# Elliptic K3 surfaces with a 6-torsion section

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Smarch, 2019

**Abstract** *n.*: a BS tract.

## 0 Introduction

[...]

## 1 Elliptic curves with a 6-torsion point

The results in this section are far from new,[1] at least outside characteristic 2 and 3.

**Lemma 1** *Let $E$ be any elliptic curve with a 6-torsion point $P$ over any field $k$. Then $E$ has an extended Weierstrass model*

$$E_6(b,c):\ y^2+(b-c)xy+bc(b+c)y = x^3-b(b+c)x^2 \tag{1}$$

*for some $b, c \in k$ with*

$$b \neq 0, \quad c \neq 0, \quad b+c \neq 0, \quad 9b+c \neq 0, \tag{2}$$

*with $nP$ ($n = 1,2,3,4,5$) having $(x,y)$ coordinates*

$$(0,0),\ \ (b(b+c), -b^2(b+c)),\ \ (bc, -b^2c),\ \ (b(b+c), 0),\ \ (0, -bc(b+c)). \tag{3}$$

*Conversely, for any $b, c \in k$ satisfying (1), the curve $E_6(b,c)$ has a 6-torsion point at $(0,0)$. If $b', c' \in k$ also satisfy (1) then there is an isomorphism $E_6(b,c) \to E_6(b',c')$ sending $(0,0)$ to $(0,0)$ if and only if $(b',c') = (\lambda b, \lambda c)$ for some $\lambda \in k^*$, in which case the isomorphism takes $(x,y)$ to $(\lambda^2 x, \lambda^3 y)$.*

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[1] Tate's expository paper [Tate 1974] already gives a similar (and necessarily more complicated) formula for 7-torsion (§7, page 195), citing [Ogg 1971] for the connection with the modular curves $X_1(m)$.

*Proof*: Given any elliptic curve $E$ with a nonzero rational point $P$, we can start from any extended Weierstrass model

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{4}$$

and translate $x$ and $y$ to put $P$ at $(0, 0)$. This gives a model (4) with $a_6 = 0$. Then the tangent line to $E$ at $P$ is $a_3y = a_4x$, so $a_3 = 0$ if and only if $2P = 0$. Assume, then, that $2P \neq 0$. Then we can translate $y$ by $(a_4/a_3)x$ to obtain a yet simpler model (4) with $a_4 = a_6 = 0$. Now the tangent line at $P$ is $y = 0$, so we obtain the following coordinates for $nP$ with $n = \pm 1, \pm 2$:

$$P : (0, 0), \quad -P : (-a_3, 0), \quad 2P : (-a_2, 0), \quad -2P : (a_1a_2 - a_3, -a_2). \tag{5}$$

Hence $3P = 0 \iff P = -2P \iff a_2 = 0$. Assume, then, that $3P \neq 0$, and thus that $a_2 \neq 0$. We then find $-3P$ by intersecting the cubic $y^2 + a_1xy + a_3y = x^3 + a_2x^2$ with the line $(a_1a_2 - a_3)x + a_2y = 0$ through $P$ and $2P$. This gives

$$3P : (x, y) = \left( \frac{a_3(a_3 - a_1a_2)}{a_2^2}, \; \frac{a_3(a_3 - a_1a_2)^2}{a_2^3} \right). \tag{6}$$

We find that $3P$ is a 2-torsion point if and only if

$$a_2^2(a_1^2 + a_2) + 2a_3^2 - 3a_1a_2a_3 = 0. \tag{7}$$

Since $a_2$ and $a_3$ are nonzero, we can write $a_3 = -ca_2$ and $a_1 = -ec$ for some $e, c \in k$; then (7) becomes

$$a_2^2\Big((e-1)(e-2)c^2 + a_2\Big) = 0, \tag{8}$$

so $a_2 = -(e-1)(e-2)/c^2$. Finally, taking $b = (e-1)c$ (so $e = 1 - (b/c)$) we recover $a_2 = -b(b+c)$, and then $a_3 = -ca_2 = bc(b+c)$ and $a_1 = ((b/c)-1)c = b - c$, identifying $E$ with the model $E_6(b, c)$ exhibited in (1). We then readily confirm that $3P$, and our formulas (5) for $\pm P$ and $\pm 2P$, agree with (3). We also compute that the curve (1) has discriminant

