

Freshman Seminar 24i: Mathematical Problem Solving

Some induction problems

1. It can be shown¹ that every planar n -gon ($n > 3$) P has an “interior diagonal” — that is, two nonconsecutive vertices V, V' such that the line segment joining V, V' is contained in the interior of P . Use this to prove that the interior angles of P total $(n - 2)180^\circ$ (a.k.a. $(n - 2)\pi$ radians). [Which version of induction is natural to use here?]

2. Recall that $\binom{n}{k}$ is the binomial coefficient (a.k.a. combinatorial coefficient) defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!};$$

if $k < 0$ or $k > n$ we set $\binom{n}{k} = 0$.

i) Given $k \geq 0$ and $n \geq k$, what is $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k}$?
 ii) [somewhat trickier] Given $k \geq 2$ and $n \geq k$, what is

$$\frac{1}{\binom{k}{k}} + \frac{1}{\binom{k+1}{k}} + \frac{1}{\binom{k+2}{k}} + \cdots + \frac{1}{\binom{n}{k}}?$$

3. For $n > 0$ and $d \geq 0$, how many monomials of total degree d are there in n variables? For example, when $d = n = 3$ the number is 10: using variables x, y, z we find the cubic monomials

$$x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3.$$

(In particular the answer is not simply $\binom{n}{d}$: that’s the number of degree- d monomials in which no variable appears to power greater than 1.)

4. Give a formula for $(\cos x)(\cos(2x))(\cos(4x))(\cos(8x)) \cdots (\cos(2^n x))$. [Especially if you’re seeing this for the first time, you might try to use the same trick to evaluate other products such as $\prod_{n=1}^{89} \cos(n^\circ) = \cos(1^\circ)\cos(2^\circ)\cos(3^\circ) \cdots \cos(89^\circ)$, or to construct a problem along the same lines involving $f(x)f(3x)f(9x)f(27x) \cdots f(3^n x)$ for some function f .]

5. [An IMO problem, but it’s from the Easiest IMO Ever, and we weren’t told there that this was an induction problem . . .] Find the integer solution (x, y) of $(x^2 + xy - y^2)^2 = 1$ that has the largest value of $x^2 + y^2$ subject to the conditions $0 \leq x \leq 1981$, $0 \leq y \leq 1981$.

(Follow-up: what can you say about the Diophantine equation² $(x^2 + 4xy - y^2)^2 = 1$?)

6. [Thanks to Sonal Jain for suggesting this one] For a S set of n (distinct) positive numbers, let $\Sigma(S) = \{\sum_{t \in T} t \mid T \subseteq S\}$; that is, $\Sigma(S)$ is the set of positive numbers that can be written as the sum of some (possibly empty) subset $T \subseteq S$. Given n , how small can the cardinality $\#(\Sigma(S))$ be? For example, if $n = 1$ or $n = 2$ then all 2^n sums are distinct, and for $n = 3$ there can be at most one coincidence among the 2^3 sub-sums (if the largest element of S is the sum of the other two); so the minimal cardinality is 2, 4, 7 for $n = 1, 2, 3$ respectively.

¹Let v be the left-most vertex (or one of them if there’s a choice), and v', v'' its neighbors along the boundary of P . If $v'v''$ is an interior diagonal, we are done. Else there is an interior diagonal vw for some other vertex w in the triangle formed by v, v', v'' ; for instance, we may choose for w the vertex in that triangle, other than v, v', v'' , that is closest to v . Thanks to Zach, our resident computational geometer, for finding this construction. Where did we use $n > 3$?

With some more care we can even use this construction to prove that P has an “interior”, that is, the fact (which I relegated in class to an application of the Jordan curve theorem) that P splits the plane into exactly two connected regions, an “interior” and an “exterior” of P .

²That is, an equation to be solved in integers; Diophantus originally worked with rational numbers, but that can always be encoded into integer solutions as well by replacing an equation in rational numbers r_1, r_2, \dots, r_n by a homogeneous equation in integers $x_0, x_1, x_2, \dots, x_n$ where $r_i = x_i/x_0$ for each $i = 1, 2, \dots, n$.