Math 122: Algebra I, Fall 2023

Homework Assignment #11 (3 December 2023): Introduction to modules, vector spaces, and linear transformations

Mathematics is the art of giving the same name to different things.

— Henri Poincaré (1854–1912),[1] quoted by Isabel Longbottom in the flier for her talk "Sylow Theorem for ∞-Groups" at the "Trivial Notions Seminar", 28 Nov. 2023

This final problem set is due *Friday*, December 8 at midnight.

Some basic facts about modules over a ring (which automatically hold also for vector spaces over a field):

1. (D&F 10.1 #1, extended) Show that if M is a left module over a ring R then $0_R m = 0_M$ for all $m \in M$. Deduce that $(-r)m = -(rm)$ for all $r \in R$ and $m \in M$. In particular, if R has a multiplicative identity 1 (and M satisfies property 2d in the definition of a left R-module on page 337) then $(-1)m = -m$ for all $m \in M$.

2. (D&F 10.1 #2) Prove that if R is a ring with a multiplicative identity and M is any left R-module then $R^\times$ and M satisfy the two axioms in D&F §1.7 for a group action of the multiplicative group $R^\times$ on M.

Linear transformations and the rank-nullity theorem:

3. (based on D&F 11.1 #1) For elements $a_1, \ldots, a_n$ of a field F define

$$V := \{(x_1, \ldots, x_n) \in F^n : a_1x_1 + \cdots + a_nx_n = 0\}.$$

Construct a linear transformation $T : F^n \to F$ such that $V = \ker T$. Deduce that V is a vector subspace of F^n, and use the rank-nullity theorem to compute $\dim V$. [Your answer shouldn't quite be the same for all choices of $a_1, \ldots, a_n$.] Explain what this has to do with duality.

4. (based on D&F 11.1 #7) Suppose U, V, W are vector spaces over the same field F. Let $\varphi : U \to V$ and $\psi : V \to W$ be linear transformations, and $f = \psi\varphi : U \to W$ their

[1] The aphorism is "well-known" but I couldn't find a reference to a primary source. It is often quoted with the reply by an unnamed poet: "Poetry is the art of giving different names to the same thing." To be sure poetry does both. So does mathematics, and we have just encountered examples of each. For example, the name "homomorphism" means something different for groups and for vector spaces (consider the $\mathbf{C}$-vector space $\mathbf{C}$, and the group homomorphism $\mathbf{C} \to \mathbf{C}$ taking any $z = x + iy \in \mathbf{C}$ to its real part x, or to the complex conjugate $\bar{z} = x - iy$, neither of which is a vector-space homomorphism). But the use of the same name suggests further analogies starting with the image and kernel. In the other direction, an F-vector space is by definition the same thing as an F-module; also we noted that a $\mathbf{Z}$-module is the same thing as an abelian group, and this point of view suggests new ways to study abelian groups, simplifying some results about their structure.

In any case neither Poincaré's statement nor the reply should be taken literally as a definition of mathematics or poetry, since there is much more to both mathematics and poetry than artful naming. Call it poetic license.

composition.

i) Show that $\operatorname{im} f \subseteq \operatorname{im}\psi$ and $\ker f \supseteq \ker\varphi$. Show that moreover if $f = 0$ then $\operatorname{im}(\varphi) \subseteq \ker(\psi)$. [This is a basic example of "diagram chasing"; here the diagram is just $U \stackrel{\varphi}{\to} V \stackrel{\psi}{\to} W$, with f being the composite map. It works equally well for group homomorphisms, whether or not the groups are abelian, and for R-module homomorphisms.]

ii) Consider now the special case $U = V = W$ and $\psi = \varphi$, so $\varphi \in \operatorname{End}(V)$ and $f = \varphi^2$. Suppose further that $\dim V = n < \infty$ and $f = \varphi^2 = 0$. Deduce that $\dim\ker(\varphi) \geq n/2$ and $\dim\operatorname{im}(\varphi) \leq n/2$. [BTW a linear operator $T \in \operatorname{End}(V)$ such that $T^n = 0$ for some $n \geq 1$ is said to be *nilpotent*.]

More about bases and dimensions of vector spaces over a given field F:

5. Let U and V be subspaces of a vector space W. Define $U+V = \{u+v : u \in U, v \in V\}$.

i) Prove that $U + V$ is a vector subspace of W and is the smallest subspace containing both U and V. [The second part means that if $W' \subseteq W$ is any subspace such that $U \subseteq W'$ and $V \subseteq W'$ then $U + V \subseteq W'$. Note that in general a vector in $U + V$ may have more than one $u + v$ representation.]

ii) Show that $\dim(U + V) < \infty$ if and only if U and V are finite-dimensional.

iii) Suppose U and V are finite-dimensional. It should be clear that $U \cap V$ is also a vector subspace. Let $\mathcal{B} = (w_1, \ldots, w_l)$ be a basis for $U \cap V$. Since $U \cap V \subseteq U$ there exist $u_1, \ldots, u_m$ such that $\mathcal{B}$ together with $u_1, \ldots, u_m$ is a basis for U. Likewise there exist $v_1, \ldots, v_n$ such that $\mathcal{B}$ together with $v_1, \ldots, v_n$ is a basis for V. Prove that $\mathcal{B}$ together with $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ is a basis for $U+V$. In particular $U+V$ has dimension $l + m + n$, whence

$$\dim(U + V) + \dim(U \cap V) = \dim U + \dim V.$$

About the dual vector space:

6. Solve Exercise #3 in D&F 11.3 (on page 435).

7. (This is basically 11.3 #4 and part of #5 made concrete.)

Let V be the F-vector space $F[x]$, consisting of polynomials $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ for some integer $n \geq 0$ and field elements $a_0, \ldots, a_n$; thus F has the countably infinite basis $1, x, x^2, \ldots, x^n, \ldots$, and for each $m \geq 0$ the associated $v_m^* \in V^*$ $(m = 0, 1, 2, \ldots)$ is the homomorphisms taking any $P \in V$ to the x^m coefficient of P.

Let F^∞ be the vector space of infinite sequences $\vec{b} = (b_0, b_1, b_2, \ldots)$ with each $b_i \in F$ and the vector-space operations defined componentwise. Define a map from F^∞ to V^* by taking any $\vec{b} = (b_0, b_1, b_2, \ldots)$ to the linear functional on V that maps any $P = \sum_{i=0}^n a_i x^i$ to $\sum_{i=0}^n a_i b_i$. Prove that this map is a vector space isomorphism $F^\infty \to V^*$. Check that the vector $(1, 1, 1, \ldots)$ maps to the functional taking any $P \in V$ to $P(1)$, and that this functional is not in the span of the v_m^*.

In fact it can be shown that F^∞ cannot have *any* countable basis for any field F, even though it is the dual of a vector space that does have a countable basis.