

THE CANONICAL GLOBAL QUANTIZATION OF SYMPLECTIC VARIETIES IN CHARACTERISTIC p

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ABSTRACT. Let X be a smooth symplectic variety over a field k of characteristic $p > 2$ equipped with a restricted structure, which is a class $[\eta] \in H^0(X, \Omega_X^1/d\mathcal{O}_X)$ whose de Rham differential equals the symplectic form. In this paper we construct a functorial in $(X, [\eta])$ formal quantization of the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X . We also construct its natural extension to a quasi-coherent sheaf of categories QCoh_h on the product $X^{(1)} \times \mathbb{S}$ of the Frobenius twist of X and the projective line $\mathbb{S} = \mathbb{P}^1$, viewed as the one-point compactification of $\mathrm{Spec} k[[h]]$. Its global sections over $X^{(1)} \times \{0\}$ is the category of quasi-coherent sheaves on X . If X is affine, QCoh_h , restricted to $X^{(1)} \times \mathrm{Spf} k[[h]]$, is equivalent to the category of modules over the distinguished “Frobenius-constant” quantization of $(X, [\eta])$ defined in [BK].

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1. INTRODUCTION

Let k be a field of characteristic $p > 2$. In [BK], Bezrukavnikov and Kaledin initiated the study of quantizations of symplectic varieties over k . More precisely, they introduced new notion of a *restricted* symplectic k -scheme: namely, they consider pairs $(X, [\eta])$, where $[\eta]$ is a global section of the sheaf $\text{coker}(\mathcal{O}_X \xrightarrow{d} \Omega_X^1)$, such that $\omega := d([\eta]) \in H^0(X, \Omega_X^2)$ is a symplectic form¹. In the case $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2, 3$, they construct a distinguished ‘‘Frobenius-constant’’ quantization of $(X, [\eta])$. Roughly speaking, such a quantization is a sheaf \mathcal{O}_h of associative $k[[\hbar]]$ -algebras on X equipped with two isomorphisms

- $\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_h/\hbar$,
- $\mathcal{O}_{X^{(1)}}[[\hbar]] \xrightarrow{\sim} Z(\mathcal{O}_h)$,

that are also compatible with the restricted structure given by $[\eta]$ in a certain way. Here $X^{(1)}$ denotes the Frobenius-twist of X and $Z(\mathcal{O}_h)$ is the center of \mathcal{O}_h . The prototypical example of such a quantization is given by the following: namely, the $k[[\hbar]]$ -version of the Rees algebra $\mathcal{D}_{Y,\hbar}$ of differential operators on Y gives the distinguished Frobenius-constant quantization of $X = T^*Y$ with the restricted structure given by the canonical 1-form η . An unfortunate feature of the construction is that it is not functorial: there is no natural way to make the distinguished quantization equivariant with respect to automorphisms of $(X, [\eta])$, and moreover the quantization exists only if we require the aforementioned cohomology vanishing for X . A solution proposed by Kontsevich ([K]) and studied further in ([VdB], [Y]) in characteristic 0 context is to replace the algebra \mathcal{O}_h by the corresponding category of modules $\mathcal{O}_h\text{-Mod}$. We develop a parallel picture in characteristic p : namely, we show that the category $\mathcal{O}_h\text{-Mod}$ is functorial in the pair $(X, [\eta])$ and glues to a canonical quantization of the category $\text{QCoh}(X)$ of quasi-coherent sheaves on X for any restricted symplectic k -scheme $(X, [\eta])$.

¹One can think of $[\eta]$ as an algebraic analogue of a ‘‘contact form’’ on X .

Moreover, our construction allows to extend the range of the quantum parameter \hbar from being a formal variable to a genuine coordinate on \mathbb{P}^1 , where the corresponding category QCoh_\hbar carries a formal resemblance with the twistor space construction appearing in Simpson’s correspondence (see Remark 1.1 below).

1.1. Plan of the paper. The idea of “formal geometry” in the sense of Gelfand-Kazhdan is to first deal with differential-geometric questions in the basic case of a formal disc, and then descend to smooth varieties via the corresponding torsor of formal coordinates. As observed by Bezrukavnikov and Kaledin in [BK] in characteristic p one can replace formal disc by the corresponding Frobenius neighborhood of zero, which makes things simpler. To be more precise, the key idea is to consider a smooth S -scheme X of dimension d as a scheme over its Frobenius twist $X^{(1)}$ via the relative Frobenius $F_{X/S}: X \rightarrow X^{(1)}$. This is not only a finite locally free map, but in fact a locally trivial bundle in flat topology: namely, Zariski-locally on X , the fiber product $X \times_{X^{(1)}} X$ splits as $X \times \mathrm{Spec} A_0$ where $\mathrm{Spec} A_0$ is the Frobenius-neighborhood of 0 in d -dimensional affine space². One then can consider the canonical torsor $\mathcal{M}_X \rightarrow X^{(1)}$ of “Frobenius frames”: a T -point $T \rightarrow \mathcal{M}_X$ is given by an isomorphism $T \times_{X^{(1)}} X \simeq T \times \mathrm{Spec} A_0$, or, in other words, an identification of (pull-back of) X with the Frobenius neighborhood of zero in an affine space of dimension d over T . The scheme $\mathcal{M}_X \rightarrow X^{(1)}$ is a torsor over the group scheme $\underline{\mathrm{Aut}}(A_0)$ of automorphisms of $\mathrm{Spec} A_0$ and many basic objects of differential-geometric nature such as³ differential operators \mathcal{D}_X , differential 1-forms Ω_X^1 , vector fields \mathcal{T}_X , or just the structure sheaf \mathcal{O}_X all come via descent from some representations of $\underline{\mathrm{Aut}}(A_0)$. This also gives a way to define differentiable structures on smooth varieties in characteristic p by considering the subgroup H of $\underline{\mathrm{Aut}}(A_0)$ that stabilizes that “structure” in the case of $X = \mathrm{Spec} A_0$. Prescribing such a structure on a given scheme X then corresponds to a reduction of the $\underline{\mathrm{Aut}}(A_0)$ -torsor $\mathcal{M}_X \rightarrow X^{(1)}$ to H . We will discuss in some detail how this picture looks in the case of restricted symplectic structure (see Section 2.11).

The above picture motivates considering things like differential operators or symplectic structures in the case of schemes like $\mathrm{Spec} A_0$, or X considered as an $X^{(1)}$ -scheme. These schemes are not smooth, but nevertheless they are pseudo-smooth: namely the sheaf $\Omega_{X/S}^1$ of relative Kähler differentials is locally free. In Section 2 we develop⁴ some basic theory of pseudo-smooth schemes. In fact pseudo-smoothness turns out to be enough to extend most of the standard features of differential geometry of smooth schemes in characteristic p : e.g. the algebra of differential operators is still an Azumaya algebra over the Frobenius twist of the total space of cotangent bundle (Corollary 2.38); there are also versions of Cartier isomorphism (Proposition 2.40) and Milne’s exact sequence (Proposition 2.48). The only difference with the smooth setting is that the notion of Frobenius twist should be slightly modified: namely, instead of Frobenius twist $X^{(1)} := X \times_{S, F_S} S$ one should consider the schematic image $X^{\{p\}} \hookrightarrow X^{(1)}$ of the relative Frobenius $F_{X/S}: X \rightarrow X^{(1)}$ (see Section 2.1 for more details). The corresponding map $F_{X/S}^{\{p\}}: X \rightarrow X^{\{p\}}$ induced by $F_{X/S}$ turns out to be

²Explicitly, $A_0 \simeq \mathbb{F}_p[x_1, \dots, x_d]/x_1^p = \dots = x_d^p = 0$.

³Strictly speaking one needs to consider their pushforwards to $X^{(1)}$ under relative Frobenius.

⁴In fact, in [BK] Bezrukavnikov and Kaledin freely assume the relevant parts of the theory in the pseudo-smooth (or, in their terminology, quasi-regular) setup, mostly without a proper justification which sometimes leads them to small inaccuracies or even false claims. So we tried to carefully setup the theory, mostly from scratch, only sometimes referencing their results.

finite locally free, and is again a locally trivial bundle with fiber given by the Frobenius neighborhood of 0 in the affine space (Lemma 2.71), however its dimension is now given by $\text{rk } \Omega_{X/S}^1$ and not the relative dimension of X over S .

The key geometric input which allows to get a good control on pseudo-smooth schemes is their local structure (Proposition 2.35) namely, Zariski locally any pseudo-smooth S -scheme X is given by the Frobenius-neighborhood of a closed subscheme in a smooth scheme. This observation allows us to extend most of the results from the smooth to pseudo-smooth setting almost for free.

Moreover, in Section 2.9 we recall the definition of restricted Poisson structure by introducing a restricted Poisson algebra monad. This section is an attempt to explain and motivate the definition of restricted Poisson structure given in [BK]. In Section 2.10 we then discuss restricted symplectic schemes (see Definition 2.65), where a restricted structure extending a symplectic form ω has an equivalent description in terms of a class $[\eta] \in \text{coker}(\mathcal{O}_X \rightarrow \Omega_X^1)$ such that $d[\eta] = \omega$ (see Remark ??). Finally, in Section 2.11 we endow $\text{Spec } A_0$ with a restricted symplectic structure and define the group $G_0 \subset \underline{\text{Aut}}(A_0)$ of its restricted Poisson automorphisms, that plays a very important role in most of the constructions below.

Construction of QCoh_h . Our goal in this paper will be to construct a certain sheaf of categories QCoh_h on $X^{\{p\}} \times \mathbb{P}^1$, given a restricted symplectic scheme $(X, [\eta])$. We want this construction to be functorial in $(X, [\eta])$ and agree⁵ with the distinguished quantization of Bezrukavnikov and Kaledin in the cases when the latter exists. We will discuss other expected properties of QCoh_h slightly later, in Section 1.2 below.

For the reader's convenience let us note right away that in Section 4 we remind the notion of a quasi-coherent sheaf of (abelian) categories on a scheme or, more generally, an algebraic stack. We then also discuss actions of group schemes on the sheaves of categories in Section 4.2.

Let $\mathbb{S} := \mathbb{P}^1$, the idea behind the notation being that \mathbb{P}^1 resembles a 2-dimensional sphere. In the case $X = T^*Y$ with the canonical 1-form η one could expect QCoh_h to come as the category of modules over the sheaf of twistor differential operators $\mathcal{D}_{Y, \mathbb{S}}$. The latter comes as an instance of a \mathbb{P}^1 -version of the Rees construction (see Section 3.1): namely, $\mathcal{D}_{Y, \mathbb{S}}$ corresponds to differential operators \mathcal{D}_Y endowed with the pair of Hodge and conjugate filtrations⁶, see Section 3.2 for more details. Below, we will define QCoh_h by certain universal property, and then check that it is indeed given by $\mathcal{D}_{Y, \mathbb{S}}\text{-Mod}$ in the case of T^*Y .

The starting point of our construction is the following. Namely, to a restricted symplectic scheme $(X, [\eta])$ one canonically associates a G_0 -torsor $\mathcal{M}_{X, [\eta]} \rightarrow X^{\{p\}}$ of its ‘‘Darboux frames’’: a T -point $T \rightarrow \mathcal{M}_X$ is given by a restricted Poisson isomorphism

$$T \times_{X^{\{p\}}} X \simeq T \times \text{Spec } A_0$$

(see Construction 2.72). This way a pair $(X, [\eta])$ produces a natural map

$$\pi: X^{\{p\}} \rightarrow BG_0.$$

A natural way to define QCoh_h would be to take the pull-back of a certain universal sheaf of categories over $BG_0 \times \mathbb{S} \simeq [\mathbb{S}/G_0]$ via the map $\pi \times \text{Id}: X^{\{p\}} \times \mathbb{S} \rightarrow BG_0 \times \mathbb{S}$. This is exactly what we do.

⁵Meaning that the restriction of QCoh_h to the formal neighborhood of 0 in \mathbb{P}^1 should be given by $\mathcal{O}_h\text{-Mod}$.

⁶That are dual to Hodge and conjugate filtrations on the de Rham complex in a certain sense.