$$\Delta(E_6(b, c)) = b^6c^2(b + c)^3(9b + c), \tag{9}$$

and is thus nonzero if and only if $b$ and $c$ satisfy (2). Finally, the model (1) is homogeneous of weight 6 under scaling $b, c, x, y$ to $\lambda b, \lambda c, \lambda^2 x, \lambda^3 y$. Conversely, for any elliptic curves $E, E'$ in extended Weierstrass form with rational points $P, P'$ at $(0, 0)$, any isomorphism taking $(E, P)$ to $(E', P')$ must take $(x, y)$ to $(\lambda^2 x, \lambda^3 y + sx)$ for some $\lambda \in k^*$ and $s \in k$; in our setting both curves are tangent to $y = 0$ at $(0, 0)$, so $s = 0$ and the isomorphism simply multiplies $a_1, a_2, a_3$ by $\lambda, \lambda^2, \lambda^3$ respectively, which multiplies both $b$ and $c$ by $\lambda$. $\square$

*Remark*: In effect this Lemma gives projective coordinates $(b : c)$ identifying the modular curve $\mathrm{X}_1(6)$ with $\mathbf{P}^1$; the geometry of the universal curve $\mathcal{X}_1(6)$ as a rational elliptic surface. The excluded $(b : c)$ in (2) are the cusps $0, \infty, -1, -9$ of $\mathrm{X}_1(6)$; these are pairwise distinct except when $k$ has characteristic 2 or 3, in which case the cusp $-9$ coincides with $-1$ or $0$ respectively. [One of the three apparent cusps on $\mathrm{X}_1(3)$ or $\mathrm{X}_1(2)$ respectively is fake. Identification of $(b : c)$ with $\mathrm{X}_1(6)$ Hauptmodul; involutions $w_2, w_3, w_6$; map to $\mathrm{X}_1(3)$: change $x$ to $x + b(b + c)$ to bring $4P$ to $(0, 0)$, then change $y$ to $y + (b + c)x$ to get a horizontal tangent, and find $(3b + c, 0, b^2(b + c), 0, 0)$; map to $\mathrm{X}_1(2)$: change $x, y$ to $x + bc, y - b^2c$ to bring $3P$ to $(0, 0)$, finding $(b - c, 2bc - b^2, 0, -b^3c, 0)$ ]

## 2 Elliptic K3's with a 6-torsion section, and their moduli

Recall that an "elliptic K3 surface" over some field $k$ is a K3 surface $S/k$ together with a nonconstant rational function $t \in k(S)$ whose generic fiber is an elliptic curve. Note that by an "elliptic curve" we mean a genus-1 curve with a choice of rational point, so an elliptic K3 surface comes with a choice of zero-section $s_0$ of the elliptic fibration $t : S \to \mathbf{P}^1$. Such a surface can be written in extended Weierstrass form (4) where each coefficient $a_i$ $(i = 1, 2, 3, 4, 6)$ is a section of $O(2i)$; we represent these coefficients by polynomials $a_i(t)$ of degree at most $2i$. It then follows from Lemma 1 that an elliptic K3 surface with a 6-torsion section has a model (1) for some $b, c \in \Gamma(O(2))$, and is determined uniquely by the degree-2 rational function $b/c : \mathbf{P}^1 \to \mathrm{X}_1(6)$. Geometrically $(S, t)$ is a quadratic base change from the rational elliptic surface $\mathcal{X}_1(6)$.[2]

[bad fibers: for $\operatorname{char} k \notin \{2, 3\}$, generically $\mathrm{I}_6$, $\mathrm{I}_3$, $\mathrm{I}_2$, $\mathrm{I}_1$ at the roots of $b$, $b + c$, $c$, $9b + c$ respectively. In characteristic 2, each $\mathrm{I}_1$ merges with a $\mathrm{I}_3$ to make a IV because $9b + c = b + c$; In characteristic 3, each $\mathrm{I}_1$ merges with a $\mathrm{I}_2$ to make a III because $9b + c = c$. But these don't change the contribution of the bad fibers to

[2] Indeed for *any* $m \geq 5$, an elliptic surface (not necessarily K3) with an $m$-torsion section is a base change from the universal elliptic curve $\mathcal{X}_1(m)$ over $\mathrm{X}_1(m)$. If we are interested in elliptic surfaces over $\mathbf{P}^1$ then $m \leq 10$ or $m = 12$, else the curve $X_1(m)$ is not rational so there is no nonconstant elliptic surface over $\mathbf{P}^1$ with an $m$-torsion section. We must exclude $m \leq 2$ because there is no universal curve $\mathcal{X}_1(2)$ due to quadratic twists; for $m = 3$ and $m = 4$ there are universal curves (excluding $j = 0$ when $m = 3$), but they have an additive fiber, so the degree of the base change does not determine its arithmetic genus.

