Math 122: Algebra I, Fall 2023

Homework Assignment #7 (19 October 2023):
The sign homomorphism and the alternating group; group actions cont'd

> Easiest color to solve on a Rubik's Cube: black.
>
> Simply remove all the little colored stickers on the cube, and each of side of the cube will now be the original color of the plastic underneath — black. According to the instructions, this means the puzzle is solved.
>
> — motd.ambians.com/quotes.php/name/freebsd_fortunes_3/toc_id/1-0-4/s/1250
> (Unix fortune #1251 of 2182; original source unknown to me)

This problem set is due Wednesday, October 25 at midnight.

Inversions and the sign homomorphism:

1. We saw in class that the permutation bringing BINGE to BEGIN, call it σ, has inversion number 5. Let σ' and σ'' be the permutations bringing BEING to BINGE and to BEGIN. Explain why $\mathrm{inv}(\sigma')$ and $\mathrm{inv}(\sigma'')$ must be of opposite parity. Verify this by calculating these inversion numbers.

Generators of A_n and homomorphisms from A_n to abelian groups:

2. i) Prove that S_n ($n \geq 2$) is generated by the $n-1$ simple transpositions $(1\ i)$ $(2 \leq i \leq n)$. [The textbook's hint for D&F 3.5, Exercise 3, applies here, though we proved that result in class in another way.]
   ii) Deduce that A_n ($n \geq 3$) is generated by the 3-cycles $(1\ i\ j)$ $(2 \leq i < j \leq n)$.
   iii) Conclude that if $h : A_n \to G$ is a homomorphism from A_n to an abelian group G then all $g \in h(A_n)$ must satisfy $g^3 = 1$.

3. [A proper normal subgroup of A_4]
   i) Prove that the $4!/2 = 12$ permutations in A_4 comprise the identity, three permutations of order 2, and eight permutations of order 3. Deduce that A_4 has no subgroup of order 6. [Since $6 = \frac{1}{2}|A_4|$ such a subgroup would have to be normal, giving a surjective homomorphism from A_4 to a 2-element group.]
   ii) Verify that the elements of order 1 or 2 in A_4 constitute a group, call it V, and prove that V is the only 4-element subgroup of A_4. Conclude that $V \triangleleft A_4$.

Thus there is a surjective homomorphism from A_4 to a 3-element group A_4/V, and this quotient group must be abelian because 3 is prime. The normal subgroup V of A_4 is called the "Klein 4-group" (V for German *vier* "four"). One can show using similar methods that for $n \geq 5$ any homomorphism from A_n to an abelian group must be trivial; but this will be subsumed by the result that in fact A_n is simple for each $n \geq 5$.

4. [The sign homomorphism and Rubik's Cube]

   i) For any integers $m, n > 1$ prove that there are five subgroups $G \subseteq S_m \times S_n$ that contain $A_m \times A_n$: the groups $S_m \times S_n$ and $A_m \times A_n$ themselves; the subgroups $A_m \times S_n$ and $S_m \times A_n$; and a fifth subgroup, call it $A_{m,n}$, consisting of all $(\sigma, \tau) \in S_m \times S_n$ such that $\epsilon(\sigma) = \epsilon(\tau)$.

   ii) Let $\mathcal{R}$ be the "Rubik's Cube group" generated by $90°$ rotations of the six faces. Since such a rotation takes corner cubelets to corner cubelets, and edge cubelets to edge cubelets, the action of $\mathcal{R}$ on the $8 + 12$ corner and edge cubelets gives a homomorphism $c : \mathcal{R} \to S_8 \times S_{12} \subset S_{8+12}$. Prove that its image $c(\mathcal{R})$ is contained in $A_{8,12}$.

It is known that in fact every permutation in $A_{8,12}$ is reachable. Note that this does not quite amount to a full solution of Rubik's Cube, because it does not account for rotations of corner cubelets and flips of edge cubelets; it turns out that each of those is subject to a further condition that allows only $1/3$ of the possible corner configurations and $1/2$ of the possible edge configurations. For example, the kernel of the homomorphism $\mathcal{R} \to A_{8,12}$ has order $2^{11}3^7$, not $2^{12}3^8$ as we might expect.

5. [A 15-puzzle variant; careful with this one!][1] The tiles of a 15 puzzle are marked with the letters of the sentence DEAD PIGS WON[']T FLY as shown in the first figure. Which of the sentences DEAD PIGS FLY TOWN and DEAD FIGS WON[']T PLY (see second and third figures) — possibly neither or both — can be reached by sliding the tiles within the 4×4 array? Can the third figure be reached from the second?

D	E	A	D
P	I	G	S
W	O	N	T
F	L	Y	

D	E	A	D
P	I	G	S
F	L	Y	
T	O	W	N

D	E	A	D
F	I	G	S
W	O	N	T
P	L	Y	

You may use the theorem of Johnson and Story (1879) that every permutation in A_{15} is reachable, even though we have not given a proof in class or in the notes.

Transitive actions and their kernels:

6. [D&F 4.1 Exercise 1] Let G act on the nonempty set A. Prove that if $a \in A$ and $g \in G$ then the stabilizer $G_{g \cdot a}$ is gG_ag^{-1}.

7. [D&F 4.1 Exercises 2–3] Let G act on the nonempty set A. Deduce that if the action is transitive then the kernel of the action is $\cap_{g \in G}\, gG_ag^{-1}$. In particular if G is a transitive subgroup of S_A acting by $g \cdot a = g(a)$ then $\cap_{g \in G}\, gG_ag^{-1} = \{1\}$. If moreover G is abelian, show that $g(a) \neq a$ for all $g \in G - \{1\}$ and $a \in A$, and deduce $|G| = |A|$. [At some point you might want to consider the stabilizer G_a.]

[1] Adapted from John Beasley's *The Mathematics of Games* (1989), Figure 7.5; the color scheme is from https://puzzleparadise.net/listing/dead-pigs-wont-fly-ipp16-exchange-puzzle/105683 : the letters of DEAD are blue, PIGS red, WONT/TOWN green, and FLY orange — not that this is likely to be of use.