Freshman Seminar 24i: Mathematical Problem Solving

Using AM–GM and other applications of Jensen's inequality

Some simple applications:

1) Of all rectangular boxes of volume V, the cube has the smallest surface area.

2) An "open box" is a rectangular box missing one of its six sides (so that we can put things in and out of the box). Given V, what is the open box of volume V that has minimal surface area? Upon further reflection, can you derive this directly from (1)?

3) Of all triangles of area A, the equilateral triangle has the smallest perimeter.

Some proofs of the inequality and generalizations:

4) Say a real-valued function f on some (possibly infinite) interval I is "midpoint-convex" upwards if

$$f\left(\frac{1}{2}(x+y)\right) \le \frac{1}{2}\Big(f(x)+f(y)\Big)$$

for all x, y in I; we say f is *strictly* midpoint-convex upwards if equality holds only for $x = y$. Prove that if f is midpoint-convex upwards then Jensen's inequality

$$f\left(\frac{1}{n}(x_1+x_2+\cdots+x_n)\right) \le \frac{1}{n}\Big(f(x_1)+f(x_2)+\cdots+f(x_n)\Big)$$

holds for all n and for all $x_1, \ldots, x_n$ in I; and that if f is strictly midpoint-convex then equality holds if and only if all the x_i are equal. [Hint: the inequality is trivial for $n = 1$ and the definition of midpoint-convexity for $n = 2$. Prove it first for $n = 4, 8, 16, \ldots$; to get the general case, try first to derive the $n = 3$ case from $n = 4$. Thanks to Zach for reminding me of the AM–GM case of this trick.]

5) Show directly (without calculus etc.) that the functions x^2 and e^x are midpoint-convex upwards on all of $\mathbf{R}$, that $1/x$ is midpoint-convex upwards on $x > 0$, and that $\sin x$ is midpoint-convex downwards on $0 \le x \le \pi$. (Hint: it can be surprisingly useful that $a \ge b$ if and only if $a - b \ge 0$.) Jensen's inequality for these functions then follows from the previous problem.

6) Suppose $w_1, w_2, \ldots, w_n$ are nonnegative numbers with $w_1 + w_2 + \cdots + w_n = 1$. The *weighted average* of $z_1, z_2, \ldots, z_n$ with weights $w_1, w_2, \ldots, w_n$ is $w_1 z_1 + w_2 z_2 + \cdots + w_n z_n$. (For instance, the usual weighted average is recovered by setting each w_i equal $1/n$, and the barycentric coordinates of a point P in triangle ABC are the weights for which P is the weighted average of A, B, C — that's one reason I didn't restrict $z_1, \ldots, z_n$ to real numbers). *Jensen's inequality for weighted averages* states that

$$f(w_1x_1 + w_2x_2 + \cdots + w_nx_n) \le w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$$

if f is convex upwards, and likewise

$$f(w_1x_1 + w_2x_2 + \cdots + w_nx_n) \ge w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$$

if f is convex downwards. (Again the inequality compares the function of the average with the average of the function.) Prove this inequality: (i) in the same graphical way that we did for the unweighted version; (ii) as a consequence of Jensen (first do the case of rational w_i, then use continuity). If f is strictly convex, when does equality hold?

7) Show that weighted Jensen still reduces to Cauchy–Schwarz for $f(x) = x^2$ or $f(x) = 1/x$.

8) Suppose P is a polynomial with positive coefficients. Prove that $P(x)P(y) \ge \big(P(\sqrt{xy}\,)\big)^2$ for all $x, y > 0$. When does equality hold? What are the functions whose convexity you can deduce from that inequality?

9) Suppose $f(x)$ is convex downwards on $a \le x \le b$ and we fix some $n \ge 1$ and s between na and nb. Consider the sum $f(x_1) + \cdots + f(x_n)$ subject to $a \le x_i \le b$ and $x_1 + \cdots + x_n = s$. Jensen tells us that this is maximized when each x_i equals s/n. What choice of $x_1, \ldots, x_n$ *minimizes* the sum? For instance, how small can $\sin\alpha + \sin\beta + \sin\gamma + \sin\delta + \sin\epsilon$ get if $0 \le \alpha, \beta, \gamma, \delta, \epsilon \le \pi/2$ and $\alpha + \beta + \gamma + \delta + \epsilon = 5$?