

**Math 213a: Complex analysis**  
Problem Set #2 (29 September 2003):  
Analytic functions, cont'd; Cauchy applications, I

Polynomial and rational functions:

1. Show directly that the functions

$$f(x, y) = x/(x^2 + y^2) \quad \text{and} \quad f(x) = \log(x^2 + y^2)$$

on  $\mathbf{R}^2 - \{(0, 0)\}$  satisfy the Laplace equation  $\Delta f = 0$ , and find their harmonic conjugates if they exist.

2. Let  $P(x, y) = \sum_{k=0}^d a_k x^k y^{d-k}$  ( $a_k \in \mathbf{C}$ ,  $x, y \in \mathbf{R}$ ) be a homogeneous polynomial of degree  $d$ . Show directly that  $P$  is an analytic function of  $z = x + iy$  if and only if  $P = a_d z^d$ . Conclude that a polynomial in  $\text{Re}(z)$  and  $\text{Im}(z)$  is an analytic function of  $z$  iff it is a polynomial in  $z$ .
- 3.\* i) Show that two polynomials  $P(z), Q(z)$  satisfy  $P'Q = PQ'$  if and only if  $aP + bQ = 0$  for some complex  $a, b$ , not both zero.  
ii) (“ABC conjecture” for polynomials.) Let  $A(z), B(z), C(z)$  be three polynomials without common factors such that  $A + B + C = 0$ . Let  $k = \max(\deg(A), \deg(B), \deg(C))$ , and assume that  $k > 0$ . Show that the number of zeros of  $ABC$ , counted without multiplicity, is at least  $k + 1$ . (Hint: consider  $A'B - AB' = B'C - BC'$ .)  
iii) (“Fermat’s Last Theorem” for polynomials.) Show that if  $A(z), B(z), C(z)$  are polynomials satisfying  $A^n + B^n = C^n$  for some integer  $n \geq 3$  then  $A, B, C$  are of the form  $aP, bP, cP$  for some polynomial  $P(z)$  and complex numbers  $a, b, c$  with  $a^n + b^n = c^n$ . Give an example showing that this result is no longer true for  $n = 2$ .

Maximum principle(s):

- 4.\* [Blaschke; Ahlfors 2.1, Ex. 4 (p.46)] What is the general form of a rational function that has absolute value 1 on the unit circle  $|z|=1$ ? In particular, how are the zeros and poles related to each other?
5. Let  $f$  be a nonconstant analytic function on (some region containing) the closed unit disc  $D = \{z : |z| \leq 1\}$ , and  $a \in D$  a complex number at which  $|f(z)|$  attains its maximum in  $D$ . Prove that  $f'(a) \neq 0$ . Construct a counterexample to show that this is no longer true if  $D$  is replaced by the square  $\{x + iy : |x| \leq 1, |y| \leq 1\}$ .
6. Let  $f$  be a real-valued  $\mathcal{C}^2$  function on an open set in  $\mathbf{R}^n$  containing the closed ball  $B = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \leq r\}$  relative to the standard (Euclidean) metric on  $\mathbf{R}^n$ . Suppose  $\Delta f(\mathbf{x}) = 0$  for  $\mathbf{x} \in B$ . Prove that  $f(\mathbf{x}_0)$  equals the average of  $f(\mathbf{x})$  over the sphere  $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| = r\}$ . [When  $n = 2$  this is a consequence of Cauchy’s theorem for an analytic function whose real part is  $f$ .] Conclude that harmonic functions on regions in  $\mathbf{R}^n$  satisfy the maximum principle.

This problem set is due Wednesday, October 8, at the beginning of class.