Math 122: Algebra I, Fall 2023

Homework Assignment #4 (28 September 2023):
Subgroups

> One basic method for unravelling the structure of any mathematical object which is defined by a set of axioms is to study *subsets* of that object which also *satisfy the same axioms*. We begin this program by discussing subgroups of a group.
>
> — D&F, start of Chapter 2.

This problem set is due Wednesday, October 4 at midnight.

Subgroup generalities:

1. [D&F 2.1 #10b] We proved in class that if H and K are subgroups of a group G then so is their intersection $H \cap K$. Now suppose we have a subgroup $G_i \leq G$ for each i in some nonempty index set I. Prove that $\cap_{i \in I} G_i$ is also a subgroup of G. (If I is finite then this follows by induction from the known case $|I| = 2$; but here I is <u>not</u> assumed finite or even countable. For instance $Z(G) = \cap_{g \in G} C_G(g)$ is a special case, and G can well be uncountable, e.g. $G = \mathrm{GL}_2(\mathbf{R})$. Likewise for any subset $A \subset G$ the subgroup generated by A can be defined as the intersection of all subgroups of G that contain A, and there might be uncountably many such subgroups; note that the collection of such subgroups is nonempty because it includes G itself.)

Unions of subgroups aren't as productive; for instance you should be able to check that if $H \leq G$ and $K \leq G$ then $H \cup K$ is not closed under multiplication unless $H \leq K$ or $K \leq H$, in which case $H \cup K$ is one of the subgroups we started with. We may, however, get a new subgroup as the union of an infinite tower of subgroups:

2. [based on D&F 2.1 #15]
   i) Suppose H_n ($n = 1, 2, 3, \ldots$) are subgroups of G such that $H_n \leq H_{n+1}$ for each n. Prove that $\cup_{n=1}^{\infty} H_n$ is a subgroup of G.
   ii) If $G = \mathbf{R}$ give an example where each H_n is cyclic and $\cup_{n=1}^{\infty} H_n = \mathbf{Q}$. (It should be clear that $\mathbf{Q}$ is not cyclic.)

Centralizers, normalizers, etc.:

3. [D&F 2.2 #6] Let H be a subgroup of the group G.
   i) Show that $H \subseteq N_G(H)$. Give an example of a group G and a subset $A \subset G$ (necessarily <u>not</u> a subgroup) such that $N_G(A)$ does not contain A.
   ii) Show that $H \subseteq C_G(H)$ if and only if H is abelian.

4. Let G be any group.

   i) Prove that $(g_1, g_2) \cdot a = g_1 a g_2^{-1}$ gives an action of $G \times G$ on G.

   ii) Prove that the kernel of this action is $\{(g, g) \mid g \in Z(G)\}$.

Cyclic groups:

5. [D&F 2.3 #9] Let $Z_{36} = \langle x \rangle$. For which integers a does the map $\psi_a : \bar{1} \mapsto x^a$ extend to a well-defined homomorphism from $\mathbf{Z}/48\mathbf{Z}$ to Z_{36}? Can ψ_a ever be a surjective homomorphism?

6. [D&F 2.3 #13] Prove that the following pairs of group are not isomorphic:

   i) $\mathbf{Z} \times Z_2$ and $\mathbf{Z}$ (hint: see D&F 2.3 #12).

   ii) $\mathbf{Q} \times Z_2$ and $\mathbf{Q}$.

7. [D&F 2.3 #16] Suppose G is a group and $x, y \in G$ have finite orders $|x| = n$ and $|y| = m$.

   i) Suppose x and y commute: $xy = yx$. Prove that $|xy|$ divides the least common multiple of m and n. (Recall that "A divides B" means "B is a multiple of A" or equivalently "A is a factor of B".)

   ii) Does the result of (i) necessarily remain true without the assumption $xy = yx$?

   iii) Give an example of G, x, y where $xy = yx$ and $|xy|$ does not equal the least common multiple of $|x|$ and $|y|$.