Freshman Seminar 24i: Mathematical Problem Solving
The two-dimensional cross product

The *cross product* of length-2 vectors $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$ is defined by the formula[1]

$$\mathbf{v}_1 \times \mathbf{v}_2 = x_1 y_2 - x_2 y_1.$$

1 Verify the identities

$$\mathbf{v}_1 \times (\mathbf{v}_2 + \mathbf{v}_2') = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \mathbf{v}_2', \quad (\mathbf{v}_1 + \mathbf{v}_1') \times \mathbf{v}_2 = \mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1' \times \mathbf{v}_2$$

(all vectors $\mathbf{v}_1, \mathbf{v}_1', \mathbf{v}_2, \mathbf{v}_2'$) and

$$(a\mathbf{v}_1) \times \mathbf{v}_2 = a(\mathbf{v}_1 \times \mathbf{v}_2) = \mathbf{v}_1 \times (a\mathbf{v}_2)$$

(all vectors $\mathbf{v}_1, \mathbf{v}_2$ and constants a) — that is, "the cross product is a bilinear pairing". Also check that $\mathbf{v} \times \mathbf{v} = 0$ for all $\mathbf{v}$ ("the cross product is alternating").

2 *Using only those identities*, prove that $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$ for all $\mathbf{v}_1$ and $\mathbf{v}_2$.

[You might wonder why I'm tying one of your hands behind your back by limiting you to using the identities of the previous problem, when the formula $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$ is easy to verify directly from the definition. There are several reasons for this. One is that alternating bilinear pairings arise in various other contexts in mathematics — see the end of this batch of problems for a few examples — and it is often easier to check the distributive laws and the identity $\mathbf{v} \times \mathbf{v} = 0$, from which the identity $\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1$ follows automaticaly, than to prove that identity directly. Another reason is that this kind of problem is typical of "functional equation" problems that tell you a function of one or more variables satisfies some identities and asks to obtain some other properties of the function; such problems appear not just in math contests but also in more structural mathematics, starting with the definitions of such basic concepts as "group", "commutative group", and "vector space".]

3 Deduce from the same identities that

$$(a\mathbf{v}_1 + b\mathbf{v}_2) \times (c\mathbf{v}_1 + d\mathbf{v}_2) = (ad - bc)(\mathbf{v}_1 \times \mathbf{v}_2),$$

for all vectors $\mathbf{v}_1, \mathbf{v}_2$ and constants a, b, c, d, whence this identity holds for any bilinear alternating pairing. (Note that this time it is more laborious, albeit still feasible, to check the identity from the definition of the cross product.)

4 Conclude that the identities "determine the cross-product up to scaling": any alternative cross-product $*$ that also satisfies the identities $\mathbf{v}_1 * (\mathbf{v}_2 + \mathbf{v}_2') = \mathbf{v}_1 * \mathbf{v}_2 + \mathbf{v}_1 * \mathbf{v}_2'$, $(\mathbf{v}_1 + \mathbf{v}_1') * \mathbf{v}_2 = \mathbf{v}_1 * \mathbf{v}_2 + \mathbf{v}_1' * \mathbf{v}_2$, and $(a\mathbf{v}_1) * \mathbf{v}_2 = a(\mathbf{v}_1 * \mathbf{v}_2) = \mathbf{v}_1 * (a\mathbf{v}_2)$ can be written as $\mathbf{v}_1 * \mathbf{v}_2 = C\mathbf{v}_1 \times \mathbf{v}_2$ for some constant C (namely $C = (1, 0) * (0, 1)$).

5 If x_1, y_1 and x_2, y_2 are the real and imaginary parts of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, define $z_1 * z_2$ to be the imaginary part of $\bar{z}_1 z_2$ (recall that if z is a complex number $x + iy$ then $\bar{z}$ is its complex conjgate $x - iy$). Show that $z_1 * z_2 = (x_1, y_1) \times (x_2, y_2)$.