We construct the universal sheaf of categories on $[\mathbb{S}/G_0]$ via descent. Namely, we first produce a certain sheaf of algebras $\mathcal{A}_{\mathbb{S}}$ over \mathbb{S} (see Section 3.4) and then construct an action of G_0 on the corresponding sheaf of categories over \mathbb{S} given by $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ (Section 5). If $(X, [\eta])$ is $\text{Spec } A_0$ equipped with the natural restricted symplectic structure then QCoh_h is equivalent $\mathcal{A}_{\mathbb{S}}\text{-Mod}$. We will call $\mathcal{A}_{\mathbb{S}}$ the “twistor reduced Weyl algebra”.

Let us very briefly explain the construction of $\mathcal{A}_{\mathbb{S}}$. Namely, in the even-dimensional case one can identify $\text{Spec } A_0$ with a subvariety in cotangent bundle of a similar Frobenius neighborhood $\text{Spec } C$, but of half the dimension. More precisely, $\text{Spec } A_0$ is just a Frobenius-neighborhood of zero section in $T^*\text{Spec } C$. The sheaf of algebras $\mathcal{A}_{\mathbb{S}}$ is defined as the corresponding central reduction of twistor differential operators $\mathcal{D}_{\text{Spec } C, \mathbb{S}}$, see Section 3.4 and Definition 3.15 in particular. $\mathcal{A}_{\mathbb{S}}$ is a locally free sheaf of algebras on \mathbb{S} with the fiber at $\{0\} \in \mathbb{S}$ is given by A_0 , and with restriction to $\mathbb{S} \setminus \{0\}$ being a split Azumaya algebra: this way one can view $\mathcal{A}_{\mathbb{S}}$ as a certain order in a matrix algebra over \mathbb{S} .

In Section 5 we construct the desired G_0 -action on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$. The key observation is that imposing certain properties on the action automatically makes it canonical. More precisely, in Theorem 1 we show that there is a unique G_0 -action on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ that restricts to the natural action of G_0 on $A_0\text{-Mod}$ over $\{0\} \in \mathbb{S}$ and is trivial over $\{\infty\} \in \mathbb{S}$. The proof of uniqueness crucially uses the description of automorphisms of $\mathcal{A}_{\mathbb{S}}$ whose restriction to $\{0\} \in \mathbb{S}$ is trivial, which we establish in Section 3.7. The construction of the existence is quite elaborate and uses an auxiliary sheaf of algebras $\mathcal{A}_{\mathbb{S}}^b$ with a natural G_0 -action (see Section 3.8), as well as a certain G_0 -action on $\mathcal{A}_{\mathbb{S}} \otimes_{\mathcal{O}_{\mathbb{S}}} \mathcal{A}_{\mathbb{S}}^{b, \text{op}}\text{-Mod}$ constructed in [BV] in the formal neighborhood of 0. In Section 5.3 we also show that the G_0 -action on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ extends to a $\mathbb{G}_m \times G_0$ -action equivariant structure where \mathbb{G}_m now acts non-trivially on \mathbb{S} . This allows to descend $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ further to a sheaf of categories over $[\mathbb{S}/(\mathbb{G}_m \times G_0)]$.

In Section 6 we discuss the properties of the resulting sheaf of categories QCoh_h . In 6.1 we translate the defining properties of our G_0 -action on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ into defining properties for QCoh_h . In Section 6.2 we give a slightly finer construction in the presence of a \mathbb{G}_m -action on X that rescales the contact form $[\eta]$ with a weight coprime to p . In this case one can descend QCoh_h further to the quotient stack $[(X^{(p)} \times \mathbb{S})/\mathbb{G}_m]$ under the diagonal action, where \mathbb{G}_m acts on \mathbb{S} by rescaling with the same weight as above. In Section 6.4 we discuss the relation of QCoh_h with the Frobenius-constant quantization \mathcal{O}_h constructed by Bezrukavnikov and Kaledin. Finally, in Section 6.5 we also construct the canonical quantization of restricted Lagrangian subvarieties extending the work of Mundinger [Mu].

1.2. Properties of QCoh_h . For simplicity, below we let X be a smooth scheme over a field of characteristic $p > 2$. In this case $X^{(p)} \simeq X^{(1)}$, and QCoh_h is a sheaf of categories over $X^{(1)} \times \mathbb{S}$; thus for each scheme $T \rightarrow X^{(1)} \times \mathbb{S}$ over $X^{(1)} \times \mathbb{S}$ we obtain a category $\text{QCoh}_h(T)$. Let us summarize the nice properties of QCoh_h that we obtain in Section 6.

- (1) If we consider the G_0 -torsor $\pi: \mathcal{M}_{X, [\eta]} \rightarrow X^{(1)}$ of Darboux frames corresponding to $[\eta]$ then the pull-back $(\pi \times \text{id})^* \text{QCoh}_h$ is equivalent to $(p_2^* \mathcal{A}_{\mathbb{S}})\text{-Mod}$, where $p_2: \mathcal{M} \times \mathbb{S} \rightarrow \mathbb{S}$ is the projection.
- (2) The global sections $\text{QCoh}_h(X' \times \{0\})$ on the fiber over $\{0\} \in \mathbb{S}$ are given by $\text{QCoh}(X)$.

- (3) In the case $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2, 3$, we have an equivalence $\mathrm{QCoh}_h|_{X^{(1)} \times \widehat{\mathbb{S}}} \simeq O_h\text{-Mod}$ where O_h is the distinguished Frobenius-constant quantization constructed in [BK].
- (4) QCoh_h is functorial in $(X, [\eta])$. In particular, for any map $T \rightarrow \mathbb{S}$ there is a natural $\mathcal{O}(T)$ -linear action of restricted Poisson automorphisms $\mathrm{Aut}((X, [\eta]))$ on the global sections $\mathrm{QCoh}_h(X^{(1)} \times T)$.
- (5) The restriction of QCoh_h to $X^{\{p\}} \times (\mathbb{S} \setminus \{0\})$ is equivalent to $\mathcal{D}_{X, [\eta]/h}\text{-Mod}$, where $\mathcal{D}_{X, [\eta]/h}$ is the central reduction of $\mathcal{D}_{X, \mathbb{S} \setminus \{0\}}$ corresponding to $[\eta]/h$ (see Construction 3.13).
- (6) The global sections $\mathrm{QCoh}_h(X' \times \{\infty\})$ on the fiber over $\{\infty\} \in \mathbb{S}$ are given by $\mathrm{QCoh}(X')$.
- (7) For a smooth Lagrangian subvariety $Y \subset X$ such that $[\eta]|_Y = 0$ there exists a canonical flat object $\mathcal{V}_{Y^{(1)} \times \mathbb{S}} \in \mathrm{QCoh}_h(Y^{(1)} \times \mathbb{S})$ whose fiber over $Y^{(1)} \times \{0\}$ is given by $(\Omega_Y^{\dim Y})^{\frac{1-p}{2}}$ (considered as a quasi-coherent sheaf on X).
- (8) If X is equipped with a \mathbb{G}_m -action which rescales $[\eta]$ with a weight 1, then QCoh_h canonically descends further to the quotient stack $(X^{(1)} \times \mathbb{S})/\mathbb{G}_m$ where \mathbb{G}_m acts diagonally.

Remark 1.1. In the case $X = T^*Y$ and η is the canonical 1-form on T^*Y , the global sections $\mathrm{QCoh}_h(X^{(1)} \times \mathbb{S})$ can be alternatively described as the category of modules over the sheaf of twisted differential operators $\mathcal{D}_{X, \mathbb{S}}$.

Also, from the above picture one gets the following properties

- $\mathrm{QCoh}_h(X^{(1)} \times \{0\})$ is given by the category of Higgs fields on Y ;
- $\mathrm{QCoh}_h(X^{(1)} \times \{1\})$ can be identified⁷ with the category $\mathcal{D}_Y\text{-Mod}$ of D -modules on Y ;
- $\mathrm{QCoh}_h(X^{(1)} \times \{\infty\})$ is the category of Higgs fields on the Frobenius twist $Y^{(1)}$.

This resembles the construction of the twistor space due to Deligne and Simpson: namely, given a smooth projective variety Y over \mathbb{C} they construct a complex-analytic space $W \rightarrow \mathbb{P}^1$ whose fiber over $\lambda \in \mathbb{P}^1(\mathbb{C})$ is the “moduli space” of λ -connections on Y . In particular, fibers W_0, W_1 and W_∞ over $0, 1$ and $\infty \in \mathbb{P}^1(\mathbb{C})$ are given by

- holomorphic Higgs fields on Y ;
- bundles with holomorphic flat connections on Y ;
- holomorphic Higgs fields on the complex conjugate \overline{Y}

correspondingly. This way the category QCoh_h can be considered as a categorified characteristic p analogue of their construction. This motivates calling $\mathrm{QCoh}_{X, [\eta], h}$ the *twistor category* associated to $(X, [\eta])$.

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⁷Using the Morita equivalence $\mathcal{D}_{X, [\eta]} \sim \mathcal{D}_Y$ ([BB]).

2. DIFFERENTIAL GEOMETRY ON PSEUDO-SMOOTH SCHEMES

2.1. Pseudo-smooth schemes. Let B be an \mathbb{F}_p -algebra. Following [BK] we consider the following notion.

Definition 2.1. A finite-type B -algebra A is called *pseudo-smooth* if $\Omega_{A/B}^1$ is a locally free A -module. A qcqs morphism $X \rightarrow S$ locally of finite type is called *pseudo-smooth* if $\Omega_{X/S}^1$ is a locally free sheaf of finite type.

Remark 2.2. In [BK], Bezrukavnikov and Kaledin call⁸ such algebras quasi-regular. However, since terms quasi-regular and quasi-smooth are quite broadly used with a different meaning, we decided to change it to pseudo-smooth.

Example 2.3. (1) Smooth schemes are pseudo-smooth.

(2) Let $A := B[x]/x^p$. Then $\Omega_{A/B}^1 = A \cdot dx$ and A is pseudo-smooth.

(3) Let $Z \hookrightarrow S$ be a closed subscheme. Then $\Omega_{Z/S}^1 = 0$ and Z is pseudo-smooth over S .

Remark 2.4. One might count Example 2.3(3) as a pathological one and impose flatness condition in Definition 2.1. This is relatively harmless for our goals: namely, pseudo-smooth schemes that will appear in main applications (e.g. quantizations of smooth schemes) will all be flat.

Remark 2.5. Pseudo-smoothness is preserved under base change: namely, having a pseudo smooth S -scheme $X \rightarrow S$ and a morphism $S' \rightarrow S$ the base change $X_{S'} = X \times_S S'$ is pseudo smooth over S' . Indeed one has $\Omega_{X_{S'}/S'}^1 = f^* \Omega_{X/S}^1$ where $f: X_{S'} \rightarrow X$ is the natural map.

Construction 2.6 (Reduced Frobenius: algebras). Let A be a B -algebra. The absolute Frobenius map $F_A: A \rightarrow A$ factors naturally through the Frobenius twist $A^{(1)} := A \otimes_{B, F_B} B$:

$$\begin{array}{ccc} A & \xrightarrow{F_A} & A \\ & \searrow W_{A/B} & \nearrow F_{A/B} \\ & & A^{(1)} \end{array}$$

with $W_{A/B}: a \mapsto a \otimes 1 \in A^{(1)}$ and $F_{A/B}: a \otimes b \mapsto a^p b \in A$. The map $F_{A/B}$ is usually called the relative Frobenius; by construction it is B -linear (while F_A and $W_{A/B}$ are F_B -linear). However, if B or A are not reduced one can factor $F_{A/B}$ even further. Namely, let $A^{\{p\}} \subset A$ be the image of $F_{A/B}$: this is the B -subalgebra in A generated by p -th powers A^p , in other words $A^{\{p\}} := B \cdot A^p \subset A$. By definition, one has a natural surjection $F_A^{int}: A^{(1)} \twoheadrightarrow A^{\{p\}}$. Moreover, we get a further factorization

$$\begin{array}{ccc} A & \xrightarrow{F_A} & A \\ & \searrow W_{A/B} & \nearrow F_{A/B} \\ & & A^{(1)} \xrightarrow{F_A^{int}} A^{\{p\}} \xrightarrow{F_A^{\{p\}}} A \end{array}$$

We have $F_{A/B} = F_A^{\{p\}} \circ F_A^{int}$ and we can put $W_A^{\{p\}} := F_A^{int} \circ W_{A/S}$. We will call $F_A^{\{p\}}$ and F_A^{int} the *reduced* and *intermediate* Frobenii of A correspondingly.

