Math 213a: Complex analysis
Problem Set #10 (8 December 2003):
Doubly-periodic functions

1. Let $L \subset \mathbf{C}$ be a lattice, F its field of doubly-periodic functions, z_j (for $j = 1, \ldots, n$) distinct points in $\mathbf{C}/L$, and $P_j \in P[X]$ polynomials with each $P_j(0) = 0$. We saw that if there exists $f \in F$ with poles only at the z_j and principal part $P_j(1/(z - z_j))$ at each z_j then $\sum_{j=1}^{n} P_j'(0) = 0$. Show that conversely if this one condition is satisfied then such f exists.

The next problem illustrates the "multiplicative" approach to elliptic functions. Fix $q \in \mathbf{C}^*$ with $|q| = 1$. We saw that if $q = \exp(\omega_1/\omega_2)$ then the doubly-periodic functions for the lattice $\mathbf{Z}\omega_1 \oplus \mathbf{Z}\omega_2$ amount to meromorphic functions on $\mathbf{C}^*/q^{\mathbf{Z}}$, that is, meromorphic functions f on $\mathbf{C}^*$ satisfying the identity $f(qz) = f(z)$. Suppose more generally that f satisfies the identity $f(qz) = cz^n f(z)$ for some $c \in \mathbf{C}^*$ and $n \in \mathbf{Z}$. While f no longer descends to a function on $\mathbf{C}^*/q^{\mathbf{Z}}$, its zeros and poles are still invariant under multiplication by q, so we may speak of zeros and poles of f in $\mathbf{C}^*/q^{\mathbf{Z}}$, each with a unique representative in a "period annulus" $\{z : |q| < |z| \le 1\}$.

2. i) Suppose these zeros and poles are z_j $(j = 1, \ldots, d)$ and z_j' $(j = 1, \ldots, d')$, listed with multiplicity. Determine $d - d'$ and

$$\prod_{j=1}^{d} z_j \Big/ \prod_{j=1}^{d'} z_j' \,.$$

ii) Show that $g(z) := \prod_{k=0}^{\infty}(1 - q^k z)\prod_{k=1}^{\infty}(1 - q^k z^{-1})$ is an analytic function on $\mathbf{C}^*$ that satisfies the identity $g(z) = -z\, g(qz)$. Check that this is consistent with your answer to (i).

iii) Show that the function

$$\sum_{m=-\infty}^{\infty} q^{(m^2-m)/2}(-z)^m = 1 - z + q(z^2 - z^{-1}) + q^3(z^{-2} - z^3) + q^6(z^{-4} - z^3) + \cdots$$

of z (an example of a theta function) satisfies the same identity as g, and deduce that this function equals cg for some $c \in \mathbf{C}^*$ depending only on q.

iv) We shall see that $c(q) = \prod_{k=1}^{\infty}(1 - q^k)$, which is equivalent to the *Jacobi triple product* identity

$$\sum_{m=\infty}^{\infty} (-1)^m q^{(m^2-m)/2} z^m = \prod_{k=0}^{\infty}(1 - q^k) \prod_{k=0}^{\infty}(1 - q^k z) \prod_{k=1}^{\infty}(1 - q^k z^{-1}).$$

Assuming this, prove that

$$c(q) = 1 - q - q^2 + q^5 + q^7 - q^{12} - + + - \cdots = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m+1)/2}$$

(Euler's celebrated pentagonal-number identity), and that

$$c(q)^3 = 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - + \cdots = \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{(m^2+m)/2}.$$

More about the Weierstrass $\wp$-function and its differential equation:

3. Prove that

$$\sum\sum_{m,n\in\mathbf{Z}}{}' \frac{1}{(m+in)^4} = \frac{B(1/4,1/2)^4}{240}$$

(where as usual the "$\prime$" in "$\sum\sum'$" indicates that the $(m,n) = (0,0)$ term is excluded from the sum). What is

$$\sum\sum_{m,n\in\mathbf{Z}}{}' \frac{1}{(m+\rho n)^6}$$

where $\rho = e^{2\pi i/3}$ is a cube root of unity? (Your answer should agree with the numerical value $5.8630\cdots$.)

4. i) Suppose L and M are lattices in $\mathbf{C}$ with $L \subseteq M$. Prove that for all $z \notin M$ we have $\wp'_M(z) = \sum_w \wp'_L(z+w)$, where w ranges over (a set of representatives of) M/L.
   ii) [2-isogeny] In the special case $[M : L] = 2$, express $\wp_M(z)$ and $\wp'_M(z)$ explicitly as rational functions of $\wp_L(z)$, $\wp'_L(z)$, and the root $\wp_L(w)$ of P_L, where $w \in M - L$. Also, determine P_M in terms of $\wp_L(w)$ and the coefficients of P_L. [The resulting identity connecting the elliptic integrals $\int dx/\sqrt{P_L(x)}$ and $\int du/\sqrt{P_M(u)}$ is known classically as "Landen's transformation".]
   iii) For at least one of the lattices $L = \mathbf{Z} \oplus \mathbf{Z}\sqrt{-2}$, $L = \mathbf{Z} \oplus \mathbf{Z}\sqrt{-3}$, and $L = \mathbf{Z} \oplus \mathbf{Z}\sqrt{-4} = \mathbf{Z} \oplus 2i\mathbf{Z}$, use the formulas of (ii) to determine a^3/b^2 where a, b are the coefficients of $P_L(X) = 4X^3 + aX + b$. [These are rational numbers whose approximate numerical values are $a^3/b^2 = -34.44$, -27.89, -27.16 respectively. It is known that in general a^3/b^2 is an algebraic number for $L = \mathbf{Z} \oplus \mathbf{Z}\sqrt{-r}$ ($r \in \mathbf{Q}$), or more generally $L = \mathbf{Z} \oplus \mathbf{Z}\tau$ for $\tau \in \mathcal{H}$ any quadratic irrational. These algebraic numbers a^3/b^2 are in effect one of the topics of the theory of "complex multiplication".]

This final problem set is due Monday, December 15, at the beginning of class.