Math 122: Algebra I, Fall 2023

Homework Assignment #1 (9 September 2023):
Introduction to groups

> Important applications of group theory are symmetries which can be found in most different connections both in nature and among the 'artefacts' produced by human beings. Group theory also has important applications in mathematics and mathematical physics. For example, the theory of elementary particles and their interactions can in essential respects be reduced to abstract symmetries.
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> — K. V. Laurikainen, *The Message of the Atoms: Essays on Wolfgang Pauli and the Unspeakable*, 11. The World of Spirit

This problem set is due ~~Wednesday, September 13~~ Friday, September 15 at midnight.

1. Prove that if G is a group then each row and each column of the multiplication table of G is a permutation of G. [Such arrays may be familiar from Sudoku or KenKen puzzles; in combinatorics they are called "Latin squares". Warning: if n is at all large ($n = 6$ is surely large enough), only a tiny fraction of all $n \times n$ Latin squares can be constructed as the multiplication table of a group.]

2. Define a binary operation $\star$ on $\mathbf{Z}$ by $x \star y = x + y + 122$. Prove that this operation makes $\mathbf{Z}$ an abelian group. Do you recognize this group?

3. In measure theory one defines the "extended real numbers" to be the set, call it $\mathbf{R}_{\text{ext}}$, consisting of $\mathbf{R}$ together with the symbol ∞ (think "$+\infty$"). Addition is defined on $\mathbf{R}_{\text{ext}}$ as follows: if $x = \infty$ or $y = \infty$ then $x + y = \infty$; otherwise x and y are real numbers and $x + y$ is defined to be the usual sum of $x + y$ in $\mathbf{R}$. Which of the group axioms is/are satisfied by $(\mathbf{R}_{\text{ext}}, +)$? Is $(\mathbf{R}_{\text{ext}}, +)$ a group?

The remaining problems are taken or adapted from the 36 Exercises for Chapter 1.1 of the D&F (Dummit and Foote) textbook. There are many other good exercises there but it is not feasible to assign more than a handful of them. Many of them introduce further properties of groups that we shall soon explore at greater depth and/or[1] generality.

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[1] To be sure, technically "A or B" already implies "...or both" unless specified otherwise, so "and/or" is equivalent to plain "or". For example, in Problem 2 we have $\infty + \infty = \infty$. But in common parlance "A or B" is often taken to be "exclusive or", that is "A or B, but not both".

Once we introduce a new mathematical structure, we often proceed to give some (hopefully familiar) examples, and constructions that let us create new examples from known ones. The next two exercises introduce two basic new-from-old constructions: subgroups and direct products. These are analogous to subspaces and direct products of vector spaces (which indeed are special cases if we consider just the group structure $(V, +)$ of a vector space).

4. [D&F 1.1#26] Suppose $(G, \star)$ is a group, and H is a nonempty subset of G that is closed under inverses and $\star$.[2] Prove that H is a group under the operation $\star$ restricted to H. (Such H is called a *subgroup* of G. Note that you must show that H contains the identity of G.)

5. [D&F 1.1#28,#29] Suppose $(A, \star)$ and $(B, \diamond)$ are groups. Let $A \times B$ be their direct product, defined in Example 6 (top of page 18). Check that $A \times B$ satisfies the group axioms by verifying each of the group axioms,[3] and prove that $A \times B$ is abelian if and only if both $(A, \star)$ and $(B, \diamond)$ are abelian.

The next exercise introduces conjugation and the commutator in a group.

6. [D&F 1.1#18,#22] Let x and y be elements of a group G. Prove that $xy = yx$ if and only if $yxy^{-1} = x$ if and only if $x^{-1}y^{-1}xy = 1$. Prove further that $|yxy^{-1}| = |x|$, and deduce that $|ab| = |ba|$ for all $a, b \in G$. [We shall call yxy^{-1} the *conjugate* of x by y, and $x^{-1}y^{-1}xy$ the *commutator* of x and y; the latter term is explained by the second part of this exercise: the commutator is the identity if and only if x and y commute.]

Finally, the easy special case $p = 2$ of Cauchy's theorem:

7. [D&F 1.1#31] Prove that every finite group G of even order contains an element of order 2. [Let $t(G)$ be the set $\{g \in G : g \neq g^{-1}\}$. Show that $t(G)$ has an even number of elements and that every element of G that is not in $t(G)$ is either the identity or an element of order 2.]

Note that this argument actually shows that the number of order-2 elements is odd if $|G|$ is even, and vice versa. We shall see before long (and even before proving Cauchy's theorem) that if $|G|$ is odd then $|G|$ cannot contain an element of order 2.

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[2]These closure assumptions mean: if $h \in H$ then $x^{-1} \in H$, and if $h \in H$ and $k \in H$ then $h \star k \in H$.

[3]The textbook spells out what each of the three axioms requires. The identity axiom is a good example of a place where the notation 1_G can be useful: it might be clearer to write $1_{A \times B} = (1_A, 1_B)$.