⁸Strictly speaking they also assume that A/B is flat and B is Noetherian.

In the case when both A and B are reduced, F_A^{int} is an isomorphism and so $F_{A/B} = F_A^{\{p\}}$. However, we will be primarily interested in the non-reduced setting, where $A^{\{p\}}$ will turn out to be more relevant than $A^{(1)}$.

Remark 2.7. Let us try to motivate consideration of the reduced Frobenius as compared to the relative one. The main reason is that when A is pseudo-smooth, $F_A^{\{p\}}$ still enjoys some properties that the relative Frobenius $F_{A/B}$ has if we assume that A is actually smooth. For example, we will see further that under the pseudo-smoothness assumption (see Remark 2.39) $F_A^{\{p\}}$ is a finite faithfully flat map, which is no longer true for the relative Frobenius $F_{A/B}$ in this generality. This is illustrated well by the example of $A := B[x]/x^p$, where one has $A^{(1)} \simeq B[x]/x^p$ and where the relative Frobenius $F_{A/B}: B[x]/x^p \rightarrow B[x]/x^p$ is the unique B -algebra map that sends x to 0. This map is clearly not flat. On the other hand, $A^{\{p\}} \subset A$ is identified with $B \subset B[x]/x^p$ and the reduced Frobenius $F_A^{\{p\}}: A^{\{p\}} \rightarrow A$ is given by the above embedding which is indeed finite and faithfully flat.

Remark 2.8. In [BK], for a pseudo-smooth A it is claimed that $A^{\{p\}}$ can be identified with the tensor product $A^p \otimes_{B^p} B$ via the natural map $a^p \otimes b \mapsto a^p b$. Unfortunately, this is not exactly true. Put $B := \mathbb{F}_p[x]$ and $A = B[t]/(t^p - x)$. One checks easily that $\Omega_{A/B}^1 \simeq A \cdot dt$, and so A is pseudo-smooth. On the other hand $B^p \simeq \mathbb{F}_p[x^p]$, $A^{\{p\}} \simeq A^p \simeq B$, and so the natural map $A^p \otimes_{B^p} B \rightarrow A^{\{p\}}$ is identified with multiplication map $\mathbb{F}_p[x] \otimes_{\mathbb{F}_p[x^p]} \mathbb{F}_p[x] \rightarrow \mathbb{F}_p[x]$ which is not an isomorphism.

Remark 2.9. Let $X = \text{Spec } A$. By [Stacks, Tag0BR8], $F_X^{\{p\}}: X \rightarrow X^{\{p\}} := \text{Spec } A^{\{p\}}$ is a universal homeomorphism. In particular, it induces an isomorphism of the underlying topological spaces. In fact all three schemes $X \rightarrow X^{\{p\}} \hookrightarrow X^{(1)} := \text{Spec } A^{(1)}$ have the same underlying topological space and the difference between them is only seen on the level of sheaves of functions.

Construction 2.10 (Reduced Frobenius: schemes). Given a base scheme S and an S -scheme X , the construction $A \rightarrow A^{\{p\}}$ globalizes, producing an S -scheme $X^{\{p\}}$. This is a scheme which has the same underlying topological space as the original X and its Frobenius twist $X^{(1)} := X \times_{S, F_S} S$, but with the sheaf of functions given by $\mathcal{O}_X^{\{p\}} := \mathcal{O}_X^p \cdot \mathcal{O}_S \subset \mathcal{O}_X$. One has maps $\mathcal{O}_X^{\{p\}} \hookrightarrow \mathcal{O}_X$ and $\mathcal{O}_{X^{(1)}} \rightarrow \mathcal{O}_X^{\{p\}}$ (given locally by $F_A^{\{p\}}$ and F_A^{int}) which induce a map $F_X^{\{p\}}: X \rightarrow X^{\{p\}}$ and a closed embedding $F_X^{int}: X^{\{p\}} \hookrightarrow X^{(1)}$. This way one can also view $X^{\{p\}}$ as the schematic image of the relative Frobenius $F_{X/S}: X \rightarrow X^{(1)}$ inside the Frobenius twist $X^{(1)} := X \times_{S, F_S} S$. We will call $F_X^{\{p\}}$ and F_X^{int} the *reduced* and *intermediate* Frobenii of X ; both are morphisms of S -schemes. We can also define a *reduced twist* map $W_X^{\{p\}}: X^{\{p\}} \rightarrow X$ (by globalizing $W_A^{\{p\}}$); this map is not a morphism of S -schemes, instead it is F_S -linear. One has relations $F_X^{int} \circ F_X^{\{p\}} = F_{X/S}$ and $W_{X/S} \circ F_X^{int} = W_X^{\{p\}}$.

Example 2.11. Assume $S = \text{Spec } B$.

- (1) Put $P := B[x_1, \dots, x_d]$ with $\text{Spec } P \simeq \mathbb{A}_S^d$. Then $P^{\{p\}} := P^p \cdot B \simeq B[x_1^p, \dots, x_d^p] \subset P$. On the other hand the Frobenius twist $P^{(1)} := P \otimes_{B, F_B} B \simeq B[y_1, \dots, y_d]$ (with $y_i := x_i \otimes 1$), and the relative Frobenius $F_{P/B}: P^{(1)} \rightarrow P$ is the unique B -algebra map that sends y_i to x_i^p . We see

that in this case the intermediate Frobenius $F_{\mathbb{A}_S^d}^{int}: (\mathbb{A}_S^d)^{\{p\}} \rightarrow (\mathbb{A}_S^d)^{(1)}$ is an isomorphism.

- (2) Consider the unique surjection $P \rightarrow B$ of B -algebras sending every x_i to 0. It corresponds to the embedding $\{0\}_S \hookrightarrow \mathbb{A}_S^d$ of 0. Put

$$Q := B \otimes_{P, F_P^{\{p\}}} P \simeq P/(x_1^p, \dots, x_d^p).$$

$X := \text{Spec } Q \hookrightarrow \mathbb{A}_S^d$ is the Frobenius neighborhood of 0: namely, $X \simeq \{0\}_S \times_{\mathbb{A}_S^d, F_{\mathbb{A}_S^d}^{\{p\}}} \mathbb{A}_S^d$. We have $Q^{\{p\}} := Q^p \cdot B \simeq B \subset Q$. However, $Q^{(1)} := Q \otimes_{B, F_B} B \simeq B[y_1, \dots, y_d]/(y_1^p, \dots, y_d^p)$, and the intermediate Frobenius $F_Q^{int}: Q^{(1)} \rightarrow Q^{\{p\}} \simeq B$ sends y_i to $x_i^p = 0$.

Remark 2.12. Since $F_X^{\{p\}}: X \rightarrow X^{\{p\}}$ is a universal homeomorphism, say by [Stacks, Tag 0BTY] it induces an equivalence of small étale sites $(X^{\{p\}})_{\text{et}} \simeq X_{\text{et}}$, by

$$(U \rightarrow X^{\{p\}}) \in (X^{\{p\}})_{\text{et}} \mapsto (X \times_{X^{\{p\}}} U \rightarrow X) \in X_{\text{et}}.$$

2.2. Restricted Lie algebras and Jacobson's formula. Let A be an associative ring such that $p \cdot A = 0$. For $a \in A$ denote by $\text{ad}(a): A \rightarrow A$ a map that sends $b \mapsto [a, b] = ab - ba$.

If A is commutative, given two elements $a, b \in A$ we have $(a + b)^p = a^p + b^p$. This formula generalizes to the general non-commutative case as follows

Lemma 2.13 (Jacobson, [GD, Chapter II, §7, Proposition 3.2]). *Let $a, b \in A$. Then*

$$(a + b)^p = a^p + b^p + \sum_{i=1}^{p-1} L_i(a, b)$$

where $L_i(a, b)$ is a Lie polynomial described as i^{-1} times the coefficient of t^{i-1} in the expression $\text{ad}(a + tb)^{p-1}(b) \in A[t]$.

Remark 2.14. In particular, $L_1(a, b) = \text{ad}(a)^{p-1}(b)$.

Remark 2.15. For an element $a \in A$ denote by $\text{ad}(a): A \rightarrow A$ the commutator map $x \mapsto [a, x]$. Note that $\text{ad}(a)^p = \text{ad}(a^p)$. Indeed, $\text{ad}(a) = \ell(a) - r(a)$ where $\ell(a), r(a): A \rightarrow A$ are left and right multiplications by a correspondingly. Since $\ell(a)$ and $r(a)$ commute we have

$$\text{ad}(a)^p = (\ell(a) - r(a))^p = \ell(a)^p - r(a)^p = \ell(a^p) - r(a^p) = \text{ad}(a^p).$$

We now remind the definition of a restricted Lie algebra.

Definition 2.16. A restricted Lie algebra over a ring B is given by a B -module L with a B -linear Lie bracket $[-, -]$ and a restricted p -power operation $-^{[p]}$ that are compatible in the following way:

- (1) $(bx)^{[p]} = b^p x^{[p]}$ for any $b \in B$ and $x \in L$;
- (2) $[x^{[p]}, y] = \text{ad}(x)^p(y)$ for any $x, y \in L$;
- (3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} L_i(x, y)$, where $L_i(x, y)$ are the Lie polynomials from Lemma 2.13.

The definition of restricted Lie algebra is mainly motivated by the following example:

Example 2.17. Let A be an associative B -algebra. Then by Lemma 2.13 and Remark 2.15, A endowed with the Lie bracket given by commutator $[-, -]$ and the restricted structure $a^{[p]} := a^p$ is a restricted Lie algebra.

Another natural source of examples of restricted Lie algebras is differential geometry in char p (see Remark 2.19 below). In particular, Lie algebra of any group scheme G is endowed with a natural restricted Lie algebra structure.

2.3. Crystalline differential operators. We essentially follow [BMR, Section 2], except we work in a relative and pseudo-smooth setting.

Let $X \rightarrow S$ be a pseudo-smooth S -scheme. Let $\mathcal{T}_X := \mathcal{T}_{X/S}$ be the sheaf of \mathcal{O}_S -linear derivations of \mathcal{O}_X . We will occasionally call (local) sections of \mathcal{T}_X vector fields.

Denote $\Omega_X^1 := \Omega_{X/S}^1$. One has $\mathcal{T}_X \simeq (\Omega_X^1)^\vee$, and, since X is pseudo-smooth, we get that \mathcal{T}_X is a locally free \mathcal{O}_X -module of finite rank.

The sheaf \mathcal{T}_X has a natural (\mathcal{O}_S -linear) Lie bracket given by commutator⁹ which endows it with the structure of a Lie algebroid over X . Let \mathcal{D}_X be the universal enveloping algebra of \mathcal{T}_X . Explicitly, \mathcal{D}_X is generated over \mathcal{O}_S by \mathcal{O}_X and \mathcal{T}_X with the relations given by $f_1 \cdot f_2 - f_2 \cdot f_1 = 0$ for $f_i \in \mathcal{O}_X$, $v \cdot f - f \cdot v = v(f) \in \mathcal{O}_X$ for $v \in \mathcal{T}_X$ and $f \in \mathcal{O}_X$, and $v_1 \cdot v_2 - v_2 \cdot v_1 = [v_1, v_2] \in \mathcal{T}_X$ for $v_i \in \mathcal{T}_X$. A data of \mathcal{O}_X -quasicoherent \mathcal{D}_X -module is equivalent to what is usually called a “D-module” on X : namely, a quasicoherent sheaf \mathcal{E} endowed with a flat connection $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$.

