Math 155: Designs and Groups

Homework Assignment #7 (13 April 2016):
About the representation $A_7 \hookrightarrow A_8 \cong \mathrm{GL}_4(\mathbf{F}_2)$;
finite subgroups of $\mathrm{PSL}_2(F)$ and related matters

This problem set is due Wednesday, April 20 in class.

Since $\mathrm{GL}_4(\mathbf{F}_2) \cong \mathrm{A}_8$ there is an index-8 subgroup of $\mathrm{GL}_4(\mathbf{F}_2)$ isomorphic with A_7. In other words, A_7 acts on the 15 nonzero vectors in $\mathbf{F}_2^4$. It turns out that the action is doubly transitive. In particular the point stabilizer has size $\#(A_7)/15 = 168$, which is suggestive…

1. Let Σ be an unstructured 7-element set. A "Π_2-structure" is a choice of 2-(7,3,1) design of subsets of Σ (i.e. an identification of Σ with the points of Π_2). Show that there are 30 such structures, and that the action of A_7 on them has two orbits of size 15. Give a combinatorial definition, similar to what we did for hyperovals and subplanes of Π_4, of an equivalence relation on the Π_2-structures for which the equivalence classes are the two A_7 orbits; and prove directly that it is an equivalence relation.

2. Let P be one of the orbits. Since we expect to identify P with the nonzero vectors of $\mathbf{F}_2^4$, there should be a distinguished set of three-element subsets of P, namely the sets of nonzero vectors in 2-dimensional subsets of $\mathbf{F}_2^4$, which are the blocks of a $(2, 3, 15)$ Steiner system. Give a combinatorial construction of such a Steiner system of subsets of P. Use this to provisionally define the structure of an $\mathbf{F}_2$ vector space on $P \cup \{0\}$. What must you check to verify that it works?

This can be used to give an alternative approach to the isomorphism $\mathrm{GL}_4(\mathbf{F}_2) \cong A_8$.

In problems 1,2 we showed that the action of A_7 on the 30 Π_2-structures on Σ has two orbits of size 15, each of which may be identified with the set of nonzero vectors in a 4-dimensional vector space over $\mathbf{F}_2$. Let V, W be these two spaces. So far we concerned ourselves with one of these; we now explore their interrelationship.

3. If P, Q are Π_2 structures in different A_7 orbits, show that they have either no lines in common or three lines through a single point. Moreover, given P and a point $p \in \Sigma$ there is a unique Q sharing with P all three lines through p.

   For $P \in V$, $Q \in W$ we now define $\langle P, Q\rangle \in \mathbf{F}_2$ as follows: If either P or Q is zero, $\langle P, Q\rangle = 0$; else $\langle P, Q\rangle$ is 0 if the Π_2-structures P, Q share three lines, and 1 if they have no lines in common. Show that for each nonzero $Q \in W$ the set $\{P \in V : \langle P, Q\rangle = 0\}$ is a subspace of V of codimension 1. Conclude that $\langle \cdot, \cdot \rangle$ is a nondegenerate pairing, so naturally identifies each of V, W with the other's dual.

About triangle groups etc.:

4. Let G be the group with the following presentation by generators and relations:

$$G = \langle b, c, d \mid b^2 = c^4 = d^4 = bcd = 1 \rangle.$$

Construct the Cayley graph of G with respect to $\{c, d\}$, and use this to identify G with a subgroup of $\mathrm{AGL}_2(\mathbf{R})$.

[There is a similar description for the groups constructed in the same way with the exponents $(2, 4, 4)$ replaced by the other solutions $(3, 3, 3)$ and $(2, 3, 6)$ of $1/e_1 + 1/e_2 + 1/e_3 = 1$, but it's a bit harder to describe those because we have more experience with square grids than we do with triangular or hexagonal ones.]

5. i) Show that the image in $\mathrm{PGL}_2(F)$ of a 2×2 matrix $A \in \mathrm{GL}_2(F)$ has order 4 if and only if $\mathrm{tr}(A)^2 = 2 \det(A)$ but $\mathrm{tr}(A) \neq 0$. Use this to prove that if F is finite then $\mathrm{PGL}_2(F)$ contains S_4 if and only if F is not of characteristic 2.

   ii) Prove on the other hand that if $\mathrm{PSL}_2(F)$ does not contain A_5 for some field F then $\mathrm{PGL}_2(F)$ does not contain A_5 either. (Hint: you don't need to figure out the analogue of part (i) for elements of order 5.)

6. Prove that $\mathrm{PGL}_2(\mathbf{R})$ does not contain A_4 (and thus does not contain S_4 or A_5 either, even though it certainly has elements of order n for any n).

7. Check directly that the centralizer of every non-identity element of A_4 or A_5 is abelian. Show however that this does not hold for S_4. Why does this <u>not</u> indicate a flaw in the proof of the classification of finite subgroups of $\mathrm{PSL}_2(F)$?