Freshman Seminar 24i: Mathematical Problem Solving

Some problems on generating functions

Some more examples using finite generating functions (we might call them "generating polynomials"):

1. Find a formula for the alternating sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - + \cdots \pm \binom{n}{n}^2$$

of the squares of the entries of the n-th row of Pascal's triangle. (We showed Monday that the sum of the squares is $\binom{2n}{n}$.) Make sure to check that your answer agrees with the first few cases such as

$$1^2 - 2^2 + 1^2 = -2, \quad 1^2 - 3^2 + 3^2 - 1^2 = 0, \quad 1^2 - 4^2 + 6^2 - 4^2 + 1^2 = 6.$$

[It's OK to give your answer in a form that depends on the parity of n, i.e. one formula for even n, another for odd n. LaTeX provides a `\cases` command for such expressions.]

2. i) Prove that if a fair coin is thrown $n > 0$ times then the total number of heads is even with probability exactly 50%.
ii) If a fair coin is thrown 100 times, is the probability that the total number of heads is a multiple of 4 equal to, greater than, or less than 25%, and (if not equal) by how much? [Hint: expand $(1+i)^n$.]

3. [From my old Chess and Math seminar] Use the expansion

$$\begin{aligned}(1+x+x^2)^7 = x^{14} + 7x^{13} + 28x^{12} + 77x^{11} + 161x^{10} + 266x^9 + 357x^8 \\ + 393x^7 + 357x^6 + 266x^5 + 161x^4 + 77x^3 + 28x^2 + 7x + 1\end{aligned}$$

to count the number of 7-move King paths from e1 to e8 (that is, from $(5,1)$ to $(5,8)$ on the standard 8×8 chessboard). How many 7-move King paths are there from e1 to f8 or g8? (Careful here!) What chess enumeration problem is solved by the constant coefficient of the "Laurent polynomial"

$$(xy + x + xy^{-1} + y + y^{-1} + x^{-1}y + x^{-1} + x^{-1}y^{-1})^6 \text{ ?}$$

4. [Another chess enumeration, this one giving rise to a power series, not a polynomial.] A Knight starts on square b1 (the middle square of the bottom row) of a $3 \times n$ chessboard, and is allowed to move only forward, so it has two options at each turn (from b1 to a3 or c3, from a3 to c4 or b5, etc.). For each $n = 1, 2, 3, \ldots$, let a_n and b_n be the number of paths from b1 to an and bn respectively. Show that for $n > 2$ we have $a_n = a_{n-1} + b_{n-2}$ and $b_n = 2a_{n-2}$. Use this to set up equations for the generating functions

$$\begin{aligned} A &= \sum_{n=1}^{\infty} a_n t^n = t^3 + t^4 + t^5 + t^6 + 3t^7 + 5t^8 + 7t^9 + 9t^{10} + 15t^{11} + \cdots, \\ B &= \sum_{n=1}^{\infty} b_n t^n = t + 2t^5 + 2t^6 + 2t^7 + 2t^8 + 6t^9 + 10t^{10} + 14t^{11} + \cdots, \end{aligned}$$

and solve them to recover A and B as rational functions of t.

More examples of using (infinite) generating functions to prove identities:

5. Let $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ be the generating function associated to the sequence $\{a_n\}_{n=0}^{\infty}$. What is the generating function associated to the sequence of partial sums $s_n = a_0 + a_1 + a_2 + \cdots + a_n$? Use this to give an alternative proof of the "hockey stick identity" $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$ and to check the formula for the sum of the first n Fibonacci numbers (if you don't know this formula you should be able to easily surmise it and then prove it by induction).

6. Let $\varphi = (1+\sqrt{5})/2$ be the Golden Ratio and $\bar{\varphi} = (1-\sqrt{5})/2$ its algebraic conjugate. Check that

$$\varphi + \bar{\varphi} = 1, \quad \varphi\bar{\varphi} = -1, \quad \varphi - \bar{\varphi} = \sqrt{5}.$$

Using these identities,[1] check that $1 - x - x^2 = (1 - \varphi x)(1 - \bar{\varphi} x)$, and find the coefficients $a, \bar{a}$ of the "partial fraction decomposition"

$$\frac{1}{1-x-x^2} = \frac{a}{1-\varphi x} + \frac{\bar{a}}{1-\bar{\varphi} x}.$$

Use this to derive the formula for the n-th Fibonacci number in terms of φ and $\bar{\varphi}$.