Remark 2.18. The algebra \mathcal{D}_X comes with a natural increasing “PBW-filtration” which is identified with the filtration by order of the differential operator. Namely, $\mathcal{D}_{X, \leq n} \subset \mathcal{D}_X$ is the \mathcal{O}_X -submodule locally generated by the image of the multiplication map $\underbrace{\mathcal{T}_X \otimes_{\mathcal{O}_S} \dots \otimes_{\mathcal{O}_S} \mathcal{T}_X}_{n} \rightarrow \mathcal{D}_X$. Note that since $[\mathcal{T}_X, \mathcal{T}_X] \subset \mathcal{T}_X$ we have

$[\mathcal{D}_{X, \leq i}, \mathcal{D}_{X, \leq j}] \subset \mathcal{D}_{X, \leq i+j-1}$. By a version of PBW-theorem for Lie algebroids (see e.g. [R]) one has a natural isomorphism

$$\mathrm{gr}_* \mathcal{D}_X \simeq \mathrm{Sym}_{\mathcal{O}_X}^* \mathcal{T}_X.$$

Consequently, if we have a trivialization $\mathcal{T}_X \simeq \mathcal{O}_X^d$ given by a frame of vector fields $\partial_1, \dots, \partial_d$ we can decompose

$$\mathcal{D}_X \simeq \bigoplus_{\alpha \in \mathbb{N}^d} \mathcal{O}_X \cdot \partial^\alpha$$

as a left \mathcal{O}_X -module. Here, for $\alpha \in \mathbb{N}^d$, ∂^α denotes the monomial $\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$. In terms of this decomposition $\mathcal{D}_{X, \leq n} \simeq \bigoplus_{\alpha \in \mathbb{N}^d} \mathcal{O}_X \cdot \partial^\alpha$ with

$$|\alpha| = \alpha_1 + \dots + \alpha_d \leq n.$$

Remark 2.19 (Restricted Lie algebra structure on \mathcal{T}_X). In char p , \mathcal{T}_X is naturally a sheaf of restricted Lie algebras over \mathcal{O}_S . Namely, note that the p -th iterate of an \mathcal{O}_S -linear derivation $v \in \mathcal{T}_X$ again defines a \mathcal{O}_S -linear derivation of \mathcal{O}_X , which we will denote $v^{[p]} \in \mathcal{T}_X$. Since $v^{[p]}$ is equal to the image of v^p in the \mathcal{O}_S -linear endomorphisms $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$ and the Lie bracket on \mathcal{T}_X is induced by the commutator in $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$, one sees from Example 2.17 that the relations in Definition 2.16 are satisfied.

The algebra \mathcal{D}_X naturally acts on \mathcal{O}_X by \mathcal{O}_S -linear endomorphisms: namely, \mathcal{O}_X acts by multiplication, while \mathcal{T}_X acts by the corresponding \mathcal{O}_S -linear derivations. Note that any \mathcal{O}_S -linear derivation acts by zero on all functions in $\mathcal{O}_X^{[p]} = \mathcal{O}_X^p \cdot \mathcal{O}_S \subset \mathcal{O}_X$.

⁹Namely, the commutator of two derivations is again a derivation.

It follows that $\mathcal{O}_X^{\{p\}} \subset \mathcal{O}_X$ lies in the center $Z(\mathcal{D}_X)$, and so one can consider \mathcal{D}_X as a sheaf of $\mathcal{O}_X^{\{p\}}$ -algebras. By adjunction, from the identification $(F_X^{\{p\}})^{-1}\mathcal{O}_{X^{\{p\}}} \simeq \mathcal{O}_X^{\{p\}}$, we get a map $\mathcal{O}_{X^{\{p\}}} \rightarrow Z(F_{X^*}^{\{p\}}\mathcal{D}_X)$ and this way we can consider $F_{X^*}^{\{p\}}\mathcal{D}_X$ as a sheaf of algebras over $X^{\{p\}}$.

Remark 2.20. By construction, the map $F_X^{\{p\}} : X \rightarrow X^{\{p\}}$ is affine. Consequently, one has a monoidal equivalence of categories of quasi-coherent sheaves on X and quasi-coherent sheaves of $F_{X^*}^{\{p\}}\mathcal{O}_X$ -modules on $X^{\{p\}}$. This then gives an equivalence of categories of quasi-coherent \mathcal{D}_X -modules on X and quasi-coherent $F_{X^*}^{\{p\}}\mathcal{D}_X$ -modules on $X^{\{p\}}$.

Remark 2.21. Note that by base change, $\Omega_{X^{(1)}}^1 \simeq W_{X/S}^*\Omega_X^1 := \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(1)}}$. In particular, if X is pseudo-smooth, then so is $X^{(1)}$. We warn that this is not necessarily true for $X^{\{p\}}$ (see Remark 2.33).

Let us also define the appropriate variants of twisted tangent and cotangent bundles.

Construction 2.22 (Twisted tangent and cotangent bundles). For the usual Frobenius twist, one can consider $\Omega_{X^{(1)}}^1 \simeq \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(1)}}$ and $\mathcal{T}_{X^{(1)}} \simeq W_{X/S}^*\mathcal{T}_X := \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(1)}}$. Analogously, we define the (reduced Frobenius) twisted tangent and cotangent bundles $\mathcal{T}_X^{\{p\}} := W_X^{\{p\}*}\mathcal{T}_X := \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\{p\}}$ and $\Omega_X^{1,\{p\}} := \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\{p\}}$ (here the map $\mathcal{O}_X \rightarrow \mathcal{O}_X^{\{p\}}$ sends $f \mapsto f^p$). One has $\Omega_X^{1,\{p\}} \simeq (\mathcal{T}_X^{\{p\}})^\vee$. We warn that it is no longer necessarily true that $\mathcal{T}_X^{\{p\}}$ (resp. $\Omega_X^{1,\{p\}}$) is isomorphic to $\mathcal{T}_{X^{\{p\}}}$ (resp. $\Omega_{X^{\{p\}}}^1$), e.g. since $X^{\{p\}}$ can happen to be not smooth. Nevertheless there is still some natural compatibility (see Corollary 2.24).

For a vector bundle \mathcal{E} on a scheme X let $\text{Tot}_X(\mathcal{E}) := \text{Spec}_X \text{Sym}_{\mathcal{O}_X}^* \mathcal{E}^\vee$ be the total space of \mathcal{E} . In the case \mathcal{E} is \mathcal{T}_X or Ω_X^1 we denote it by TX and T^*X correspondingly. We denote by $\mathcal{E}^{\{p\}} := W_X^{\{p\}*}\mathcal{E} \simeq \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\{p\}}$ the pull-back of \mathcal{E} to $X^{\{p\}}$.

Lemma 2.23. *One has a natural isomorphism*

$$\text{Tot}_{X^{\{p\}}}(\mathcal{E}^{\{p\}}) = (\text{Tot}_X(\mathcal{E}))^{\{p\}}.$$

Proof. Note that $\text{Tot}_{X^{\{p\}}}(\mathcal{E}^{\{p\}}) \simeq \text{Tot}_X(\mathcal{E}) \times_X X^{\{p\}}$ under the map $W_X^{\{p\}} : X^{\{p\}} \rightarrow X$. Indeed, $\text{Sym}_{\mathcal{O}_{X^{\{p\}}}}^*(\mathcal{E}^{\{p\}}) \simeq (\text{Sym}_{\mathcal{O}_X}^* \mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\{p\}}$ and the isomorphism is obtained by taking $\text{Spec}_{X^{\{p\}}}$ of both sides. By taking $W_{\text{Tot}_X(\mathcal{E})}^{\{p\}} : (\text{Tot}_X(\mathcal{E}))^{\{p\}} \rightarrow \text{Tot}_X(\mathcal{E})$ and the projection $(\text{Tot}_X(\mathcal{E}))^{\{p\}} \rightarrow X^{\{p\}}$ we obtain a map

$$(\text{Tot}_X(\mathcal{E}))^{\{p\}} \rightarrow \text{Tot}_X(\mathcal{E}) \times_X X^{\{p\}} \simeq \text{Tot}_{X^{\{p\}}}(\mathcal{E}^{\{p\}}).$$

To check that it is an isomorphism we can work locally on X and S . So we can assume $X = \text{Spec } A$, $Y = \text{Spec } B$ and \mathcal{E} is trivial. Then we need to check that the natural map

$$A[x_1, \dots, x_n] \otimes_A A^{\{p\}} \rightarrow (A[x_1, \dots, x_n])^{\{p\}}$$

given by $f \otimes s \mapsto f^p s$ is an isomorphism. But it is clear that the right hand side is the polynomial ring on x_1^p, \dots, x_n^p over $A^{\{p\}}$ and so this map is indeed an isomorphism. \square

Plugging \mathcal{E} to be \mathcal{T}_X or Ω_X^1 we get

Corollary 2.24. *One has isomorphisms*

$$\mathrm{Tot}_{X^{[p]}}(\mathcal{T}_X^{\{p\}}) \simeq (TX)^{\{p\}} \quad \text{and} \quad \mathrm{Tot}_{X^{[p]}}(\Omega_X^{1,\{p\}}) \simeq (T^*X)^{\{p\}}.$$

2.4. Differential operators on smooth schemes. For this section we assume that X is a smooth scheme over S . We start with identifying the reduced Frobenius in this setting.

Assume that $X = \mathbb{A}_S^d$ is the affine space over S of dimension d . Then locally over S we have $S = \mathrm{Spec} B$ and $X = \mathrm{Spec} B[x_1, \dots, x_d]$. Let $P := B[x_1, \dots, x_d]$. We have $P^{\{p\}} = B[x_1^p, \dots, x_d^p] \subset P$ and it is clear that P is a free $P^{\{p\}}$ -module of rank p^d . In particular, we see that $F_X^{\{p\}}: X \rightarrow X^{\{p\}}$ is a finite faithfully flat map.

Moreover, following Example 2.11(1), F_P^{int} induces an isomorphism $P^{(1)} \xrightarrow{\sim} P^{\{p\}}$. This shows that $F_X^{\mathrm{int}}: X^{(1)} \xrightarrow{\sim} X^{\{p\}}$ is an isomorphism, and that under this identification $F_X^{\{p\}} = F_{X/S}$.

In general smooth setting, Zariski locally X has an étale map $q: X \rightarrow \mathbb{A}_S^d$ for some d . Since q is étale, the relative Frobenius F_{X/\mathbb{A}_S^d} is an isomorphism, and so one has a fibered square

$$\begin{array}{ccc} X & \xrightarrow{F_{X/S}} & X^{(1)} \\ q \downarrow & & \downarrow q^{(1)} \\ \mathbb{A}_S^d & \xrightarrow{F_{\mathbb{A}_S^d/S}} & (\mathbb{A}_S^d)^{(1)} \end{array} .$$

Since q is flat, the pull-back $F_{X/S}^{-1}(\mathcal{O}_{X^{(1)}}) \rightarrow \mathcal{O}_X$ is locally injective, and so $\mathcal{O}_X^{\{p\}} \simeq F_{X/S}^{-1}(\mathcal{O}_{X^{(1)}})$. This shows that $F_X^{\mathrm{int}}: X^{(1)} \xrightarrow{\sim} X^{\{p\}}$ is an isomorphism and $F_X^{\{p\}} = F_{X/S}$, as in the case of affine space. From the latter we also get that $F_X^{\{p\}}$ is a finite faithfully flat map of rank $p^{\dim_S X}$.

Given the above natural identification $X^{(1)} \xrightarrow{\sim} X^{\{p\}}$, it follows that in the smooth case the vector bundle $\mathcal{T}_X^{\{p\}}$ can be identified with $\mathcal{T}_{X^{(1)}}$ and, consequently, $(T^*X)^{\{p\}}$ can also be identified with $T^*X^{(1)}$.

Construction 2.25 (p -curvature map). Let X be a smooth S -scheme. Consider the map $\theta := \mathcal{T}_X \rightarrow \mathcal{D}_X$ of sheaves on X given by the following formula

$$v \mapsto \theta(v) := v^p - v^{[p]},$$

where $-^{[p]}$ is the restricted power (see Remark 2.19). We claim that

- θ is additive: $\theta(v_1 + v_2) - (\theta(v_1) + \theta(v_2)) = 0$;
- θ is Frobenius-linear: $\theta(fv) - f^p \cdot \theta(v) = 0$;
- $\theta(v)$ lies in the center $Z(\mathcal{D}_X)$ for any $v \in \mathcal{T}_X$: $[\theta(v), f] = [\theta(v), w] = 0$ for any $f \in \mathcal{O}_X$ and $w \in \mathcal{T}_X$.