Give an analogous formula for "dot product" $(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$ (a.k.a. the "inner product") of (x_1, y_1) and (x_2, y_2)). What does this tell you about $(\mathbf{v}_1 \times \mathbf{v}_2)^2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2$?

6 Now define $*$ geometrically as follows: $\mathbf{v}_1 * \mathbf{v}_2$ is the area of the triangle with vertices $0, \mathbf{v}_1, \mathbf{v}_2$ if $\mathbf{v}_2$ is to the left of $\mathbf{v}_1$, or -1 times this area if $\mathbf{v}_2$ is to the right. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are on a line through the origin, the triangle is "degenerate" and has area zero. Show that this is again an alternating bilinear

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[1] The LaTeX for $\times$ is \times .

form, and thus that $\mathbf{v}_1 * \mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 \times \mathbf{v}_2$ for all $\mathbf{v}_1, \mathbf{v}_2$. (This is not as easy as the $z_1\bar{z}_2$ formula to check directly...)

7 Use this to show that any triangle with vertices A, B, C has area $\frac{1}{2}|A \times B + B \times C + C \times A|$. Give a cross-product formula for the area of a quadrilateral with vertices A, B, C, D in this order, and generalize to arbitrary polygons with vertices $A_1, A_2, \ldots, A_n$.

Finally, a few other appearances of this two-dimensional cross product:

8 For a function f of one variable t, with derivative f', define $T_{x,y}f$ to be the function

$$(T_{x,y}f)(t) = xf'(t) + ytf(t).$$

If $\mathbf{v}$ is the vector (x, y) we abbreviate $T_{x,y}$ by $T_\mathbf{v}$. Note the linearity properties: $T_\mathbf{0}f$ is the zero function; $cT_\mathbf{v}f = T_{c\mathbf{v}}f$ for all constants c and vectors $\mathbf{v}$; and

$$T_{\mathbf{v}_1}f + T_{\mathbf{v}_2}f = T_{\mathbf{v}_1+\mathbf{v}_2}f$$

for any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. If f' is itself differentiable, it makes sense to apply $T_{\mathbf{v}_1}$ to $T_{\mathbf{v}_2}f$ and vice versa, and we can measure the difference using the commutator

$$[T_{\mathbf{v}_1}, T_{\mathbf{v}_2}]f = (T_{\mathbf{v}_1}T_{\mathbf{v}_2} - T_{\mathbf{v}_2}T_{\mathbf{v}_1})f = T_{\mathbf{v}_1}T_{\mathbf{v}_2}f - T_{\mathbf{v}_2}T_{\mathbf{v}_1}f.$$

Show that this is just $\mathbf{v}_1 \times \mathbf{v}_2$ multiplied by f.

9 The *Heisenberg group* consists of 3×3 matrices of the form

$$h(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the identity matrix is $h(0, 0, 0)$, and the group operations are given by the formulas

$$h(x_1, y_1, z_1)h(x_2, y_2, z_2) = h(x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1y_2)$$

and

$$h(x, y, z)^{-1} = h(-x, -y, xy - z).$$

[You need not check these: this would be almost trivial if you have already seen matrix multiplication, and distracting if you haven't, so I'm giving you the formulas to help put everybody on the same footing.]

Define the (multiplicative) *commutator* of invertible matrices A, B to be $ABA^{-1}B^{-1}$: this is the identity if and only if $AB = BA$, that is, if and only if A commutes with B. What is the commutator of $h(x_1, y_1, z_1)$ and $h(x_2, y_2, z_2)$? What does this have to do with the previous problem?

10 If now f is a function of $t > 0$, and a, b are constants with $b > 0$, define $H_{a,b}f$ to be the function

$$(H_{a,b}f)(t) = t^a f(bt).$$

How do $H_{a_1,b_1}H_{a_2,b_2}f$ and $H_{a_2,b_2}H_{a_1,b_1}f$ compare? Can you relate this with the Heisenberg group as well?