Math 213a: Complex analysis
Problem Set #0 (15 September 2003)

A few straightforward exercises in thinking about complex numbers and functions:

1. (i) Let z be the complex number $(1+\sqrt{7}\,i)/2$. Verify that $z^4 = (1-3\sqrt{7}\,i)/2$, and thus find another complex number w such that $w^4 + z^4 = 1$.
   (ii) Find two complex numbers w and z such that $w^n + z^n + 1 = 0$ for every positive integer n not divisible by 3.

2. (i) Give necessary and sufficient conditions on complex numbers z_1, z_2, z_3 to be vertices of an equilateral triangle traversed in the "positive" (counterclockwise) direction. [Hint: the algebra in this exercise will be much easier and more pleasant if you use the **R**-basis $\{1, \omega\}$ for **C**, rather than the usual $\{1, i\}$, where $\omega := (-1+\sqrt{3}\,i)/2$ is a cube root of unity.]
   (ii) Let A, B, C be any distinct points in the Euclidean plane, and A', B', C' the points such that triangles $A'BC$, $AB'C$, ABC' are equilateral in the same orientation. Prove that — with what one exception? — the line segments AA', BB', CC' have the same length, and make $60°$ angles with each other (extended if necessary).

[It is also known that these three lines are concurrent; when the equilateral triangles are external to $\triangle ABC$, the point of intersection is known as the *Fermat point* of $\triangle ABC$ — yes, the same Fermat that Problem 1 should bring to mind. I do not ask that you prove the concurrence, which cannot be easily obtained using the arithmetic of complex numbers.]

3. Let $a \neq 1$ be a positive real number. Show that the image of the **unit circle** (the set of complex numbers $z = x + iy$ such that $x^2 + y^2 = 1$) under the function $z \mapsto z + a/z$ is an ellipse. What happens when $a = 1$, or a is a complex number?

4. Show that the area A enclosed by a simple closed curve C in the complex plane is given by the **contour integral**

$$A = \frac{1}{2i} \oint_C \bar{z}\, dz.$$

   (An integral $\int f\,dz$ over some path in the complex plane is interpreted as the line integral $\int f\,dx + i\int f\,dy$ where x and y are the real and imaginary parts of $z = x + iy$ — imagine that $dz = dx + i\,dy$. The integrand $\bar{z}$ is the **complex conjugate** $x - iy$ of $z = x + iy$. We'll soon have a lot more to say about such integrals.)

This problem set is due Monday, September 22, at the beginning of class.