Indeed, note that $\theta(v) \in \mathcal{D}_{X, \leq p}$ and that its image in $\mathrm{gr}_p \mathcal{D}_X \simeq \mathrm{Sym}_{\mathcal{O}_X}^p(\mathcal{T}_X)$ is v^p . It is easy¹⁰ to see that all expressions above have 0 image in $\mathrm{gr}_p \mathcal{D}_X$, and so belong to $\mathcal{D}_{X, \leq p-1}$. Thus, by Lemma 2.26 below, they are zero if and only of their images in $\mathrm{End}_{\mathcal{O}_S}(\mathcal{O}_X)$ are. However, $\theta(v)$ acts by 0 on any function f in \mathcal{O}_X since by definition

¹⁰For all cases except $[\theta(v), w]$ this follows from the fact that the associated graded of \mathcal{D}_X is commutative. For $[\theta(v), w]$ one can use that $[-, w]$ induces a derivation of $\mathrm{gr}_* \mathcal{D}_X$ by sending $x \in \mathrm{gr}_i \mathcal{D}_X$ to the class of $[\tilde{x}, w] \in \mathcal{D}_{X, \leq i}$ in $\mathrm{gr}_i \mathcal{D}_X$, and that any derivation is 0 on the p -th power v^p .

$v^{[p]}$ acts as v^p , and so the image of all of the expressions above in $\text{End}_{\mathcal{O}_S}(\mathcal{O}_X)$ is identically 0.

This way we obtain a map

$$\theta': \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^p \rightarrow Z(\mathcal{D}_X)$$

of \mathcal{O}_X^p -modules. Since $Z(\mathcal{D}_X)$ is in fact an $\mathcal{O}_X^{\{p\}} \simeq \mathcal{O}_X^p \cdot \mathcal{O}_S$ -algebra we can extend this to a map $\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\{p\}} \rightarrow Z(\mathcal{D}_X)$ of $\mathcal{O}_X^{\{p\}}$, or in other words a map of sheaves on $X^{\{p\}}$

$$(2.1) \quad \theta: \mathcal{T}_X^{\{p\}} \rightarrow Z(F_{X^*}^{\{p\}} \mathcal{D}_X).$$

which we call the “ p -curvature map”.

Lemma 2.26. *Let X be smooth over S . Then the natural \mathcal{O}_S -linear action of \mathcal{D}_X on \mathcal{O}_X induces an embedding $\mathcal{D}_{X, \leq p-1} \hookrightarrow \text{End}_{\mathcal{O}_S}(\mathcal{O}_X)$.*

Proof. The statement is Zariski-local, so we can assume that we have an étale map $f: X \rightarrow \mathbb{A}_S^n$ given by functions f_1, \dots, f_n . Let z_1, \dots, z_n be the coordinates on \mathbb{A}^n . We have an isomorphism $\mathcal{T}_X \simeq f^* \mathcal{T}_{\mathbb{A}_S^n}$ and can consider the trivialization of \mathcal{T}_X given by $\partial_i := f^{-1}(\partial_{z_i})$. We have $f_i = f^*(z_i)$ and this way $\partial_i(f_j) = \delta_{ij}$. \mathcal{T}_X is trivial with basis ∂_i , $i = 1, \dots, n$ where $n = \text{rk}(\Omega_X^1)$. Let $D \in \mathcal{D}_{X, \leq p-1}$; by Remark 2.18 it can be written as a sum

$$D = \sum_{\alpha \in \mathbb{N}^d} g_\alpha \cdot \partial^\alpha$$

where $g_\alpha \in \mathcal{O}_X$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is such that $|\alpha| \leq p-1$. Take a maximal α with respect to the lexicographic order on \mathbb{N}^d such that $g_\alpha \neq 0$; then it is easy to check that $D(f_1^{\alpha_1} \dots f_n^{\alpha_n}) = \alpha_1! \dots \alpha_n! \cdot g_\alpha$, which is non-zero since all $\alpha_i < p$. \square

Recall the twisted tangent and cotangent bundles (Construction 2.22 $\mathcal{T}_X^{\{p\}}, \Omega_X^{1, \{p\}}$). By Corollary 2.24 we have

$$(T^*X)^{\{p\}} \simeq \text{Spec}_{X^{\{p\}}}(\text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}})$$

be the total space of $\Omega_X^{1, \{p\}}$. The natural projection $(T^*X)^{\{p\}} \rightarrow X^{\{p\}}$ is affine and so one has an equivalence of categories of quasi-coherent sheaves on $(T^*X)^{\{p\}}$ and quasi-coherent $\text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}}$ -modules on $X^{\{p\}}$. The map θ above extends to a map of sheaves of $\mathcal{O}_X^{\{p\}}$ -algebras

$$\theta: \text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}} \rightarrow Z(F_{X^*}^{\{p\}} \mathcal{D}_X),$$

and this way $F_{X^*}^{\{p\}} \mathcal{D}_X$ defines a sheaf of algebras over $(T^*X)^{\{p\}}$.

Then, following [BMR] (or rather [OV, Section 2.1] in the relative setting) we have the following structural result for $F_{X^*}^{\{p\}} \mathcal{D}_X$.

Proposition 2.27. *Let X be a smooth S -scheme. Then*

- (1) *The p -curvature map (defined in the end of Section 2.3)*

$$\theta: \text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_{X^{\{p\}}} \rightarrow Z(F_{X^*}^{\{p\}} \mathcal{D}_X)$$

is an isomorphism.

- (2) *The sheaf of algebras over $(T^*X)^{\{p\}}$ defined by $F_{X^*}^{\{p\}} \mathcal{D}_X$ is an Azumaya algebra of rank $2p^{\dim_S X}$.*

Example 2.28. Assume that $S = \text{Spec } B$ is affine and that $X := \mathbb{A}_S^d$ with coordinates z_1, \dots, z_d . In this case global sections of \mathcal{D}_X are given by the Weyl algebra

$$W_d(B) := B\langle z_1, \dots, z_d, \partial_{z_1}, \dots, \partial_{z_d} \rangle / ([z_i, z_j] = [\partial_{z_i}, \partial_{z_j}] = 0, [\partial_{z_j}, z_i] = \delta_{ij}).$$

The p -th power $\partial_{z_i}^p$ acts by zero on \mathcal{O}_X , so $\partial_{z_i}^{[p]} = 0$ and $\theta(\partial_{z_i}) = \partial_{z_i}^p$. Then proposition 2.27 tells us that the center $Z(\mathcal{D}_X)$ is isomorphic to the polynomial ring $B[z_i^p, \partial_{z_i}^p] \subset \mathcal{D}_X$ and that \mathcal{D}_X is an Azumaya algebra over it.

We finish with some remarks that will be useful in the next section:

Remark 2.29. Consider the homomorphism $\text{Sym}_{\mathcal{O}_X^{[p]}}^* \mathcal{F}_X^{[p]} \rightarrow \mathcal{O}_X^{[p]}$ sending $\mathcal{F}_X^{[p]}$ to 0. It corresponds to the embedding of the zero section $i_0: X^{[p]} \hookrightarrow (T^*X)^{[p]}$. Then the pull-back $\mathcal{D}_{X,0} := i_0^*(F_{X^*}^{[p]} \mathcal{D}_X)$ is an Azumaya algebra on $X^{[p]}$ which is canonically split. Indeed, $F_{X^*}^{[p]} \mathcal{O}_X$ is a vector bundle of rank $p^{\dim_S X}$ and $\mathcal{F}_X^{[p]} \subset Z(F_{X^*}^{[p]} \mathcal{D}_X)$ acts on it by 0 (since $v^{[p]}$ by definition acts as v^p): so the action of $F_{X^*}^{[p]} \mathcal{D}_X$ on $F_{X^*}^{[p]} \mathcal{O}_X$ factors through $\mathcal{D}_{X,0}$. Comparing the ranks we get that $F_{X^*}^{[p]} \mathcal{O}_X$ is a splitting bundle for $\mathcal{D}_{X,0}$, and so

$$\mathcal{D}_{X,0} \simeq \text{End}_{\mathcal{O}_X^{[p]}}(F_{X^*}^{[p]} \mathcal{O}_X).$$

Remark 2.30 (Cartier's equivalence). From the above remark one gets an equivalence of categories of quasi-coherent sheaves of $\mathcal{D}_{X,0}$ -modules and quasi-coherent sheaves on $X^{[p]}$. The corresponding pair of quasi-inverse functors is given by $-\otimes_{\mathcal{O}_X^{[p]}} F_{X^*}^{[p]} \mathcal{O}_X$ and $\mathcal{H}om_{\mathcal{D}_{X,0}}(F_{X^*}^{[p]} \mathcal{O}_X, -)$. Under this equivalence the $\mathcal{D}_{X,0}$ -module $F_{X^*}^{[p]} \mathcal{O}_X$ corresponds to $\mathcal{O}_X^{[p]}$.

Remark 2.31. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals that is invariant under the action of \mathcal{D}_X . Note that $F_X^{[p]}$ is affine, so the pushforward $F_{X^*}^{[p]}$ is exact, and for any quasi-coherent sheaf \mathcal{F} on X the map $(F_X^{[p]})^{-1}(\mathcal{F}_{X^*}^{[p]}) \rightarrow \mathcal{F}$ is an equivalence. The pushforward $\mathcal{F}_{X^*}^{[p]}(\mathcal{J})$ then gives a sub- $\mathcal{D}_{X,0}$ -module of $\mathcal{F}_{X^*}^{[p]} \mathcal{O}_X$ which by the equivalence in Remark 2.30 is necessarily of the form $\mathcal{J} \otimes_{\mathcal{O}_X^{[p]}} \mathcal{F}_{X^*}^{[p]} \mathcal{O}_X \subset \mathcal{F}_{X^*}^{[p]} \mathcal{O}_X$ for some ideal $\mathcal{J} \subset \mathcal{O}_X^{[p]}$. Under the equivalence of categories between quasi-coherent modules over $\mathcal{F}_{X^*}^{[p]} \mathcal{O}_X$ and quasi-coherent sheaves on X the module $\mathcal{J} \otimes_{\mathcal{O}_X^{[p]}} \mathcal{F}_{X^*}^{[p]} \mathcal{O}_X$ exactly corresponds to $(F_X^{[p]})^* \mathcal{J}$, and so we get that $\mathcal{J} \simeq F_X^{[p]*}(\mathcal{J})$ for some ideal $\mathcal{J} \subset \mathcal{O}_X^{[p]}$.

2.5. Local structure of a pseudo-smooth scheme. The goal of this section is to give a local description of a general pseudo-smooth scheme which will allow us to make a reduction to smooth case. We start with the following lemma:

Lemma 2.32. *Let Y be a smooth S -scheme, and let $Z \subset Y^{[p]}$ be a subscheme. Then the preimage $(F_Y^{[p]})^{-1}(Z) = Z \times_{Y^{[p]}} Y \subset Y$ is a pseudo-smooth scheme.*

Proof. Let $\mathcal{J} \subset \mathcal{O}_Y^{[p]}$ be the sheaf of ideals defining Z . Then $\mathcal{J} := F_Y^{[p]*}(\mathcal{J}) \subset \mathcal{O}_X$ is the sheaf of ideals defining $X := (F_Y^{[p]})^{-1}(Z)$. Note that \mathcal{J} is locally generated by elements in $\mathcal{O}_Y^{[p]} \simeq (F_Y^{[p]})^{-1} \mathcal{O}_{Y^{[p]}}$. Let $i: X \rightarrow Y$ be the corresponding embedding. We have an exact sequence

$$\dots \rightarrow i^{-1}(\mathcal{J}/\mathcal{J}^2) \xrightarrow{d} i^* \Omega_Y^1 \rightarrow \Omega_X^1 \rightarrow 0$$

of sheaves on X where the left map is induced by the map that sends $[f] \in \mathcal{J}/\mathcal{J}^2$ to $df \in \Omega_Y^1$. However, for any $f \in \mathcal{O}_Y^{\{p\}}$ we have $df = 0$. Since \mathcal{J} is locally generated by elements in $\mathcal{O}_Y^{\{p\}}$ we get that the map d above is equal to 0 and so $\Omega_X^1 \simeq i^*\Omega_Y^1$. \square

Remark 2.33. We claim that in the above case the reduced Frobenius twist $X^{\{p\}}$ is given by Z . Indeed, \mathcal{O}_X (or rather $i_*\mathcal{O}_X$) is described as the reduction $\mathcal{O}_Y \otimes_{\mathcal{O}_Y^{\{p\}}} \mathcal{O}_Y^{\{p\}}/\mathcal{J}$.

