Math 122: Algebra I, Fall 2023

Homework Assignment #2 (14 September 2023):
Examples of groups

$$i^2 = j^2 = k^2 = ijk = -1$$

Sir William Rowan Hamilton, 1843 graffiti on Broom Bridge in Dublin, Ireland; see Chapter 1.5 and Problem 7.

This problem set is due Wednesday, September 20 at midnight.

1. [from D&F 1.2 #1] Determine the order of each of the elements of the dihedral groups D_8 and D_{10}.

2. [slightly extending D&F 1.2 #4] Suppose for some $n \geq 3$ that $z \in D_{2n}$ is a non-identity element such that $zg = gz$ for all $g \in D_{2n}$. Prove that n is even, say $n = 2k$, and $z = r^k$ where r is our order-n generator of D_{2n}.

Can you prove that for any group G the set of $z \in G$ that commute with every $g \in G$ is an abelian subgroup of G? (This subgroup is called the *center* of G, and often denoted by $Z(G)$ because German [*Zentrum*].)

3. [D&F #9,#11] Prove that the group of rigid motions of a regular tetrahedron in $\mathbf{R}^3$ has order 12, and that the group of rigid motions of a regular octahedron in $\mathbf{R}^3$ has order 24.

These and related groups are relevant to the physical and chemical properties of substances such as methane (CH_4) and uranium hexafluoride (UF_6), whose molecules consist of one central atom linked to 4 or 6 identical smaller atoms located at the vertices of a regular tetrahedron or octahedron.

4. In this Rubik's Cube problem Ω is the set of cubelets, not stickers; and we consider only two faces meeting at an edge (this edge consists of cubelets 3, 8, 13 in the following diagram):

1	**2**	**3**	3	4	5
6	**7**	**8**	8	9	10
11	**12**	**13**	13	14	15

Let σ be the permutation of $\{1, 2, 3, \ldots, 15\}$ obtained by a 180° turn of the left face (columns 1–3), and let τ be the permutation of $\{1, 2, 3, \ldots, 15\}$ obtained by a 180° turn of the right face (the other 3 columns). For example, $\sigma(3) = 11$ and $\tau(3) = 15$.

i) Determine the cycle decompositions of σ and τ.

ii) Use your answer to (i) to determine the cycle decomposition of $\sigma\tau$, and deduce that $|\sigma\tau| = 6$ (possibly using the result of Problem 6 below).

iii) What does $(\sigma\tau)^3$ do? (You might want to check your answer with a physical or virtual Rubik's Cube. Happily the result remains true even when we keep track of the cubelets' orientations, not just their locations.)

5. [D&F 1.3 #8] Prove that if $\Omega = \{1, 2, 3, \ldots\}$ then the group S_Ω is infinite. (Warning: it is not enough to say "$\infty! = \infty$", because the proof that $|S_n| = n!$ assumes that n is finite — never mind that strictly speaking "$\infty!$" is not defined.)

6. [D&F 1.3 #15] Prove that the order of an element in S_n equals the least common multiple of the lengths of the cycles in its cycle decomposition. [D&F refer here to Exercises 1.3 #10 (which we more-or-less did in class) and 1.1 #24 (for which you'll need only the $n > 0$ case, and possibly not even that).]

7. [adapted from D&F 1.4 #10] Let G consist of the invertible upper-triangular real 2×2 matrices, i.e. all matrices of the form $g(a, b, c) := \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ with $a, c \in \mathbf{R}^\times$ and $b \in \mathbf{R}$).

i) Compute the matrix product $g(a_1, b_1, c_1)\, g(a_2, b_2, c_2)$ to show that G is closed under multiplication.

ii) Compute the inverse of $g(a, b, c)$ and deduce that G is closed under inverses.

iii) Deduce that G is a subgroup of $\mathrm{GL}_2(\mathbf{R})$.

iv) Prove that G is not commutative, but its subset consisting of $g(a, b, c)$ with $b = 0$ is a commutative subgroup.

All but the last part works for matrices with entries in any field, not just real matrices. Do you see what might go wrong with part (iv) if we replace $\mathbf{R}$ by an arbitrary field?

You may postpone the final problem till Problem Set 3 without penalty.

8. Let $M_{\pm 1}, M_{\pm i}, M_{\pm j}, M_{\pm k}$ be the following 2×2 matrices with complex entries:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ M_{-1} = -M_1; \quad M_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ M_{-i} = -M_i;$$

$$M_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ M_{-j} = -M_j; \quad M_k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ M_{-k} = -M_k.$$

(Such matrices arise naturally in quantum computing and elsewhere in mathematics and quantum physics.) Note that M_1 is the identity matrix in $\mathrm{GL}_2(\mathbf{C})$. Check that these matrices satisfy the Q_8 identities

$$M_{-1}^2 = M_1, \quad M_i^2 = M_j^2 = M_k^2 = M_{-1}, \quad M_i M_j = M_k,\ M_j M_i = M_{-k},$$

etc. (you're welcome to use a computer algebra program for this). Deduce that these eight matrices constitute a subgroup of $\mathrm{GL}_2(\mathbf{C})$ with the same multiplication table as Q_8, and thus in particular that Q_8 is a group.