NS, which remains $A_1^2 A_2^2 A_5^2$.]

Choose one root of each of $b, c, b+c$, and use the coordinate $t$ on $\mathbf{P}^1$ so that those roots are at $t = \infty, 1, 0$ respectively (there is a unique such choice because $\mathrm{PGL}_2$ acts sharply 3-transitively on $\mathbf{P}^1$). Then $b$ is a polynomial in $t$ of degree 1; multiply $b$ and $c$ by the same nonzero scalar so that $b$ is monic, say $b = t + B$. Then $b + c = A_0 t^2 + A_1 t$ for some $A_0, A_1 \in k$, and the condition that $c$ vanishes at $t = 1$ determines $B$. We find:

$$B = A_0 + A_1 - 1; \quad b = t + B, \;\; c = (t-1)(A_0 t + B), \;\; b + c = A_0 t^2 + A_1 t. \quad (10)$$

This gives an explicit birational parametrization of the moduli surface $\mathcal{M}_{36}$ by $A_0, A_1$. This moduli surface has three commuting involutions, because for each of $b, c, b+c$ we could have chosen the other root to place at $t = \infty, 1, 0$ respectively, giving rise to alternative $A_0, A_1$ that parametrize the same surface $S$ but with another polarization. Call these involuations $\iota_\infty, \iota_1, \iota_0$. We might expect those to generate a group $(\mathbf{Z}/2\mathbf{Z})^3$ of automorphisms of $\mathcal{M}_{36}$, but in fact the group is only $(\mathbf{Z}/2\mathbf{Z})^2$:

**Lemma 2** *The product $\iota_0 \iota_1 \iota_\infty$ is the identity automorphism of $\mathcal{M}_{36}$.*

*Proof*: The degree-2 map $b/c : \mathbf{P}^1 \to \mathrm{X}_1(6)$ determines an involution $\jmath$ of $\mathbf{P}^1$ that takes any $t \in \mathbf{P}^1$ to the other root of $(b/c)(t)$. In particular, the other roots of $b, c, b+c$ are $\jmath(\infty)$, $\jmath(1)$, and $\jmath(0)$ respectively. To compute the action of $\iota_0 \iota_1 \iota_\infty$, we need the element of $\mathrm{PGL}_2$ that moves these points to $\infty, 1, 0$ respectively; but that is just $\jmath$, and since $\jmath$ is an involution it also takes $\infty, 1, 0$ to $\jmath(\infty)$, $\jmath(1)$, and $\jmath(0)$. Thus each of $b, c, b+c$ changes to some scalar multiple, say $\lambda_1 b$, $\lambda_2 c$, $\lambda_3(b+c)$; and then $\lambda_1 b + \lambda_2 c = \lambda_3(b+c)$ forces $\lambda_1 = \lambda_2 = \lambda_3$ (else $b, c$ would satisfy two independent linear equations, which would make them scalar multiples of each other). Thus after scaling the new $b, c$ to make $b$ monic we obtain the same $b, c$ as in (10), and thus the same $A_0$ and $A_1$. $\Box$

For our $b/c$ we compute

$$\jmath(t) = -\frac{B(A_0 t + A_1)}{A_0(t + B)}. \quad (11)$$

Using this we determine the action of $\iota_0, \iota_1, \iota_\infty$ on our coordinates $A_0, A_1$ of $\mathcal{M}_{36}$, finding the following simple formulas.

**Lemma 3** *The involutions $\iota_0, \iota_1, \iota_\infty$ act on $\mathcal{M}_{36}$ by*

$$\begin{aligned} \iota_0(A_0, A_1) &= (A_0 + A_1, -A_1), \\ \iota_1(A_0, A_1) &= (-B, A_1) = (1 - A_0 - A_1, A_1), \\ \iota_\infty(A_0, A_1) &= (1 - A_0, -A_1). \end{aligned} \tag{12}$$

*Proof*: for $\iota_1$, we take $\infty$ to $\infty$ and 0 to 0, while 1 goes to $-B/A_0$; thus $t$ goes to $-Bt/A_0$. We apply this substitution to (10), and then multiply the resulting $b, c$ by $-A_0/B$ to make the new $b$ monic. The new $b, c, b + c$ are then

$$t - A_0, \;\; -(t-1)(Bt + A_0), \;\; -Bt^2 + A_1 t, \tag{13}$$

so we read $\iota_1(A_0, A_1)$ off the coefficients of the new $b+c$, finding that $\iota_1(A_0, A_1) = (-B, A_1)$ as claimed. (Note that since $-B = (1 - A_1) - A_0$, applying $\iota_1$ again brings $(A_0, A_1)$ back to $(A_0, A_1)$ as expected.)