Indeed, $\mathcal{O}_X^{\{p\}}$ is the subalgebra in \mathcal{O}_X generated by \mathcal{O}_X^p and \mathcal{O}_S , which is easily seen to be isomorphic to $\mathcal{O}_Y^{\{p\}}/\mathcal{J} \simeq \mathcal{O}_Z$. Moreover, this way the corresponding embedding $i^{\{p\}}: X^{\{p\}} \hookrightarrow Y^{\{p\}}$ is identified with the original $Z \hookrightarrow Y^{\{p\}}$. Consequently, we also see that the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_X^{\{p\}}} & Z \\ i \downarrow & & \downarrow i^{\{p\}} \\ Y & \xrightarrow{F_Y^{\{p\}}} & Y^{\{p\}} \end{array}$$

is in fact a fiber square. In particular, $F_X^{\{p\}}$ is a finite locally free map of degree $p^{\dim_S Y}$.

Note that Z here was an arbitrary closed scheme of Y ; in particular, it could be very singular.

Remark 2.34. Recall that in the course of proof of Lemma 2.32 we showed that $i^*\Omega_Y^1 \simeq \Omega_X^1$ via the natural map. It then formally follows that $i^*\Omega_Y^j \simeq \Omega_X^j$ for all $j \geq 0$. By considering the commutative diagram

$$\begin{array}{ccc} X^{\{p\}} & \xrightarrow{W_X^{\{p\}}} & X \\ i^{\{p\}} \downarrow & & \downarrow i \\ Y^{\{p\}} & \xrightarrow{W_Y^{\{p\}}} & Y \end{array}$$

it also follows that the natural map $i^{\{p\}*}\Omega_Y^{j,\{p\}} \rightarrow \Omega_X^{j,\{p\}}$ induced by the identification $i^{\{p\}*} \circ W_Y^{\{p\}} = W_X^{\{p\}} \circ i$ is an isomorphism for any $j \geq 0$.

In fact, all pseudo-smooth schemes Zariski-locally look like Frobenius neighborhoods in smooth schemes:

Proposition 2.35. *Let X be a pseudo-smooth S -scheme. Then Zariski-locally it is of the form $(F_Y^{\{p\}})^{-1}(Z) = Z \times_{Y^{\{p\}}} Y \subset Y$ with Y smooth over S and $Z \hookrightarrow Y$ is a closed subscheme (see Lemma 2.32).*

Proof. Let $x \in X$ be a point. We will show that there is a neighborhood of x of the form above.

Since X is locally of finite type, passing to an open we can assume that there is an embedding $i: X \hookrightarrow \mathbb{A}^n$; the natural map $i^*: i^*\Omega_{\mathbb{A}^n}^1 \rightarrow \Omega_X^1$ is a surjection. Let $\mathcal{J} \subset \mathcal{O}_{\mathbb{A}^n}$ be the ideal defining X . As before, we have an exact sequence

$$\dots \rightarrow i^{-1}(\mathcal{J}/\mathcal{J}^2) \xrightarrow{d} i^*\Omega_{\mathbb{A}^n}^1 \xrightarrow{i^*} \Omega_X^1 \rightarrow 0.$$

Since X is pseudo-smooth, the kernel of i^* is a vector bundle. Let $d := \text{rk } H^0(X, \Omega_X^1)$. After passing to a neighborhood U of x in \mathbb{A}^n we can assume that $\text{Ker}(i^*)$ is a trivial vector bundle with basis given by differentials df_1, \dots, df_{n-d} of $(n-d)$ functions

$f_1, \dots, f_{n-d} \in \mathcal{J}$. Consider the map $s: \mathcal{O}_U^{\oplus n-d} \rightarrow \Omega_U^1$ defined by $(n-d)$ sections df_1, \dots, df_{n-d} . The support of its kernel is a closed subset which doesn't intersect X (and in particular doesn't contain x) thus considering the complement we can assume that s is an embedding. Moreover, the dimension of the stalk of the cokernel is an upper-semicontinuous function bounded below by $n - (n-d) = d$. Since it is equal to d on X we see that it is also equal to d on some neighborhood, and so restriction of the cokernel to this neighborhood is a vector bundle of rank d . Shrinking the open further we can assume that Ω_U^1 is trivial with basis given by differentials $df_1, \dots, df_{n-d}, dg_1, \dots, dg_d$ (with f_i 's being the functions we considered before, and g_i 's some other functions).

Now, let $i_Y: Y \hookrightarrow U$ be the subscheme defined by $\mathcal{J}_Y := (f_1, \dots, f_{n-d}) \subset \mathcal{J}$. Then we claim that the map $q: Y \rightarrow \mathbb{A}^d$ given by g_1, \dots, g_d is étale. Indeed, one can realize Y as a subscheme in $U \times \mathbb{A}^d \subset \mathbb{A}^n \times \mathbb{A}^d$ cut out by equations $f_1 = \dots = f_{n-d} = t_1 - g_1 = \dots = t_d - g_d = 0$ where t_i 's are coordinates on \mathbb{A}^d . Since $df_1, \dots, df_{n-d}, dg_1, \dots, dg_d$ are linearly independent over \mathcal{O}_U fiberwise, the Jacobian of this system of equations (relative to \mathbb{A}^d) is a unit, and so projection to \mathbb{A}^d is étale.

This way we get that Y is in fact smooth over S . Let $\mathcal{J}' \subset \mathcal{O}_Y$ be the ideal defining $i_X: X \hookrightarrow Y$. We have an exact sequence

$$\dots \rightarrow i^{-1}(\mathcal{J}'/\mathcal{J}'^2) \xrightarrow{d} i_X^* \Omega_Y^1 \xrightarrow{i_X^*} \Omega_X^1 \rightarrow 0.$$

However, note that the map i_X^* is now an isomorphism. Indeed, $i_X^* \Omega_Y^1$ is exactly the quotient of $i^* \Omega_U^1$ by df_i , which are spanning the kernel of i_X^* . Thus, we get that the map d is 0, or, in other words, that $d(\mathcal{J}') \subset \mathcal{J}'$. This is equivalent to \mathcal{J}' being invariant under the action of \mathcal{D}_Y , from which it follows by Remark 2.31 that $\mathcal{J}' \simeq F_Y^{[p]*} \mathcal{J}$ for some $\mathcal{J} \subset \mathcal{O}_{Y^{[p]}}$. Then, putting $Z \subset Y^{[p]}$ to be the subscheme defined by \mathcal{J} we exactly get that $X \simeq (F_Y^{[p]})^{-1}(Z)$. \square

Proposition 2.35 allows to extend the discussion in Section 2.4 to the pseudo-smooth setting. We start with an extension of Lemma 2.26:

Corollary 2.36. *Let X be a pseudo-smooth scheme over S . Then the natural map*

$$\mathcal{D}_{X, \leq p-1} \rightarrow \text{End}_{\mathcal{O}_S}(\mathcal{O}_X)$$

induced by the action of \mathcal{D}_X on \mathcal{O}_X is an embedding.

Proof. The statement is local, so we can assume that $X = (F_Y^{[p]})^{-1}(Z) \hookrightarrow Y$ for a closed subscheme $Z \hookrightarrow Y^{[p]}$ in smooth scheme. We can also assume that Y has an étale map $f: Y \rightarrow \mathbb{A}^n$. Let ∂_i and f_i be as in the proof of Lemma 2.26. Let $i: X \rightarrow Y$ be the embedding. Recall (see proof of Lemma 2.32) that $i^* \Omega_Y^1 \xrightarrow{\sim} \Omega_X^1$, and, consequently, $\mathcal{T}_X \simeq i^* \mathcal{T}_Y$. Let $\partial'_i := i^{-1}(\partial_i) \in \mathcal{T}_X$ and let $f'_i := i^*(f_i) \in \mathcal{O}_X$. We have $\partial'_i(f'_j) = \delta_{ij}$. The same argument as in Lemma 2.26 then shows that for any $D \in \mathcal{D}_{X, \leq p-1}$ there is a function $g \in \mathcal{O}_X$ such that $D(g) \neq 0$. \square

Construction 2.37 (p -curvature map in the pseudo-smooth setting). As in the smooth case, consider the map $\theta := \mathcal{T}_X \rightarrow \mathcal{D}_X$ of sheaves on X given by the following formula

$$v \mapsto \theta(v) := v^p - v^{[p]}.$$

Using Corollary 2.36 in place of Lemma 2.26 as in Construction 2.25 one shows that

- θ is additive: $\theta(v_1 + v_2) - (\theta(v_1) + \theta(v_2)) = 0$;
- θ is Frobenius-linear: $\theta(fv) - f^p \cdot \theta(v) = 0$;

- $\theta(v)$ lies in the center $Z(\mathcal{D}_X)$ for any $v \in \mathcal{T}_X$: $[\theta(v), f] = [\theta(v), w] = 0$ for any $f \in \mathcal{O}_X$ and $w \in \mathcal{T}_X$.

This way we obtain a map

$$\theta': \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^p \rightarrow Z(\mathcal{D}_X)$$

of \mathcal{O}_X^p -modules. Since $Z(\mathcal{D}_X)$ is in fact an $\mathcal{O}_X^{\{p\}} \simeq \mathcal{O}_X^p \cdot \mathcal{O}_S$ -algebra we can extend this to a map $\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\{p\}} \rightarrow Z(\mathcal{D}_X)$ of $\mathcal{O}_X^{\{p\}}$, or, equivalently, a map of sheaves on $X^{\{p\}}$

$$(2.2) \quad \theta: \mathcal{T}_X^{\{p\}} \rightarrow Z(F_{X^*}^{\{p\}} \mathcal{D}_X).$$

which we still call the “ p -curvature map”.

