

Math 165: Designs and Groups

Homework Assignment #1 (28 January 2025):

More about Π_2 , the Petersen graph, and all that

This problem set is due midnight between Monday and Tuesday, February 3 and 4.

1. Let $X = \mathbf{Z}/7\mathbf{Z}$ and let $\mathcal{B}$ be the family of 3-element subsets of X of the form $\{a+1, a+2, a+4\}$ ($a = 0, 1, 2, \dots, 6$). Show that $(X, \mathcal{B})$ is a 2-(7,3,1) design: i) combinatorially, ii) by identifying it with our picture of the finite projective plane Π_2 . Conclude that Π_2 has an automorphism of order 7.
[The “order” of an automorphism g , or more generally of an element of any group, is the smallest positive integer c such that the c -th power (iterate) g^c is the identity. If there is no such c then g is said to be of “infinite order”, but this cannot happen in a finite group (why?).]
2. More generally let $X = \mathbf{Z}/N\mathbf{Z}$ and let $\mathcal{B}$ be the family of translates of some subset $B_0 \subset X$ (in the previous problem N was 7 and B_0 could be $\{1, 2, 4\}$). Under what conditions is $(X, \mathcal{B})$ a 2-(N, b, λ) design for some λ , and what is that λ as a function of N and the size of B_0 ? Determine the next N after 7 that allows $\lambda = 1$, and find a subset B_0 producing such a Steiner system.
[This Steiner system turns out to be the unique finite projective plane of order 3. In general such B_0 are “perfect difference sets mod N ”.]
3. Show that taking $N = 7$ and $B_0 = \{0, 3, 5, 6\}$ in the last problem gives rise to a 2-(7,4,2) design. How is this design related to Π_2 ? What general rule does this suggest?
4. i) Let Z be any subset of the set X of points of Π_2 . Show that either Z or its complement, but not both, contains a line of Π_2 .
ii) Y was it a good idea to use the letter Y as one of the seven letters in the picture of Π_2 on our first handout?
[You might have fun looking for other such collections of seven words, in English or some other language.]
5. Show that the graph obtained from the Petersen graph by deleting any one of its ten vertices (together with all edges through that vertex) cannot be drawn in the plane without extraneous intersections. It follows *a fortiori* that the Petersen graph itself cannot so be drawn. [Hints: Use Euler’s formula $V - E + F = 2$. It might help that the ten vertices are all equivalent under graph automorphisms.]
6. Verify that the Petersen graph may be obtained as follows: start with the 20 vertices and 30 edges of the regular dodecahedron (the “Platonic solid”¹ bounded by twelve pentagons), and identify opposite vertices and edges. Show that this yields 60 of the automorphisms of the Petersen graph (which will turn out to constitute exactly half of its automorphism group).

¹Apparently mentioned, but not mathematically investigated, by Plato. This kind of misattribution is alas more common than it ought to be. See also “Stigler’s law of eponymy”.