Math 155: Designs and Groups

Homework Assignment #2 (1 February 2016):
More about designs

This problem set is due Wednesday, Feb.10 in class.

1. i) In Theorem 1.20 (page 6), suppose that $\mathcal{D}$ is not just a 2-design but even a 3-design. Give a formula for

$$\sum_i i^3 n_i = \sum_{B' \neq B} \#(B \cap B')^3$$

analogous to what we found for $\sum_i i n_i$ and $\sum_i i^2 n_i$. (Note that in the $\sum_i i^2 n_i$ formula λ must now be replaced by λ_2.)

   ii) Suppose now that $\mathcal{D}$ is a 3-$(22, 6, 1)$ design. Show that, for any block B' other than B, the intersection $B \cap B'$ is either empty or contains exactly 2 points. How many B' are disjoint from B? (You may be able to do this without part (i), using only the formulas from Theorem 1.20. At the end of this problem set we shall see that there exists a 3-$(22, 6, 1)$ design; later in the semester we shall prove that it is unique up to isomorphism, with an interesting automorphism group that contains the sporadic Mathieu group M_{22} with index 2.)

2. Prove Proposition 1.39 (the complement of a t-design is a t-design) for $t = 2$ using the incidence matrix and verify in that case that $\overline{\lambda} = b - 2r + \lambda$ [formula (1.38)].

3. For which q, n does the square 2-design of points and hyperplanes in $\mathbf{F}_q\mathbf{P}^n$ have an even number of points? Verify that the criterion of Bruck-Ryser-Chowla (Thm. 1.21a) holds for these designs.

4. [CvL p.25 #2] The Kronecker product or tensor product of matrices $A = (a_{ij})$ and $B = (b_{kl})$ is the matrix $A \otimes B$ with $((i, k), (j, l))$ entry $a_{ij} b_{kl}$. Prove that the Kronecker product of Hadamard matrices of orders n_1 and n_2 is a Hadamard matrix of order $n_1 n_2$. Prove also that the Kronecker product of a number of copies of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is a Sylvester matrix.

5. [CvL p.25 #3, extended] Suppose that H is a Hadamard matrix of order $n > 1$ with constant row and column sums d. Prove that $n = d^2$ with d even. Prove also that $(H + J)/2$ [i.e. H with its -1 entries replaced by zeros] is the incidence matrix of a 2-$(4u^2, 2u^2 + u, u^2 + u)$ design where $d = 2u$ (NB: d may be positive or negative). Show that the Kronecker product of two such H's is again of the same form, and use this to construct a square 2-(16,6,2) design. is a $(t + 1)$-design extending $\mathcal{D}$.

6. Prove that the square 2-design coming from a perfect difference set (see problem 2 of the previous homework set) is self-dual and in fact admits a polarity.

7. [CvL p.26 #8a, extended] Prove that the Paley square 2-(11,5,2) design is the unique design with these parameters up to isomorphism. How large is its automorphism group?

8. [CvL p.26 #8b; Assmus-Mezzaroba-Salwach 1977] Let X be the set of points *and* blocks of the 2-(11,5,2) design. Let $\mathcal{B}$ be the family of subsets of X of the following types:

   - a point and the five blocks containing it;
   - a block and the five points contained in it;
   - an oval and its tangents.

   Prove that $(X, \mathcal{B})$ is a 3-(22,6,1) design (a.k.a. $S(3, 6, 22)$ Steiner system).