For $\iota_0$, we need the $\mathrm{PGL}_2$ transformation that fixes 1 and $\infty$ and moves 0 to the nonzero root $-A_1/A_0$ of $b+c$. This transformation is $t \mapsto ((A_0+A_1)t - A_1)/A_0$; applying this to (10), and multiplying the resulting $b, c$ by $A_0/(A_0 + A_1)$ to make the new $b$ monic, we find

$$t + A_0 - 1, \;\; (t-1)((A_0 + A_1)t + A_0 - 1), \;\; (A_0 + A_1)t^2 - A_1 t, \tag{14}$$

whence $\iota_1(A_0, A_1) = (A_0 + A_1, -A_1)$, again as claimed. For $\iota_\infty$, we can carry out a similar albeit somewhat harder computation (using the transformation $t \mapsto -Bt/(t-(A_0+A_1))$ in this case); more simply, we can compose $\iota_0$ with $\iota_1$, which gives $\iota_\infty$ by Lemma 2. $\Box$

Some other simple parametrizations of $\mathcal{M}_{36}$ can be obtained by choosing a coordinate $t'$ so that $t' = 0$ and $t' = \infty$ are the roots of one of $b, c, b + c$ while $t' = 1$ is a root of one of the other two. In recent work of [Weinstein 2019] on Drinfeld moduli spaces in characteristic 2, an elliptic K3 surface $E_6(b, c)$ arises with

$$b = (t'+P)(t'+Q), \quad c = -(t'-1)(t'-PQ), \quad b+c = (P+1)(Q+1)t', \tag{15}$$

so $\jmath$ takes $t'$ to $PQ/t'$. (The minus signs in (15) do not matter in characteristic 2 but are needed in characteristic 0 to make the formula for $b+c$ correct.) To convert this to our form (10), we write $t' = Pt/(P + 1 - t)$ to move the roots at $t' = -P, 0, 1$ to $t = \infty, 0, 1$. Multiplying the resulting $b, c$ by $(P + 1 - t)^2/(P^2 + P)$ gives

$$b = (P - Q)t + (P + 1)Q, \quad c = (1 - t)\big((Q + 1)t - (P + 1)Q\big), \tag{16}$$

and $b + c = (Q + 1)t(P + 1 - t)$. We divide this by $P - Q$ to make $b$ monic, producing $c$ with coefficients

$$A_0 = -\frac{Q+1}{P-Q}, \quad A_1 = \frac{(P+1)(Q+1)}{P-Q}. \tag{17}$$

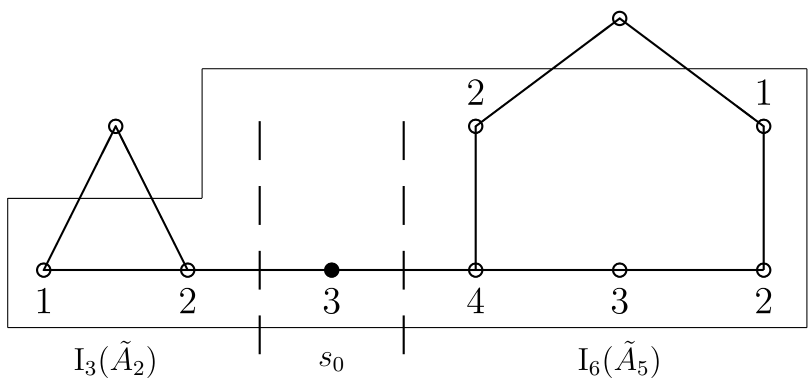
This transformation from $(P, Q)$ to $(A_0, A_1)$ is birational, with inverse

$$P = -\frac{A_0 + A_1}{A_0}, \quad Q = \frac{B}{1 - A_0} \tag{18}$$

(still using $B = A_0 + A_1 - 1$ as in (10)). In these coordinates, the involutions $\iota_0, \iota_1$ take $(P, Q)$ to $(1/Q, 1/P)$ and $(Q, P)$ respectively, so $\iota_\infty = \iota_0\iota_1$ takes $(P, Q)$ to $(1/P, 1/Q)$.

