

Math 155: Designs and Groups

Homework Assignment #6 (21 March 2016):
Some finite group theory

This short problem set is due Monday, March 28 in class.

1. Let G be a finite group and H a (necessarily) normal subgroup of index 2 in G . Prove that, of the G -conjugacy classes, each class disjoint from H is permuted transitively under conjugation by elements of H , while each class contained in H is either permuted transitively (and thus constitutes a conjugacy class of H) or splits into two H -conjugacy classes of equal size. For $(G, H) = (S_n, A_n)$, which cycle structures (i.e. which (a_1, a_2, \dots, a_n) with $\sum_i ia_i = n$) correspond to split conjugacy classes?
[Hint: Exercise 3.22 (page 51) of Rotman may help.]
2. Verify assertions (1), (2), (3) on p.22 of the Cameron-van Lint textbook. That is, show that the group-theoretical partition of the 168 (hyper)ovals in Π_4 into three equal classes of 56 has the following combinatorial description: two ovals O, O' are in the same class iff $\#(O \cap O')$ is even. Do the same for the $3 \cdot 120 = 360$ Baer subplanes S (this time $S \sim S'$ iff $\#(S \cap S')$ is odd), and show that the parity of $\#(O \cap S)$ depends only on the classes of O, S , and given O or S is even for exactly one class of S 's or O 's. [There are various symmetries of Π_4 , ovals, and subplanes that you can exploit to greatly reduce the number and complexity of cases you'll need to check.]
3. i) Find a 24-element subgroup G of $\text{PGL}_2(\mathbf{F}_5)$ and use its existence to obtain a different proof of the isomorphism $\text{PGL}_2(\mathbf{F}_5) \cong S_5$.
ii) Find a 24-element subgroup G of $\text{PSL}_2(\mathbf{F}_7)$ to obtain an embedding of $\text{PSL}_2(\mathbf{F}_7)$ into A_7 .
[Hint: S_4 has presentation $\langle \sigma, \tau \mid \sigma^2 = \tau^3 = (\sigma\tau)^4 = 1 \rangle$.]