Math 122: Algebra I, Fall 2023

Homework Assignment #10 (17 November 2023):
Introduction to rings

I will <u>not</u> quote Beyoncé song lyrics here. —NDE

This problem set is due Wednesday, November 29 at midnight.

Some basic properties and examples of rings.

1. (D&F 7.1 #1,2) Suppose R is a ring with 1.
   i) Show that $(-1)^2 = 1$.
   ii) Show that u is a unit of R if and only if $-u$ is.

2. (D&F 7.1 #5) Which of the following subsets A of $\mathbf{Q}$ are subrings of $\mathbf{Q}$?
   i) The set of rational numbers with odd denominators (when written in lowest terms)
   ii) The set of rational numbers with even denominators (when written in lowest terms)
   iii) The set of rational numbers with odd numerators (when written in lowest terms)
   iv) The set of rational numbers with even numerators (when written in lowest terms)
   v) The set of nonnegative rational numbers
   vi) The set of squares of rational numbers

3. (D&F 7.1 #6abce) Let R be the ring of all functions from from the closed interval $[0, 1]$ to $\mathbf{R}$. For each of the following subsets $A \subset R$, determine whether A is a subring.
   i) The set of all functions $f(x)$ such that $f(q) = 0$ for all $q \in \mathbf{Q} \cap [0, 1]$. [NB f is not assumed continuous.]
   ii) The set of all polynomial functions.
   iii) The set of all functions that have only a finite number of zeros.
   iv) The set of functions f such that $\lim_{x\to 1^-} f(x) = 0$.

Reminder for problems 2 and 3: if the answer is "no" then you can prove it by exhibiting one counterexample; e.g. if A does not contain 0, or contains 1 but not 2, then A is not a subring (in the latter case because it is not closer under addition: 1 is in A but $1 + 1$ is not).

Definition. The *center* of a ring R is the set $\{z \in R : \forall r \in R, zr = rz\}$ of all ring elements that commute with every element of R.

4. (based on D&F 7.1 #7)
   i) Prove that the center of a ring R is a commutative subring of R, which contains 1_R if R

has an identity.

ii) Prove that the center of the matrix ring $M_2(\mathbf{R})$ consists of real multiples of the identity matrix, i.e. matrices of the form $cI_2 = \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)$ for $c \in \mathbf{R}$. [Try using for r matrices such as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ with a single nonzero entry which is 1. You're welcome to use a computer algebra package to work out $rz - zr$. Much the same proof shows that if R is any ring with identity then for every n the matrix ring $M_n(R)$ has center $\{cI_n : c \in R\}$.]

About integral domains:

5. (D&F 7.1 #12,11)
   i) Suppose F is a field and $A \subseteq F$ is a subring that contains 1_F. Prove that A is an integral domain.
   ii) Suppose R is an integral domain. Show that if $x^2 = 1$ in R then $x = 1$ or $x = -1$.

A warning about identity elements in subrings:

6. i) Prove that if n is a positive integer and m is a factor of n then $m\mathbf{Z}/n\mathbf{Z}$ is a subring of $\mathbf{Z}/n\mathbf{Z}$, and that these are all the subrings of $\mathbf{Z}/n\mathbf{Z}$.
   ii) Give an example of n and $m|n$ for which the ring $m\mathbf{Z}/n\mathbf{Z}$ has no identity element. Give an example of n and $m|n$ (with $m \neq 1$ and $m \neq n$) for which the ring $m\mathbf{Z}/n\mathbf{Z}$ has an identity (necessarily different from the identity element $\bar{1}$ of $\mathbf{Z}/n\mathbf{Z}$!), and exhibit this identity element.
   iii) Extra credit: Find a necessary and sufficient condition on a positive integer n and a factor $m|n$ under which the ring $m\mathbf{Z}/n\mathbf{Z}$ has an identity element.

Hamilton's quaternions and the quaternion group Q_8:

7. Two months ago (HW2 #8) you found that the matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

   in $\mathrm{GL}_2(\mathbf{C})$, together with their negatives, form a subgroup isomorphic with Q_8. Use this to show that the matrices of the form $aM_1 + bM_i + cM_j + dM_k$ $(a, b, c, d \in \mathbf{R})$ form a subring of $M_2(\mathbf{C})$ isomorphic with the Hamilton quaternions $\mathbf{H}$. (Since multiplication in $M_2(\mathbf{C})$ is already known to be associative, this shows in particular that $\mathbf{H}$ is indeed a ring.)