# 3 From $(A_1^2 A_2^2 A_5^2)^{+6}$ to $(A_1^2 E_7^2)^{+2}$

[...]




The $\mathrm{III}^*(\widetilde{E}_7)$ divisor $D$ supported on $s_0$ and the $\mathrm{I}_3, \mathrm{I}_6$ fibers at $t = 0, \infty$

Since $0 < D < 3s_0 + 2f_0 + 4f_\infty$ we have

$$\Gamma(0) \subseteq \Gamma(D) \subseteq \Gamma(3s_0 + 2f_0 + 4f_\infty), \tag{19}$$

and $\Gamma(0)$ consists of the constant functions on $S$, while $\Gamma(3s_0 + 2f_0 + 4f_\infty)$ is the 11-dimensional space generated by the functions $t^k$ $(-2 \le k \le 4)$, $x$, $x/t$, $x/t^2$, and $y/t^2$. [Check: $3s_0 + 2f_0 + 4f_\infty$ has self-intersection $(3s_0 + 6f) \cdot (3s_0 + 6f) = 36 - 18 = 18$, and indeed $11 = \frac{1}{2}18 + 2$.] So we need a nonconstant function $T \in \Gamma(3s_0 + 2f_0 + 4f_\infty)$ in this space with $(T)_\infty \le D$. It turns out to be enough to require that $T$ be regular on the one component of each of the $t = 0$ and $t = \infty$ fibers that is not contained in $D$. At $t = 0$, the non-identity components have $x = O(t)$, and the component that meets $4P$ (and $P$) has $y = O(t^2)$; so we seek a linear combination of $t^k$ $(0 \leq k \leq 4)$, $x$, $x/t$, and $y/t^2$. The situation at $t = \infty$ is somewhat more complicated: we set $x = b(b + c) + O(t^2)$ (since $b(b + c)$ is the $x$-coordinate of $4P$), and solve for $y$ to within $O(t^2)$ on the component containing $4P$ rather than $2P$. We calculate that on this component

$$y = (b + c)(x - b(b + c)) + O(t^2). \tag{20}$$

Thus $\Gamma(D)$ must be generated by 1 and

$$u := t^{-2}(y - (b + c)(x - b(b + c))). \tag{21}$$

(Note that this is indeed in the span of $t^k$ $(0 \leq k \leq 4)$, $x$, $x/t$, and $y/t^2$, because $b, c$ are polynomials in $t$ of degree at most 2, and $b + c$ is a multiple of $t$.) To write $S$ as a pencil of genus-1 curves over the $u$-line, we solve (21) for $y$:

$$y = (b + c)(x - b(b + c)) + ut^2, \tag{22}$$

and substitute this into the defining equation of $S$. The further change of variables

$$x = b(b + c) + tx' \tag{23}$$

lets us remove a factor of $t^3$, leaving a cubic in $x'$ and $t$ with coefficients in $\mathbf{Z}[A_0, A_1][u]$, namely

$$\begin{aligned} x'^3 \;=\; & u\bigl(A_0t^2 + (A_1 + 2)t + 2B\bigr)x' + A_0ut^3 + (2A_0B + A_1)ut^2 \\ & + \bigl(u^2 + B(A_0B + 2A_1)u\bigr)t + uA_1B^2. \end{aligned} \tag{24}$$

Let $\widetilde{S}$ be the elliptic fibration over the $u$-line whose generic fiber is the Jacobian of the cubic (24). Then we get a degree-9 map $S \to \widetilde{S}$ by taking any $(u, x', t)$ to the class of $3(x', t) - H$ on the $u$-fiber of $\widetilde{S}$ (where $H$ is the divisor of any hyperplane [i.e. line] section of the cubic). We compute an equation for $\widetilde{S}$ using the formulas of [A-RV-T 2005, equation 1.6]: the Jacobian of any plane cubic

$$Ax_1^3{+}Bx_2^3{+}Cx_3^3 + Px_1^2x_2{+}Qx_2^2x_3{+}Rx_3^2x_1 + Tx_1x_2^2{+}Ux_2x_3^2{+}Vx_3x_1^2 + Mx_1x_2x_3 = 0 \tag{25}$$

has an extended Weierstrass equation with coefficients[3]

