Freshman Seminar 23j (Spring [2005–]2006): Chess and Mathematics

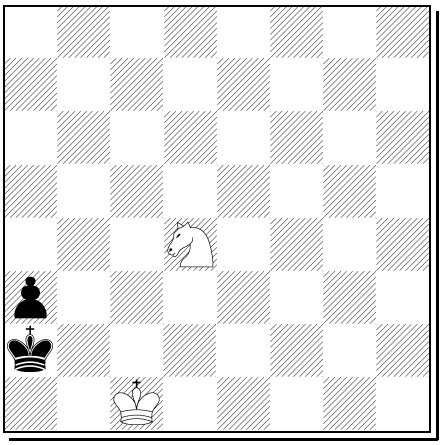
Preliminary Puzzle

Here is a mathematical chess puzzle (see Section 3 below) that introduces some of the ideas, techniques, and questions that we'll consider in the Chess and Math seminar. We begin with a chess endgame with only four units on the board (Diagram 1 below). We then modify the position by replacing the Knight by an unorthodox chess piece, the "Nightrider" (Section 2). Finally we also allow an arbitrary rectangular board in place of the orthodox 8×8 square, and ask how the outcome of the position depends on the board size (Section 3).

See Section 4 for instructions on how to communicate your solutions, partial solutions, generalizations, conjectures, queries, and other comments to me.

1. A warmup endgame. Consider the position shown in Diagram 1 (White King on c1, Knight on d4; Black King on a2, Pawn on a3). What is the outcome with best play by White and Black?

Diagram 1



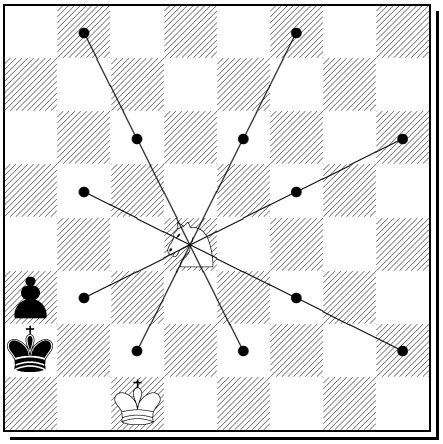
What outcome?

2. Introducing the Nightrider. We'll now study the same position but with the Knight on d4 replaced by a stronger piece, the **Nightrider**. This is an unorthodox chess piece, introduced by T.R. Dawson (1889–1951); its move consists of one or more knight moves *along a straight line*. For instance, an unobstructed Nightrider on d4 can play to any of the eight

squares controlled by a Knight on the same square, plus h2, h6, b8, and f8 (see Diagram 2). We say "unobstructed" because, for instance, if the square e6 is occupied then the Nightrider cannot move from d4 to f8 (though like a Knight it can still capture the unit on e6 if it is the opponent's).

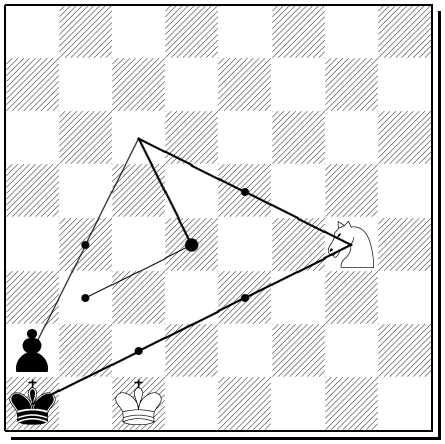
Consider first what could happen in Diagram 1 with Black to move. The only legal move is Ka1 (since Nd4 controls b3). Now White can move the Nd4 to c6, controlling a2 via b4. Then Black's only legal move is to push the pawn from a3 to a2. Now White can checkmate by moving the Nc6 to g4, the only available square that attacks a1. This produces the position in Diagram 3, which also shows the Nightrider's route and the lines on which it controlled the key squares b3, a2, and a1.

Diagram 2



squares controlled by Nightrider d4

Diagram 3



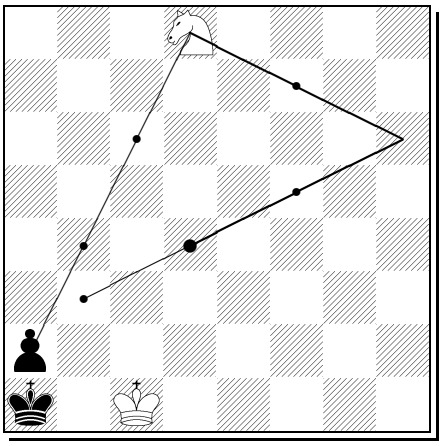
after 1. . . Ka1 2 Nc6 a2 3 Ng4#

Now what happens if it is White's move in Diagram 1? If allowed to pass the move, White could force mate in 3 (that is, in reply to Black's second move) as shown in Diagram 3. But passing the move is not allowed in chess,[1] and White has no suitable "waiting move": playing Ne2 or Nb8, to keep control of c6 along the e2–b8 line (shown on Diagram 2), would lose control of b3 and let Black's King escape; and playing Kc2 would obstruct the g4–a1 line that White needs for the checkmate (see Diagram 3 again). White can try 1 Nh6, keeping control of b3 via f5-d4, and after Black's forced reply Ka1 White can control a2 by playing Nd8 to force the pawn push, producing Diagram 4; but this, too, is in vain: even though the Black

[1]This is just as well for White: if passing were allowed, Black could pass too instead of playing Ka1, and White would have made no progress anyway. . .

