

## Math 122: Algebra I, Fall 2023

### Homework Assignment #3 (21 September 2023): Group homomorphisms and actions

The modern treatment of abstract algebra begins with the disarmingly simple abstract definition of a *group*.

— D&F, top of page 13; quoted by Wyatt Reeves in the flier for his talk at the “Trivial Notions Seminar”, 19 September 2023

This problem set is due Wednesday, September 27 at midnight.

0. If you have not done it yet, solve Problem 8 of the previous problem set. [We now see that the map taking each element of  $Q_8$  to the corresponding  $2 \times 2$  matrix is an injective homomorphism from  $Q_8$  to  $GL_2(\mathbf{C})$ ; cf. D&F 1.6 #13 (contained in Problem 3 below) and the final exercise for D&F 1.6 (#26 on page 41).]
1. [D&F 1.6 #1] Prove that if  $\varphi : G \rightarrow H$  is a homomorphism and  $x \in G$  then  $\varphi(x^n) = (\varphi(x))^n$  for all integers  $n$ . [First prove the case  $n > 0$  by induction. The case  $n = 0$  was in effect done in class. Do the case  $n = -1$  and deduce that  $\varphi(x^n) = (\varphi(x))^n$  holds also for all  $n < 0$ .]
2. [conjugation cont'd, D&F 1.7 #16,17] For an element  $g$  of a group  $G$  define  $c_g : G \rightarrow G$  by  $c_g(x) = gxg^{-1}$ .
  - i) Prove that  $c_g$  is a group isomorphism. [This may look familiar from D&F 1.1 #22 ( $\subset$  PS1 #6); see also 1.6 #2.]
  - ii) Prove that  $g \cdot x = c_g(x)$  defines a group action of  $G$  on  $G$ .

The next pair of exercises introduces the image and kernel of a group homomorphism, which are analogous to the corresponding notions for a linear transformations — and then preview an important phenomenon that did not arise in the context of linear transformations.

3. [based on D&F 1.6 #13] Suppose  $G, H$  are groups and  $\varphi : G \rightarrow H$  is a homomorphism. Prove that  $\varphi(G)$ , the *image* of  $G$ , is a subgroup of  $H$ , and if  $\varphi$  is injective then  $G \cong \varphi(G)$ . Show that for every subgroup  $H' \subseteq H$  there is a group  $G$  and a homomorphism from  $G$  to  $H$  whose image is  $H'$ .

4. [based on D&F 1.6 #14] Suppose  $G, H$  are groups and  $\varphi : G \rightarrow H$  is a homomorphism. The *kernel* of  $\varphi$  is defined to be  $\varphi^{-1}(1_H)$ ; that is,  $\ker \varphi = \{g \in G \mid \varphi(g) = 1_H\}$ . Prove that  $\ker \varphi$  is a subgroup of  $G$  (cf. D&F 1.1 #26  $\cong$  PS1 #4). Prove that  $\varphi$  is injective if and only if  $\ker \varphi = \{1_G\}$ . Show that there can be a group  $G$  and a subgroup  $G'$  that is not the kernel of any homomorphism  $\varphi : G \rightarrow H$ ; in fact we can take  $G = D_{2n}$  (any  $n \geq 3$ ) and choose for  $G'$  the two-element subgroup  $\{1, s\}$ . [If  $s \in \ker \varphi$ , what is  $\varphi(rs^{-1})$ ?]

We can already use group actions to prove Lagrange's theorem: *if  $G$  is a finite subgroup and  $H$  is a subgroup of  $G$  then  $|H|$  is a factor of  $|G|$ .*

5. [D&F 1.7 #18] Let  $H$  be a group acting on a set  $A$ . Prove that the relation  $\sim$  on  $A$  defined by

$$a \sim b \text{ if and only if there exists } h \in H \text{ such that } a = hb$$

is an equivalence relation. (See p.3, item (3) of D&F Chapter 0 if you need to review equivalence relations. For  $x \in A$  the equivalence class  $[x]$  is called the *orbit* of  $x$  under this action.)

6. [D&F 1.7 #19] Let  $H$  be a subgroup of the finite group  $G$ , and let  $H$  act on  $G$  (here  $A = G$ ) by left multiplication (that's why we in the previous Exercise we called the group  $H$  rather than  $G \dots$ ). Let  $x \in G$ , and let  $\mathcal{O}$  be the orbit of  $x$  under the action of  $H$ . Prove that the map  $H \rightarrow \mathcal{O}$  defined by  $h \mapsto hx$  is a bijection. Deduce that  $\mathcal{O}$  has cardinality  $|H|$ . Since this is true for every orbit, Lagrange's theorem follows from the fact that the orbits partition  $G$ .

In particular it follows that  $|x|$  is a factor of  $|G|$  for all  $x \in G$ . Note, however, that not all factors of  $|G|$  need arise as the orders of group elements, or even the orders of subgroups; for example, we shall see that  $S_5$  has no subgroup of order 15, 30, or 40, even though each of these is a factor of  $|S_5| = 5! = 120$ .

Finally, a geometric construction of a nontrivial homomorphism  $S_4 \rightarrow S_3$  (assuming you already believe that  $S_4$  is isomorphic with the group of rigid motions of a cube):

7. [D&F #1.23] Explain why the action of the group of rigid motions of a cube on the set of three pairs of opposite faces is not faithful. Find the kernel of this action.