Generating functions and partitions:

7. A "partition" of a natural number n is a decomposition of n as a sum of natural numbers (the "parts"), without regard to order. For example, $3+1+1$ and $1+3+1$ are the same partition of 5; equivalently, we may require the parts to be listed in non-increasing order, as we do later in this and the next problem. The number of partitions of n is denoted by p_n. For example: $p_5 = 7$, the seven partitions of 5 being

$$5, \;\; 4+1, \;\; 3+2, \;\; 3+1+1, \;\; 2+2+1, \;\; 2+1+1+1, \;\; 1+1+1+1+1.$$

We deem p_0 to equal 1. Prove that the generating function $\sum_{n=0}^{\infty} p_n x^n$ equals[2]

$$\prod_{m=1}^{\infty} \frac{1}{1-x^m} = \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3}\,\frac{1}{1-x^4}\cdots.$$

More generally, for any $a_1, a_2, a_3, \ldots$ the number of partitions of n each of whose parts is one of $a_1, a_2, a_3, \ldots$ is the coefficient of x^n in $\prod_i 1/(1 - x^{a_i})$. For example, the number of different ways to settle an n-cent debt in U.S. coins is the x^n coefficient in

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})(1-x^{100})}$$

(but good luck finding a 50-cent piece). How would you adjust this formula to reflect the fact that there are actually two kinds of dollar coins in circulation? What if you cannot use more than 15 pennies (in accordance with the urban myth that 16 or more pennies are not legal tender)?

8. Prove that the number of partitions of an integer n with no part divisible by 3 is the same as the number of partitions of n with no part repeated more than twice. (For example, when $n = 6$ the first count omits only the four partitions 6, $3+3$, $3+2+1$, $3+1+1+1$, and the second count only $2+2+2$, $3+1+1+1$, $2+1+1+1+1+1$, $1+1+1+1+1+1$, so both counts are $p_6 - 4$.) Generalize.

[1] Trust me, this is easier than working directly with the explicit form $(1 \pm \sqrt{5})/2$ of φ and φ'.

[2] $\prod = \Pi$roduct, used in the same way as $\sum = \Sigma$um.

Exponential generating functions:

9. For sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ let $A(x) = \sum_{n=0}^{\infty} \alpha_n x^n/n!$ and $B(x) = \sum_{n=0}^{\infty} \beta_n x^n/n!$ be the exponential generating functions. What is the "exponential convolution" sequence, call it $\{\gamma_n\}_{n=0}^{\infty}$, whose exponential generating function $\sum_{n=0}^{\infty} \gamma_n x^n/n!$ is $A(x)B(x)$? (This should give you a sense of one kind of context where you might prefer to use exponential rather than ordinary generating functions.)

10. Define numbers β_n by the exponential generating function

$$\beta_0 + \beta_1 x + \beta_2 \frac{x^2}{2!} + \beta_3 \frac{x^2}{3!} + \cdots = \sum_{n=0}^{\infty} \beta_n \frac{x^n}{n!} = \frac{t}{e^t - 1}.$$

Compute at least β_0 through β_7 (you're welcome to use a computer package, but please make sure you get answers such as $\beta_4 = -1/30$ rather than $\beta_4 = -0.03333\ldots$).

i) What pattern do you observe for the odd-order coefficients β_{2m+1}? Prove it.

ii) Define polynomials $b_n(t)$ by

$$b_n(t) = \sum_{m=0}^{n} \binom{n}{m} \beta_m t^{n-m} = t^n + n\beta_1 t^{n-1} + \binom{n}{2} \beta_2 t^{n-2} + \cdots + \binom{n}{n} \beta_n.$$

[This expression is sometimes denoted by "$(\beta+t)^{[n]}$": expand $(\beta+t)^n$ binomially and replace each factor β^m by β_m.] Find a formula for the exponential generating function $\sum_{n=0}^{\infty} b_n(t) x^n/n!$, and use it to prove the identity $b_n(t+1) - b_n(t) = nt^{n-1}$. It follows that for $t = 1, 2, 3 \ldots$ we have

$$1 + 2^n + 3^n + \cdots + t^n = \frac{1}{n+1}(b_{n+1}(t) - \beta_{n+1}).$$

Thus these β_n let us generalize the familiar formulas for the sums of the first t natural numbers and their squares.