Corollary 2.38. *Let X be a pseudo-smooth scheme over S . As before, consider $(T^*X)^{\{p\}} \simeq \text{Spec}_{X^{\{p\}}}(\text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}})$.*

(1) *The map $\theta: \mathcal{T}_X^{\{p\}} \rightarrow Z(F_{X^*}^{\{p\}} \mathcal{D}_X)$ induces an isomorphism*

$$\theta: \text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}} \xrightarrow{\simeq} Z(F_{X^*}^{\{p\}} \mathcal{D}_X).$$

(2) *$F_{X^*}^{\{p\}} \mathcal{D}_X$, considered as a sheaf of algebras over $(T^*X)^{\{p\}}$, is an Azumaya algebra of rank¹¹ $p^{2 \text{rk} \Omega_X^1}$.*

Proof. Both statements are local (e.g. in Zariski topology) and so we can assume $X \simeq (F_Y^{\{p\}})^{-1}(Z)$ where Y is smooth over S and $Z \hookrightarrow Y$ is a closed subscheme. In this case, \mathcal{D}_X is a central reduction of \mathcal{D}_Y via an embedding $T^*Y^{(1)}|_Z \hookrightarrow T^*Y^{(1)} \simeq (T^*Y)^{\{p\}}$. More precisely, let $i: X \hookrightarrow Y$ be the embedding. Following the proof of Lemma 2.32 in this situation we have isomorphisms $\Omega_X^1 \simeq i^* \Omega_Y^1$, and consequently, $\mathcal{T}_X \simeq i^* \mathcal{T}_Y$. By naturality, the latter is an isomorphism of Lie algebroids, and induces an isomorphism $\mathcal{D}_X \simeq i^* \mathcal{D}_Y$. Also

$$\text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}} \simeq \text{Sym}_{\mathcal{O}_Y^{\{p\}}}^* \mathcal{T}_Y^{\{p\}} \otimes_{\mathcal{O}_Z^{\{p\}}} \mathcal{O}_Z$$

and so $(T^*X)^{\{p\}}$ can be identified with the restriction of $(T^*Y)^{\{p\}}$ to $Z \hookrightarrow Y^{\{p\}}$. Moreover, if we denote by $i_Z: Z \hookrightarrow Y^{\{p\}}$ the corresponding embedding, applying F_{X^*} we get

$$F_{X^*}^{\{p\}} \mathcal{D}_X \simeq F_{X^*}^{\{p\}}(i^* \mathcal{D}_Y) \simeq i_Z^* F_{Y^*}^{\{p\}} \mathcal{D}_Y \simeq F_{Y^*}^{\{p\}} \mathcal{D}_Y \otimes_{\mathcal{O}_Z^{\{p\}}} \mathcal{O}_Z \simeq F_{Y^*}^{\{p\}} \mathcal{D}_Y \otimes_{\mathcal{O}_{(T^*Y)^{\{p\}}}} \mathcal{O}_{(T^*X)^{\{p\}}},$$

where in the last isomorphism we considered $F_{Y^*}^{\{p\}} \mathcal{D}_Y$ as an Azumaya algebra over $(T^*Y)^{\{p\}}$. Thus we get that $F_{X^*}^{\{p\}} \mathcal{D}_X$ as a sheaf of algebras over $(T^*X)^{\{p\}}$ is a restriction of an Azumaya algebra, and as such is an Azumaya algebra itself. Its center then also is necessarily isomorphic to $\mathcal{O}_{(T^*X)^{\{p\}}} \simeq \text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}}$. \square

Remark 2.39. From the proof of Proposition 2.35 it also follows that $F_X^{\{p\}}: X \rightarrow X^{\{p\}}$ is a finite faithfully flat map of degree $p^{\text{rk} \Omega_X^1}$. Indeed, it is locally obtained as a base change of relative Frobenius for a smooth scheme Y of (relative) dimension $\text{rk} \Omega_X^1$ (see Remark 2.33).

¹¹A priori this is a locally constant function on X .

2.6. De Rham complex and Cartier operation. Let X be a pseudo-smooth scheme over S . Put $r := \text{rk } \Omega_X^1$; this is a locally constant function on X . Consider the de Rham complex

$$\text{dR}_X := [0 \rightarrow \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow \dots \rightarrow \Omega_X^r \rightarrow 0]:$$

this is a complex of sheaves of \mathcal{O}_S -modules on X with differential given by the (relative) de Rham differential. Since the relative de Rham differential is $\mathcal{O}_X^p \cdot B$ -linear, the pushforward $F_{X^*}^{\{p\}} \text{dR}_X$ defines a complex of coherent sheaves on $X^{\{p\}}$.

Let $\mathcal{H}_{\text{dR}, X}^* := \mathcal{H}^*(F_{X^*}^{\{p\}} \text{dR}_X)$ denote the sheaf cohomology algebra of $F_{X^*}^{\{p\}} \text{dR}_X$. Recall the notation $\Omega_X^{1, \{p\}} := W_X^{\{p\}*} \Omega_X^1$. One can define the inverse Cartier morphism

$$C^{-1}: \wedge_{\mathcal{O}_X^{\{p\}}}^*(\Omega_X^{1, \{p\}}) \rightarrow \mathcal{H}_{\text{dR}, X}^*$$

as follows. Namely in degree 0 it is given by $(F_X^{\{p\}})^{-1}: \mathcal{O}_{X^{\{p\}}} \rightarrow F_{X^*} \mathcal{O}_X$, while in degree 1 locally one sends $(W_X^{\{p\}})^{-1}(df) \in \Omega_X^{1, \{p\}}$ to $[f^{p-1}df] \in \mathcal{H}_{\text{dR}, X}^1$. By a standard computation the latter gives a well-defined additive map to $\mathcal{H}_{\text{dR}, X}^1$, which is $F_X^{\{p\}}$ -linear. For higher degrees, C^{-1} uniquely extends by multiplicativity.

The classical result of Cartier states that C^{-1} is an equivalence if X over S is smooth. This remains true in the pseudo-smooth setting as well.

Proposition 2.40. *Let X be a pseudo-smooth S -scheme. Then the inverse Cartier map is an isomorphism.*

Proof. We deduce the statement from the smooth case. The statement is Zariski local, so we can assume $X = (F_Y^{\{p\}})^{-1}(Z)$ is as in Lemma 2.32 (so Y is smooth and $Z \hookrightarrow Y^{\{p\}}$). In this case $X^{\{p\}} \simeq Z$. Let $i: X \hookrightarrow Y$ and $i^{\{p\}}: Z \hookrightarrow Y^{\{p\}}$. By Remarks 2.33, 2.34 and base change, $F_{X^*}^{\{p\}} \text{dR}_X$ can be naturally identified with $i^{\{p\}*} F_{Y^*}^{\{p\}} \text{dR}_Y$. By Cartier isomorphism in the smooth case, the cohomology sheaves $\mathcal{H}_{\text{dR}, Y}^*$ are isomorphic to $\wedge_{\mathcal{O}_Y^{\{p\}}}^* \Omega_Y^{1, \{p\}}$; in particular, they are vector bundles and thus are flat over $\mathcal{O}_Y^{\{p\}}$. From this and Remark 2.34 we get that

$$\mathcal{H}_{\text{dR}, X}^* \simeq \mathcal{H}^*(i^{\{p\}*} F_{Y^*}^{\{p\}} \text{dR}_Y) \simeq i^{\{p\}*} \mathcal{H}_{\text{dR}, Y}^* \simeq i^{\{p\}*} (\wedge_{\mathcal{O}_Y^{\{p\}}}^* \Omega_Y^{1, \{p\}}) \simeq \wedge_{\mathcal{O}_{X^{\{p\}}}^*}^* \Omega_X^{1, \{p\}}. \quad \square$$

Let C be the inverse to C^{-1} . In each degree i , C gives an isomorphism $\mathcal{H}_{\text{dR}, X}^i \simeq \Omega_X^{i, \{p\}}$ which we can precompose with the projection $F_{X^*}^{\{p\}} \Omega_{X, \text{cl}}^i \rightarrow \mathcal{H}_{\text{dR}, X}^i$ from the closed i -forms. We will call the resulting map $C^i: F_{X^*}^{\{p\}} \Omega_{X, \text{cl}}^i \rightarrow \Omega_X^{i, \{p\}}$ the *Cartier operation*. For $i = 1$ one gets a short exact sequence

$$(2.3) \quad 0 \rightarrow F_{X^*}^{\{p\}} \mathcal{O}_X / \mathcal{O}_X^{\{p\}} \xrightarrow{d} F_{X^*}^{\{p\}} \Omega_{X, \text{cl}}^1 \xrightarrow{C} \Omega_X^{1, \{p\}} \rightarrow 0,$$

of sheaves on $X^{\{p\}}$ where the term on the left is identified with the exact 1-forms via d .

Remark 2.41. The map $C: F_{X^*}^{\{p\}} \Omega_{X, \text{cl}}^1 \rightarrow \Omega_X^{1, \{p\}}$ dually induces a map $\mathcal{T}_X^{\{p\}} \otimes F_{X^*}^{\{p\}} \Omega_{X, \text{cl}}^1 \rightarrow \mathcal{O}_X^{\{p\}}$ which lifts the pairing map $\mathcal{T}_X^{\{p\}} \otimes \Omega_X^{1, \{p\}} \rightarrow \mathcal{O}_X^{\{p\}}$. Substituting a local section $\xi \in \mathcal{T}_X^{\{p\}}$ we obtain the corresponding ‘‘contraction’’ maps

$$\tilde{\iota}_\xi: F_{X^*}^{\{p\}} \Omega_{X, \text{cl}}^1 \rightarrow \mathcal{O}_X^{\{p\}} \quad \text{and} \quad \iota_\xi: \Omega_X^{1, \{p\}} \rightarrow \mathcal{O}_X^{\{p\}}.$$

Following [BK, Lemma 2.1], we have the following explicit formula for $\widetilde{\iota}_\xi$. Namely, given a closed 1-form $\alpha \in \Omega_{X,cl}^1$ and a vector field $\xi \in \mathcal{T}_X$ (giving a section $\xi^{\{p\}} := (W_X^{\{p\}})^{-1}(\xi) \in \mathcal{T}_X^{\{p\}}$) we have the following formula

$$(2.4) \quad \widetilde{\iota_{\xi^{\{p\}}}}(\alpha) := \iota_{\xi^{\{p\}}}(\mathbb{C}(\alpha)) = \iota_{\xi^{\{p\}}}\alpha - \xi^{p-1}(\iota_\xi \alpha) \in \mathcal{O}_X^{\{p\}} \subset \mathcal{O}_X$$

for the corresponding contraction.

2.7. The relative dualizing sheaf of the Frobenius morphism. As an application of the Cartier isomorphism we prove the following.

Proposition 2.42. *Let X be a pseudo-smooth S -scheme, $d = \text{rk } \Omega_X^1$. Then the relative dualizing sheaf for the Frobenius morphism $F_X^{\{p\}} : X \rightarrow X^{\{p\}}$ is given by $(\Omega_X^d)^{\otimes 1-p}$.*

Proof. Since $F_X^{\{p\}}$ is finite flat it is enough to construct an isomorphism of $F_{X^*}^{\{p\}} \mathcal{O}_X$ -modules

$$(2.5) \quad F_{X^*}^{\{p\}}((\Omega_X^d)^{\otimes 1-p}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X^{\{p\}}}}(F_{X^*}^{\{p\}} \mathcal{O}_X, \mathcal{O}_{X^{\{p\}}}).$$

Define the trace map

$$\text{tr} : F_{X^*}^{\{p\}}((\Omega_X^d)^{\otimes 1-p}) \rightarrow \mathcal{O}_{X^{\{p\}}}$$

to be the composition

$$F_{X^*}^{\{p\}}((\Omega_X^d)^{\otimes 1-p}) \xrightarrow{\sim} (\Omega_X^{d,\{p\}})^{\otimes -1} \otimes F_{X^*}^{\{p\}} \Omega_X^d \rightarrow (\Omega_X^{d,\{p\}})^{\otimes -1} \otimes \mathcal{H}_{\text{dR},X}^d \xrightarrow{\sim} \mathcal{O}_{X^{\{p\}}}.$$

Here the first and the last isomorphisms come from the projection formula (using $F_X^{\{p\}*} \Omega_X^{d,\{p\}} \xrightarrow{\sim} (\Omega_X^d)^{\otimes p}$) and Proposition 2.40 respectively. The morphism in the middle is given by the projection from the sheaf of top degree forms to the de Rham cohomology sheaf. Define a morphism of $F_{X^*}^{\{p\}} \mathcal{O}_X$ -modules

$$(2.6) \quad F_{X^*}^{\{p\}}((\Omega_X^d)^{\otimes 1-p}) \rightarrow \mathcal{H}om_{\mathcal{O}_{X^{\{p\}}}}(F_{X^*}^{\{p\}} \mathcal{O}_X, \mathcal{O}_{X^{\{p\}}})$$

sending a local section a of $F_{X^*}^{\{p\}}((\Omega_X^d)^{\otimes 1-p})$ and a local section f of $F_{X^*}^{\{p\}} \mathcal{O}_X$ to $\text{tr}(fa)$. We claim that (2.6) is an isomorphism. Indeed, (2.6) is morphism of vector bundles. Thus, it suffices to check the isomorphism property on fibers. Using Proposition 2.35 and base change we reduce the problem to $X = \text{Spec } \mathbb{F}_p[x_1, \dots, x_d]/(x_i^p, 1 \leq i \leq d)$, $S = \text{Spec } \mathbb{F}_p$, where it is verified directly. \square

Remark 2.43. For every section ξ of \mathcal{T}_X , we have that

$$(2.7) \quad \text{tr} \circ L_\xi = 0.$$

Indeed, for a local generator μ of $\Omega_{X/S}^d$,

$$(\text{tr} \circ L_\xi)(f\mu^{\otimes 1-p}) = \text{tr}(\mu^{\otimes -p} d\iota_\xi(f\mu)) = 0$$

because the Cartier morphism vanishes on exact forms. Observe that the \mathcal{O}_X -module $(\Omega_X^d)^{\otimes 1-p}$ has a right \mathcal{D}_X -module structure: the right action of ξ is given by $-L_\xi$. (More generally, a left \mathcal{D}_X -module structure on M yields a right \mathcal{D}_X -module structure on $M \otimes_{\mathcal{O}_X} \Omega_X^d$. In particular, this can be applied to $M = (\Omega_X^d)^{\otimes -p} \xrightarrow{\sim} F_X^{\{p\}*}((\Omega_X^{d,\{p\}})^{\otimes -1})$ with its Frobenius descent connection.) Also, the left action of \mathcal{D}_X on \mathcal{O}_X makes $\mathcal{H}om_{\mathcal{O}_{X^{\{p\}}}}(F_{X^*}^{\{p\}} \mathcal{O}_X, \mathcal{O}_{X^{\{p\}}})$ a right $F_{X^*}^{\{p\}} \mathcal{D}_X$ -module. Equation (2.7) implies that (2.5) is an isomorphism of right $F_{X^*}^{\{p\}} \mathcal{D}_X$ -modules.

