Math 122: Algebra I, Fall 2023

Homework Assignment #9 (2 November 2023):
Simplicity of A_n ($n > 5$); Sylow's theorem(s)

> Q: How do I prove that every group of order 4 is simple?
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> A: [. . .] Case 1: G is isomorphic to Z4 (the cyclic group of order 4):
> In this case, G is generated by a single element a such that a^4 = e (the identity element). Since G is cyclic, the subgroup generated by a is e, a, a^2, a^3. Any proper subgroup of G would have order less than 4, and therefore, it cannot be normal in G. Thus, G is simple. [. . .]
>
> —Asher Auel (Q) and ChatGPT (A)
> (https://math.dartmouth.edu/~auel/courses/71f23/docs/AI_simple_groups_order_4.pdf)
>
> So we can't yet rely on ChatGPT to solve simple group theory problems.

This problem set is due Wednesday, November 8 at midnight.

Variations and application of the simplicity of A_n for $n \geq 5$:

In the first two problems we prove that for $n \geq 5$ the only normal subgroups of S_n are $\{1\}$, A_n, and S_n itself. Call this claim $\mathtt{C}(n)$.

1. [Induction step] Suppose for some $n > 5$ we already know that $\mathtt{C}(n-1)$ is true. Adapt our proof of the simplicity of A_n to deduce that $\mathtt{C}(n)$ is true as well. [The end is somewhat easier because S_n is generated by the conjugacy class of simple transpositions, and the product of any two simple transpositions has a fixed point once $n > 4$, so there's no special case of $n = 6$ to worry about.]

2. [Proof of $\mathtt{C}(5)$] We know already that the even conjugacy classes in S_5, namely the classes of the identity, 3-cycles, double transpositions, and 5-cycles (partitions 11111, 113, 122, 5) have sizes $1, 20, 15, 24$ respectively. Prove that the odd classes (partitions 1112, 14, 23 — see table at the top of D&F page 127) have sizes $10, 30, 20$ respectively. Use this to deduce $\mathtt{C}(5)$ and complete the proof of $\mathtt{C}(n)$ for all $n \geq 5$. [Hint: the calculation in the proof of $\mathtt{C}(5)$ will be made easier by using a computer algebra system to expand $x(1 + x^{20})(1 + x^{15})(1 + x^{24})(1 + x^{10})(1 + x^{30})(1 + x^{20})$ — do you see why?]

This is D&F Exercise 4.6, made easier by spelling out some of the steps. (NB $\mathtt{C}(n)$ happens to be true also for $n \leq 3$ but it is false for $n = 4$.)

An application of the simplicity of A_n for $n \geq 5$:

3. [D&F 4.6 Exercise 1, extended] Prove that if $n \geq 5$ then A_n does hot have a proper subgroup of index $< n$. Deduce that any action of A_n on a set of size less than n is trivial. What happens for $n = 4$?

See Exercise 3 for the corresponding statements for S_n.

Uses and examples of the Sylow theorem(s):

4. [adapted from D&F 4.5, Exercise 1] Let G be a finite group, p a prime factor of $|G|$, and $P \in Syl_p(G)$. Prove that if H is a group with $P \leq H \leq G$ then $P \in Syl_p(H)$. Prove that conversely any p-Sylow subgroup of H is also a p-Sylow subgroup of G if and only if $|G : H|$ is not a multiple of p.

5. [D&F 4.5, Exercise 36] For each of $p = 2, 3, 5$ determine $n_p(A_5)$ and $n_p(S_5)$. [It may help that $A_4 \leq A_5$ and $S_4 \leq S_5$.]

6. [D&F 4.5, Exercises 4 and 5]
   i) For each of $p = 2, 3$ find all p-Sylow subgroups of D_{12} and $S_3 \times S_3$.
   ii) Prove that if p is an odd prime factor of n then D_{2n} has a unique (and thus normal) p-Sylow subgroup, and that this subgroup is cyclic.

7. [D&F 4.5, Exercise 30] How many elements of order 7 must there be in a simple group of order 168?

Such a group does exist; it is unique, and remarkably isomorphic to both[1] $\mathrm{GL}_3(\mathbf{Z}/2\mathbf{Z})$ and $\mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$, but this probably wouldn't help you count its elements of order 7.

[1]When F is the 2-element field $\mathbf{Z}/2\mathbf{Z}$, the groups $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$, $\mathrm{PGL}_n(F)$, $\mathrm{PSL}_n(F)$ are all isomorphic for each n.