Math 122: Algebra I, Fall 2023

Homework Assignment #5 (5 October 2023):
Subgroups $\langle A \rangle$; normal subgroups

> **normal** adjective
> [...]
> 5 : PERPENDICULAR, *especially*: perpendicular to a tangent (see TANGENT entry 1 sense 3) at a point of tangency
> 6 **of a subgroup** : having the property that every coset produced by operating on the left by a given element is equal to the coset produced by operating on the right by the same element
> 7 : relating to, involving, or being a normal curve or normal distribution
> 8 **of a matrix** : having the property of commutativity under multiplication by the [Hermitian] transpose of the matrix [...]
>
> From the online Merriam–Webster Dictionary's definition of "normal"; senses 4a–4d are technical terms in chemistry (see www.m-w.com/dictionary/normal).

This problem set is due Wednesday, October 11 at midnight.

Subgroups generated by subsets:

1. [D&F 2.4 #1, extended] Prove that if A is a subset of a group G then $A = \langle A \rangle$ <u>iff</u> A is a subgroup of G.

2. [A subgroup of a 2-generator group need not be finitely generated] Let $G \leq \mathrm{GL}_2(\mathbf{R})$ be the group of invertible upper-triangular matrices, and let $x, y \in G$ be the elements $x = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let $H \leq G$ be the subgroup $\langle x, y \rangle$.
   i) Prove that H consists of all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ where $a = 2^f$ for some $f \in \mathbf{Z}$ and b is a "dyadic rational" (a rational number m/n where n is a power of 2).
   ii) Prove that the subgroup of H consisting of matrices of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ is not finitely generated. [A group G is "finitely generated" if $G = \langle A \rangle$ for some finite subset $A \subseteq G$; cf. D&F 2.4 Exercise 14 on page 65. Hint: the matrices of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b \in \mathbf{R}$ constitute a subgroup of $\mathrm{GL}_2(\mathbf{R})$, call it U, and the map $U \to (\mathbf{R}, +)$ taking $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ to b is an isomorphism of groups.]

Homomorphisms and normal subgroups:

3. [based on D&F 3.1 #6,8,9] For each of the following maps $\varphi : G \to \mathbf{R}^\times$, prove φ is a homomorphism, find its image and kernel, and describe the fibers of image elements $c \in \varphi(G)$ geometrically. Note that in this problem $|x|$ denotes the ordinary absolute value of x.
   i) $G = \mathbf{R}^\times$, $\varphi(x) = x/|x|$.
   ii) $G = \mathbf{R}^\times$, $\varphi(x) = |x|$.
   iii) $G = \mathbf{C}^\times$, $\varphi(a + bi) = a^2 + b^2$.

4. [D&F 3.1 #12,13; see Figure 5 on p.86] Let $G = (\mathbf{R}, +)$ and $H = \{z \in \mathbf{C}^\times \mid z\bar{z} = 1\}$ (so H is regarded as a subgroup of $\mathbf{C}^\times$). Let $\varphi : G \to H$ be the homomorphism defined by $\varphi(r) = e^{2\pi i r}$. Find the kernel of φ, and the fibers of φ above the points -1, i, and $-(1 + \sqrt{3}\, i)/2$ of H. Repeat this with φ defined by $\varphi(r) = e^{4\pi i r}$.

5. Let $G \leq \mathrm{GL}_2(\mathbf{R})$ be the group of matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$. Which of the following subgroups $H_i \leq G$ is normal? if H_i is normal, describe the quotient group G/H_i.
   i) H_1 consists of the matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
   ii) H_2 consists of the matrices of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$.
   iii) H_3 consists of the matrices with $a = 2^f$ for some $f \in \mathbf{Z}$.

6. [based on D&F 3.1 #16] Let G be a group, N a normal subgroup, and $\overline{G} = G/N$.
   i) Prove that if a subgroup $H \leq G$ contains N then $N \trianglelefteq H$.
   ii) Therefore we have a subgroup $\overline{H} := H/N$ of $\overline{G}$. Prove that if $H = \langle A \rangle$ for some subset $A \subseteq H$ then $\overline{H} = \langle \overline{A} \rangle$.

Here $\overline{A}$ is the subset of $\overline{H}$ consisting of cosets of elements of $A \bmod N$; you might find it helpful to start from the cases where A consists of two elements (as in the D&F exercise), or even just one.