[3]We give these formulas in computer-readable form so readers can copy-and-paste them for their own use.

```
a1 = M

a2 = -(P*U + Q*V + R*T)

a3 = 9*A*B*C - (A*Q*U + B*R*V + C*P*T) - (T*U*V + P*Q*R)

a4 = (A*R*Q^2 + B*P*R^2 + C*Q*P^2 + A*T*U^2 + B*U*V^2 + C*V*T^2)
  + (P*Q*U*V + Q*R*V*T + R*P*T*U) - 3*(A*B*R*U + B*C*P*V + C*A*Q*T)

a6 = -27*(A*B*C)^2
 + 9*(A^2*B*C*Q*U + B^2*C*A*R*V + C^2*A*B*P*T)
 + 3*A*B*C*(T*U*V + P*Q*R)
 - (A*B*Q*R*U*V + B*C*R*V*P*T + C*A*P*Q*T*U)
 - (A^2*C*Q^3 + B^2*A*R^3 + C^2*B*P^3 + A^2*B*U^3 + B^2*C*V^3 + C^2*A*T^3)
 - P*Q*R*T*U*V
 + 2 * (A*C*Q^2*T*V + B*A*R^2*U*T + C*B*P^2*V*U
      + A*C*Q*R*T^2 + B*A*R*P*U^2 + C*B*P*Q*V^2)
 - (A*Q*T*V*U^2 + B*R*U*T*V^2 + C*P*V*U*T^2
  + A*P*Q^2*R*U + B*Q*R^2*P*V + C*R*P^2*Q*T)
 - (A*Q^2*R^2*T + B*R^2*P^2*U + C*P^2*Q^2*V
  + A*R*T^2*U^2 + B*P*U^2*V^2 + C*Q*V^2*T^2)
 + M * (A*B*U^2*V + B*C*V^2*T + C*A*T^2*U + A*B*R^2*Q + B*C*P^2*R + C*A*Q^2*P)
 + M * (A*Q*R*T*U + B*R*P*U*V + C*P*Q*V*T - 3*A*B*C*(Q*V+R*T+P*U))
 - M^2 * (A*B*R*U + B*C*P*V + C*A*Q*T) + M^3*A*B*C
```

In our case we find that $\widetilde{S}$ has a 2-torsion section at

$$\begin{aligned} x &= -u^2(2A_0B-1), \\ y &= u^2((A_0-1)^2B(B+1)^2+uA_0B^2) - u^3((A_0-4)A_0B+1). \end{aligned} \tag{26}$$

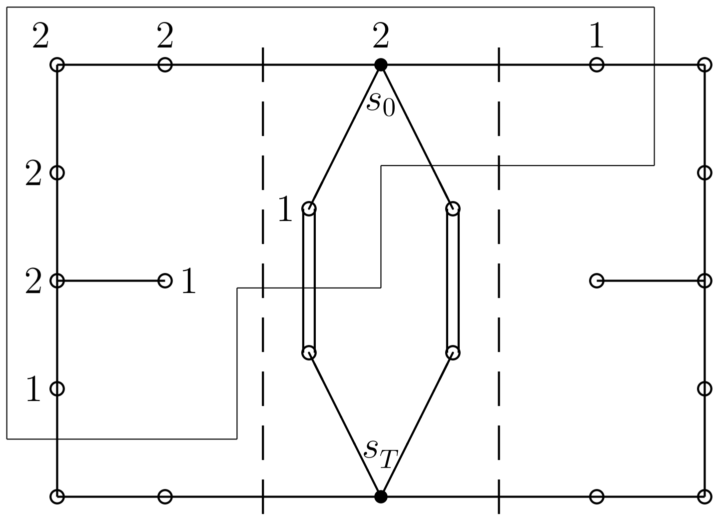
Translating this section to $(0,0)$ we obtain the equation

$$\begin{aligned} y^2+(A_1+2)uxy &= x^3+(4A_0B-3)u^2x^2 \\ &\quad - u^3\big(u-(A_0+A_1)^3\big)\big(A_0u-(A_0-1)^3B\big)\,x \end{aligned} \tag{27}$$

