problem set is due Wendesday, October 15, at the beginning of class.