**Math 213a: Complex analysis**
Problem Set #1 (22 September 2003):
Inversions and $\mathrm{PSL}_2(\mathbf{C})$

1. [Inversion in higher-dimensional Euclidean space] In Euclidean space $\mathbf{R}^n$ of arbitrary dimension, **inversion** $I$ with respect to the sphere of radius $R$ centered at the origin is the map taking any vector $\mathbf{x} \in \mathbf{R}^n$ to $I(\mathbf{x}) := (R^2/|\mathbf{x}|^2)\mathbf{x}$. We take $I(\mathbf{0}) = \infty$ and $I(\infty) = \mathbf{0}$, as in the case $n = 2$.
   i) Show that $I$ preserves hyperspheres (sets of the form $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| = r\}$), with the convention that a hyperplane is a degenerate special case of a hypersphere, namely a hypersphere passing through $\infty$.
   ii*) What is the group generated by inversions $I$ and the isometries of $\mathbf{R}^n$?

2. Show that the "stereographic projection" from the north pole of a sphere in $\mathbf{R}^3$ to $\mathbf{C}$ can be realized as inversion with respect to a sphere centered at that north pole. Noting that a circle in $\mathbf{R}^3$ can be obtained as the intersection of two spheres and thus that circles are preserved by inversion, conclude that circles or lines in $\mathbf{C}$ correspond to circles on the sphere.

3. [More about the group $\mathrm{PGL}_2(\mathbf{C})$ acting on the extended complex plane $\mathbf{P}^1(\mathbf{C})$]
   i) Show that any two circles or lines in $\mathbf{P}^1(\mathbf{C})$ are equivalent under the action of $\mathrm{PGL}_2(\mathbf{C})$.
   ii) Show that the subgroup of $\mathrm{PGL}_2(\mathbf{C})$ preserving a given circle is isomorphic to $\mathrm{PGL}_2(\mathbf{R})$.

4. i) Show that any fractional linear transformation $f$ such that $f(w) = w'$ and $f(w') = w$ for two *distinct* complex numbers $w, w'$ is an involution (i.e., satisfies $f(f(z)) = z$ for all $z$).
   ii) Conclude that for any four distinct complex numbers $z_1, z_2, z_3, z_4$ there exists a unique fractional linear transformation $f$ such that $f(z_1) = z_2$, $f(z_2) = z_1$, $f(z_3) = z_4$, and $f(z_4) = z_3$.

5. Determine the images under the fractional linear transformations

$$\alpha : z \mapsto \frac{1+z}{1-z}, \qquad \beta : z \mapsto iz$$

   of the real axis, the imaginary axis, and the unit circle. Show that $\alpha$ and $\beta$ generate a finite subgroup of $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})/\{\pm 1\}$, and interpret its action on $\mathbf{P}^1(\mathbf{C})$ in terms of the geometry of the Riemann sphere.

6*. The **Hermitian transpose** of an $n \times n$ complex matrix $(a_{ij})_{i,j=1}^n$ is the complex conjugate $(\bar{a}_{ji})$ of its (ordinary) transpose. An invertible $n \times n$ matrix is said to be **unitary** if its inverse is equal to its Hermitian transpose.
   i) Show that the unitary $n \times n$ complex matrices form a group.
   ii) Show that an $n \times n$ matrix is unitary if and only if it represents a linear transformation: $\mathbf{C}^n \to \mathbf{C}^n$ preserving the norm $\|(z_1, \ldots, z_n)\| = (\sum_{i=1}^n |z_i|^2)^{1/2}$.
   iii) Show that a $2 \times 2$ unitary matrix of determinant 1 is a matrix $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$ for some $a, b \in \mathbf{C}$ with $|a|^2 + |b|^2 = 1$. (The set of such $(a, b)$ is the unit hypersphere in $\mathbf{R}^4$; thus this hypersphere is endowed with a noncommutative group structure.)
   iv) Identify $\mathbf{P}^1(\mathbf{C})$ with the unit sphere in $\mathbf{R}^3$ by the usual projection, taking the unit circle of $\mathbf{C}$ to the equator of the sphere. Show that the group of orientation-preserving isometries of this sphere is then identified with the quotient of the group $\mathrm{SU}_2(\mathbf{C})$ of $2 \times 2$ unitary matrices of determinant 1 by its normal subgroup $\{\pm \mathbf{1}\}$.

This problem set is due Monday, September 29, at the beginning of class.