for $\widetilde{S}$. We see that $\widetilde{S}$ has the expected reducible fiber of type $\mathrm{III}^*(\widetilde{E}_7)$ at $u=\infty$. There's also a second such fiber at $u=0$, coming from the section $4P$ of $S$ and most of the $\tilde{A}_5$ and $\tilde{A}_2$ fibers above $t=\jmath(\infty)$ and $t=\jmath(0)$; plus two $\mathrm{I}_2(\tilde{A}_1)$ fibers at $u=(A_0+A_1)^3$ and $u=(A_0-1)^3B/A_0$, each sharing a component with the image of one of the $\mathrm{I}_2(\tilde{A}_1)$ fibers of $S$. This accounts for a sublattice of $\mathrm{NS}(\widetilde{S})$ of rank $2+7+7+1+1 = 18$, so our elliptic fibration of $\widetilde{S}$ has rank zero, and we readily see that $(0,0)$ is the only nonzero torsion point, so $\mathrm{disc}(\mathrm{NS}(\widetilde{S}) = -\mathrm{disc}(A_1^2E_7^2)/|T|^2 = -2^4/2^2 = -4$.

# 4 From $(A_1^2E_7^2)^{+2}$ to $(D_8^2)^{+2}$

We next connect $\widetilde{S}$ with a Kummer surface $\mathrm{Km}(E\times E')$ by finding another elliptic parametrization of $\widetilde{S}$ with two $I_4^*$ $(D_8)$ fibers and a 2-torsion section. We form a $\widetilde{D}_8$ divisor support on the zero-section $s_0$, most of the $\mathrm{III}^*$ fiber at $u=\infty$, and the identity components of the reducible fibers at $u = 0$ and $u = (A_0 + A_1)^3$. The next diagram shows the configuration of zero-section $s_0$, torsion section $s_T$, and the components of the two $\tilde{A}_1$) and two $\widetilde{E}_7$ fibers of the $(A_1^2 E_7^2)^{+2}$ fibration, together with the $\widetilde{D}_8$ divisor:




A $\mathrm{I}_4^*(\widetilde{D}_8)$ divisor supported on $s_0$, most of the III$^*$ fiber at $u = \infty$, and the identity components of the III$^*$ fiber at $u = 0$ and the $\mathrm{I}_2$ fiber at $u = (A_0 + A_1)^3$

We find that the space of sections of this divisor is generated by 1 and

$$v = \frac{x}{u(u - (A_0 + A_1)^3)} \tag{28}$$

So we substitute $x = u(u - (A_0 + A_1)^3)v$ in (27). We remove most factors of $u$ and $u-(A_0+A_1)^3$ by replacing $y$ by $u(u-(A_0+A_1)^3)y$ and dividing through by $u^2(u - (A_0 + A_1)^3)^2$. The right-hand side is then a cubic in $u$ with leading coefficient $-A_0 v$, so we bring our fibration into Weierstrass form by writing $u = -x/(A_0 v)$, changing $y$ to $y/(A_0 v)$, and multiplying the resulting equation by $(A_0 v)^2$. We obtain

$$\begin{aligned} y^2 - (A_1+2)vxy \;&=\; x^3 + (v^3 + (4A_0B-3)v^2 + (A_0-1)^3Bv)x^2 \\ &\quad + A_0(A_0+A_1)^3v^4x, \end{aligned} \tag{29}$$

with the $I_4^*$ $(D_8)$ fibers at $v = 0$ and $v = \infty$, and 2-torsion at $(x, y) = (0, 0)$.

## 5 Identifying $\widetilde{S}$ with $\mathrm{Km}(E \times E')$ in characteristic 2

Recall that over any field we can construct $\mathrm{Km}(E \times E')$ as a twist of the constant curve $E \times \mathbf{P}^1$ by a quadratic extension $E'/\mathbf{P}^1$, or of $E' \times \mathbf{P}^1$ by a quadratic extension $E/\mathbf{P}^1$, where the quadratic extensions map the origin of $E'$ or $E$ to a branch point. In odd characteristic it is well known that each of these elliptic fibrations has four reducible fibers of type $\mathrm{I}_0^*$ ($\widetilde{D}_4$) and Mordell–Weil group with $(\mathbf{Z}/2\mathbf{Z})^2$ torsion, which is the full Mordell–Weil group if and only if $E, E'$ are not isogenous; for example, in the twist of $E \times \mathbf{P}^1$ the singular fibers come from $E[2]$ and the torsion sections from $E'[2]$. (See the next section for a diagram of configuration of fiber components and torsion sections on such a Kummer surface.) It follows that in this generic case $\mathrm{Km}(E \times E')$ has Néron–Severi rank 18 and discriminant $-16$. But in characteristic 2, an ordinary elliptic curve has only two 2-torsion points, and a supersingular curve has trivial 2-torsion. If $E, E'$ are ordinary non-isogenous curves then $\mathrm{Km}(E \times E')$ still has Néron–Severi rank 18, but this time only two reducible fibers, each of type $\mathrm{I}_4^*(\widetilde{D}_8)$, and torsion group only $\mathbf{Z}/2\mathbf{Z}$ [Shioda 1974]. Since our surface $\widetilde{S}$ has a $(D_8^2)^{+2}$ fibration (29), we thus expect it to be $\mathrm{Km}(E \times E')$. We next confirm this by identifying (29) with the quadratic twist of a constant curve, and thus find $E$ and $E'$.