King is stuck without a single legal move, the Nightrider on d8 cannot move to any of the six squares b3, c5, d7, c2, e3, g4 from which it would have given checkmate.

Diagram 4



after 1 Nh6 Ka1 2 Nd8 a2: no checkmate

So, while White *can* win the Nightrider version of Diagram 1 even if it is White's turn,[2] it takes more than 3 moves.

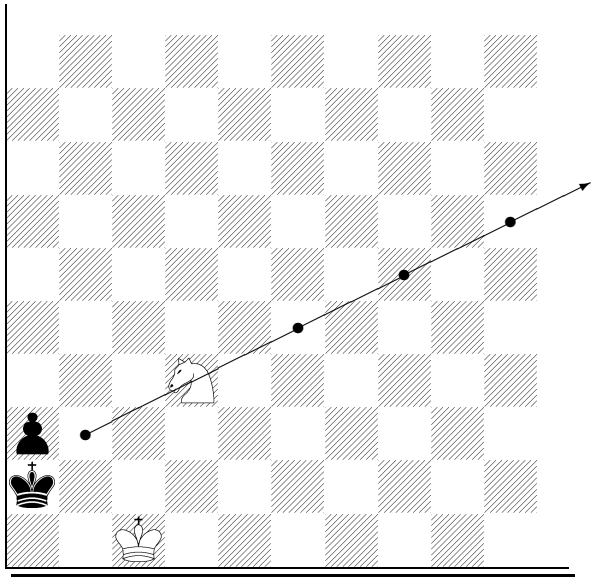
3. Diagram 1 on bigger boards. As long as we're being unorthodox, we should not be constrained by the traditional 8×8 size of the board: while this size makes chess a very good game in practice, mathematically speaking the 8×8 board has no particular advantage over a 5×5 or 32×32 or even 113×355 board. In fact, one natural choice is to impose no limit at all and imagine a "quarter-infinite" board with only one corner, at a1. The bottom left 10×10 corner of such a board, with the now-familiar Diagram 1 setup, is shown in Diagram 5.

On such a board an unobstructed Nightrider has infinitely many moves; Diagram 5 shows the first few of them along the line from b3 to d4, f5, h6, j7 [sic], and beyond. Since we'll run out of letters after z15, let us use a fully algebraic notation, with each square specified by two numerical coordinates instead of one letter and one number. Thus Diagram 5 (and 1) can be described as: White King $(3, 1)$, Nightrider $(4, 4)$; Black King $(1, 2)$, pawn $(1, 3)$. Then the moves east-by-northeast shown in Diag. 5 are $(4, 4)$ to $(6, 5)$, $(8, 6)$, $(10, 7)$, and in general the

[2]One way is to start 1 Nf5 Ka1 2 Nh6 Ka2 — pushing the Pawn to a2 would allow 3 Nb3, checkmate — and now 3 Nd4 returns to Diagram 1 but with Black to move, and we already know how White can win in this case. Note that White used the Nightrider to in effect pass the turn to Black with the 3-move maneuver Nd4-f5-h6-d4. We shall see early in the semester that the orthodox Knight, unless the Nightrider, cannot do this trick in any odd number of moves, even without the requirement of keeping control of b3.

n-th such move takes the Nightrider from $(4,4)$ to $(4+2n, 4+n)$ for each $n = 1, 2, 3, \ldots$. (For instance, z15 becomes $(26, 15)$, for $n = 11$.) Each of these moves keeps control of b3 — oops, I mean $(2, 3)$ — and forces Black to play the King to (1,1) as before.

Diagram 5



quarter-infinite board: Mate in 3

Okay, now that we have some feel for Nightrider geometry we're finally ready to state our puzzle. It turns out that in Diagram 5 white does have a forced mate in 3, in fact infinitely many such.

i) Give an example of one such mate in 3. Can you show that there are infinitely many? Can you find one or more formulas that describe *all* possible mates in 3?

ii) What is the *smallest* rectangular board on which White can force mate in 3?

iii) What is the smallest rectangular board on which White has a choice between at least two mates in 3? [A problemist would say that on such a board the mate-in-3 problem has a "cook" and is thus unsound.]

iv) What is the smallest rectangular board on which White has a choice between at least two mates in 3 *which start with the same first move*? [A problemist would call this a "dual".]

You're welcome to use any tools to solve this puzzle, short of trying to contact problemists outside Harvard or looking for this puzzle in the literature; in particular, trial and error (whether in the form of hand calculation or computer experimentation), mathematical analysis, or any combination is legitimate.

4. e-mail instructions. You can e-mail your solutions, partial solutions, generalizations, conjectures, queries, and other comments to me at elkies@math.harvard.edu . Please use text only if at all possible; do not gratuitously MIME your message, "attach" it, or duplicate it in HTML: I do not use a browser to read my e-mail, and mathematical writing is particularly hard to read when encrusted in MIME or HTML incantations. Whatever you do, **do not e-mail me a Micro$oft Word (.doc) file**: I will not be able to read it. If you need to include a diagram, please either convert it to .pdf, or just write up your answer legibly by hand and then send or bring it to my math department mailbox or office (both in the 3rd floor of the Science Center). Thank you.

Good luck!

—Noam D. Elkies