In characteristic 2, the formula (29) is more simply

$$y^2 + A_1 vxy = x^3 + (v^3 + v^2 + (A_0+1)^3 Bv)x^2 + A_0(A_0+A_1)^3 v^4 x. \quad (30)$$

Replacing $(x, y)$ by $(v^2, v^3 y)$ and dividing by $v^6$ we obtain the birational model

$$y^2 + A_1 xy = x^3 + (v + 1 + (A_0+1)^3 Bv^{-1})x^2 + A_0(A_0+A_1)^3 x, \quad (31)$$

revealing that for any $\alpha \in k$ this elliptic fibration is a quadratic twist of the constant curve

$$E : y^2 + A_1 xy = x^3 + \alpha x^2 A_0(A_0+A_1)^3 x \quad (32)$$

by

$$E' : \eta^2 + A_1 \eta = v + (1+\alpha) + (A_0+1)^3 Bv^{-1}, \quad (33)$$

which we bring to Weierstrass form

$$E' : Y^2 + A_1 vY = v^3 + (1+\alpha)v^2 + (A_0+1)^3 Bv \quad (34)$$

by writing $\eta = Y/v$ and multiplying by $v^2$. [Changing $\alpha$ to say $\alpha + \delta$ twists both $E$ and $E'$ by $\eta^2 + A_1\eta = \delta$; thus this applies a quadratic twist to $E \times E'$, but as expected it does not change $\mathrm{Km}(E \times E')$.] By (10), the coefficient $A_1$ in (34) is also $(A_0+1)+B$ in characteristic 2.

We next find a choice of $\alpha$ for which $E$ depends only on $P$ and $E'$ depends only on $Q$, where $P, Q$ are the parameters of [Weinstein 2019]. We gave in (18) formulas for $P, Q$ in terms of our coordinates $A_0, A_1$ on $\mathcal{M}_{36}$; in characteristic 2, these are

$$P = \frac{A_0 + A_1}{A_0}, \quad Q = \frac{B}{A_0 + 1}, \tag{35}$$

and the inverse map (17) is

$$A_0 = \frac{Q+1}{P+Q}, \quad A_1 = \frac{(P+1)(Q+1)}{P+Q}, \tag{36}$$

which makes

$$B = A_0 + A_1 + 1 = \frac{(P+1)Q}{P+Q}. \tag{37}$$

For $E$, scale $(x, y)$ to $(A_0^2 x, A_0^3 y)$ in (32), finding

$$y^2 + (P+1)xy = x^3 + A_0^{-2}\alpha x^2 + P^3 x = x^3 + \left(\frac{P+Q}{Q+1}\right)^2 \alpha x^2 + P^3 x. \tag{38}$$

For $E'$, scale $(x, y)$ to $((A_0+1)^2 x, (A_0+1)^3 y)$ in (34), finding

$$\begin{aligned} Y^2 + (Q+1)vY &= v^3 + \frac{1+\alpha}{(A_0+1)^2} v^2 + Qv \\ &= v^3 + \left(\frac{P+Q}{P+1}\right)^2 (1+\alpha) v^2 + Qv. \end{aligned} \tag{39}$$

Computations reported in [Weinstein 2019] suggest that $\alpha$ be chosen so that the coefficient of $x^2$ in (38) equal $P^2$, so that (38) matches the model $y^2 + (P+1)xy = x^3 + P^2x^2 + P^3x$ of $E_6(P, 1)$ that puts the 2-torsion point at the origin. This gives

$$\alpha = \left(\frac{P(Q+1)}{P+Q}\right)^2, \qquad \alpha + 1 = \left(\frac{(P+1)Q}{P+Q}\right)^2. \tag{40}$$

This simplifies the equation (39) for $E'$ to $Y^2 + (Q+1)vY = v^3 + Q^2v^2 + Qv$, which is the model of $E_6(1, Q)$ that puts the 2-torsion point at the origin.

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