2.8. Line bundles with connection and Milne's exact sequence. Let (\mathcal{E}, ∇) , with $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$ be a quasi-coherent sheaf with flat connection on a pseudo-smooth S -scheme X . The connection ∇ induces a map $\mathcal{D}_X \rightarrow \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$, whose restriction to $Z(\mathcal{D}_X)$ factors through the subalgebra of \mathcal{O}_X -linear endomorphisms $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})^{\nabla'=0} \subset \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$ that are flat with respect to the natural connection¹² on $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. Indeed, central elements commute both with multiplication on functions and action by vector fields.

Construction 2.44 (p -curvature map for line bundles). Let $\mathcal{E} := \mathcal{L}$ in (\mathcal{E}, ∇) be a line bundle. In this case, the endomorphism sheaf $(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L}), \nabla')$ with its natural connection is given by (\mathcal{O}_X, d) . In particular, $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})^{\nabla'=0} \subset \mathcal{O}_X$ is given by $\ker d$, which is identified with $\mathcal{O}_X^{\{p\}} \subset \mathcal{O}_X$ by Proposition 2.40. Using the p -curvature map (2.2) we obtain a map of sheaves $\mathcal{T}_X^{\{p\}} \rightarrow \mathcal{O}_X^{\{p\}}$, or, in other words, a section of $\Omega_X^{1,\{p\}}$, which we will denote $c_p(\mathcal{L}, \nabla) \in \Omega_X^{1,\{p\}}$. It is not hard to see from the definition (and the way $\mathcal{T}_X^{\{p\}} \subset Z(F_{X^*}^{\{p\}} \mathcal{D}_X)$ acts on the tensor product), that

$$c_p((\mathcal{L}_1, \nabla_1) \otimes (\mathcal{L}_2, \nabla_2)) = c_p(\mathcal{L}_1, \nabla_1) + c_p(\mathcal{L}_2, \nabla_2).$$

If we are given a trivialization $\mathcal{L} \simeq \mathcal{O}_X$, ∇ can be written in the form $d + \alpha$ for some closed 1-form $\alpha \in \Omega_{X,cl}^1$. Restricting to line bundles of this form we get a map of sheaves

$$c_p: F_{X^*}^{\{p\}} \Omega_{X,cl}^1 \rightarrow \Omega_X^{1,\{p\}}, \quad c_p: \alpha \mapsto c_p(\mathcal{O}_X, d + \alpha).$$

Since $(\mathcal{O}_X, d + \alpha_1) \otimes (\mathcal{O}_X, d + \alpha_2) \simeq (\mathcal{O}_X, d + \alpha_1 + \alpha_2)$ the above map is a map of sheaves of abelian groups.

Remark 2.45. One has an explicit formula for the map c_p in terms of the Cartier operation. Namely, given $\alpha \in \Omega_{X,cl}^1$ let's compute the pairing $\langle \xi^{\{p\}}, c_p(\alpha) \rangle$ for any $\xi \in \mathcal{T}_X$, $\xi^{\{p\}} \in (W_X^{\{p\}})^{-1}(\xi) \in \mathcal{T}_X^{\{p\}}$: this will define $c_p(\alpha)$ uniquely. By definition, we need to compute the action of $\theta(\xi) = \xi^p - \xi^{\{p\}}$ on line bundle $(\mathcal{O}_X, d + \alpha)$. This is given by

$$(\xi + \langle \xi, \alpha \rangle)^p - \xi^{\{p\}} - \langle \xi^{\{p\}}, \alpha \rangle = \langle \xi, \alpha \rangle^p - \langle \xi^{\{p\}}, C(\alpha) \rangle \in \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{O}_X)$$

where for the equality we used that $\xi^p - \xi^{\{p\}}$ acts by 0 on \mathcal{O}_X , formula (2.4) and also Jacobson's formula. Note that $\langle \xi, \alpha \rangle^p = (W_X^{\{p\}})^{-1}(\langle \xi, \alpha \rangle) = \langle \xi^{\{p\}}, \alpha^{\{p\}} \rangle$, where $\alpha^{\{p\}} := (W_X^{\{p\}})^{-1}\alpha$. This gives

$$c_p(\alpha) = \alpha^{\{p\}} - C(\alpha).$$

We now consider a non-commutative counterpart of the above p -curvature map.

Construction 2.46 (Central reductions of \mathcal{D}_X associated to 1-forms). Let $\alpha \in H^0(X^{\{p\}}, \Omega_X^{1,\{p\}})$ be a global section. Let $i_\alpha: X^{\{p\}} \hookrightarrow (T^*X)^{\{p\}}$ be the graph of α . Under the identification $(T^*X)^{\{p\}} \simeq \text{Spec}_X \text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}}$, the image $\Gamma_\alpha = \text{Im}(i_\alpha) \subset (T^*X)^{\{p\}}$ of i_α is defined by the sheaf of ideals generated by linear expressions

$$\mathcal{J}_\alpha := (\xi - \langle \xi, \alpha \rangle)_{\xi \in \mathcal{T}_X^{\{p\}}} \subset \text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}}.$$

¹²Here the connection ∇' on $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ is given by $(\nabla'(\psi))(s) := \nabla(\psi(s)) - \psi(\nabla(s))$ for $s \in \mathcal{E}$ and $\psi \in \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. Thus an endomorphism $\psi \in \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ is flat with respect to ∇' if and only if it commutes with the action of \mathcal{T}_X induced by ∇ .

We denote by $\mathcal{D}_{X,\alpha} := i_\alpha^*(F_{X^*}\mathcal{D}_X)$ the corresponding pull-back of $F_{X^*}^{[p]}\mathcal{D}_X$ considered as a sheaf of algebras over $(T^*X)^{\{p\}}$. For any α the sheaf $\mathcal{D}_{X,\alpha}$ defines an Azumaya algebra over $X^{\{p\}}$ of rank $p^{2\text{rk}\Omega_X^1}$.

Construction 2.47 (Line bundles with connection vs splittings.). Suppose we are given a line bundle with flat connection (\mathcal{L}, ∇) and put $\alpha := c_p(\mathcal{L}, \nabla)$. Then by definition of c_p we have that $\xi \in \mathcal{T}_X^{\{p\}}$ acts on $F_{X^*}^{[p]}\mathcal{L}$ by $\langle \xi, \alpha \rangle \in \mathcal{O}_X^{\{p\}}$. It follows that the action of $F_{X^*}^{[p]}\mathcal{D}_X$ on $F_{X^*}^{[p]}\mathcal{L}$ induced by ∇ factors through $\mathcal{D}_{X,\alpha}$. Moreover, by Remark 2.39 $F_{X^*}^{[p]}\mathcal{L}$ is a vector bundle of rank $p^{\text{rk}\Omega_X^1}$ on $X^{\{p\}}$ and thus is a splitting bundle for $\mathcal{D}_{X,\alpha}$: namely, ∇ induces an isomorphism

$$\nabla: \mathcal{D}_{X,\alpha} \xrightarrow{\sim} \text{End}_{\mathcal{O}_X^{\{p\}}}(F_{X^*}^{[p]}\mathcal{L}).$$

Vice versa, let \mathcal{E} be a splitting bundle $\mathcal{D}_{X,\alpha}$ -module on $X^{\{p\}}$. Then \mathcal{E} is a vector bundle of rank $p^{\text{rk}\Omega_X^1}$. However \mathcal{E} is also a quasi-coherent $F_{X^*}\mathcal{O}_X$ -module (with the action via the composition $F_{X^*}^{[p]}\mathcal{O}_X \rightarrow F_{X^*}^{[p]}\mathcal{D}_X \rightarrow \mathcal{D}_{X,\alpha}$) and so comes as a push-forward $(F_{X^*}^{[p]})\mathcal{F}$ of some quasi-coherent sheaf \mathcal{F} on X . Since $F_{X^*}^{[p]}$ is a finite locally free map of degree $p^{\text{rk}\Omega_X^1}$ the only option is that $\mathcal{F} \simeq (F_{X^*}^{[p]})^{-1}(\mathcal{E})$ is a line bundle. The action of $\mathcal{D}_X \simeq (F_{X^*}^{[p]})^{-1}(F_{X^*}^{[p]}\mathcal{D}_X)$ on \mathcal{F} induces a natural flat connection, whose p -curvature is given by α . Moreover, isomorphisms between two line bundles $(\mathcal{L}_1, \nabla_1)$ and $(\mathcal{L}_2, \nabla_2)$ with the same p -curvature α exactly correspond to isomorphisms between the corresponding $\mathcal{D}_{X,\alpha}$ -modules $F_{X^*}^{[p]}\mathcal{L}_1$ and $F_{X^*}^{[p]}\mathcal{L}_2$. In particular, any two such line bundles à étale (and, in fact even Zariski) locally isomorphic.

This gives an equivalence of two groupoids: line bundles with flat connection on X with p -curvature α , and splitting bundles¹³ for $\mathcal{D}_{X,\alpha}$ on $X^{\{p\}}$ correspondingly.

Consider the map $d \log: \mathcal{O}_X^\times \rightarrow \Omega_{X,cl}^1$ of étale sheaves on X sending f to df/f . It induces a map of étale sheaves $F_{X^*}^{[p]}\mathcal{O}_X^\times \rightarrow F_{X^*}^{[p]}\Omega_{X,cl}^1$ on $X^{\{p\}}$.

Proposition 2.48. *There is a short exact sequence of étale sheaves on $X^{\{p\}}$*

$$0 \rightarrow F_{X^*}^{[p]}\mathcal{O}_X^\times / (\mathcal{O}_X^{\{p\}})^\times \xrightarrow{d \log} F_{X^*}^{[p]}\Omega_{X,cl}^1 \xrightarrow{c_p} \Omega_X^{1,\{p\}} \rightarrow 0,$$

where c_p is the map from Construction 2.44.

Proof. The idea is to use the equivalence of groupoids described in Construction 2.47. First of all, the kernel of $d \log: F_{X^*}^{[p]}\mathcal{O}_X^\times \rightarrow F_{X^*}^{[p]}\Omega_{X,cl}^1$ is given by the intersection of $F_{X^*}^{[p]}\mathcal{O}_X^\times$ and $\ker d \subset F_{X^*}^{[p]}\mathcal{O}_X$. The latter is isomorphic to $\mathcal{O}_X^{\{p\}}$ by Proposition 2.40, so the intersection is exactly $(\mathcal{O}_X^{\{p\}})^\times$. This shows exactness from the left. To show that c_p is surjective, for any local section $\alpha \in \Omega_X^{1,\{p\}}$ it is enough to show that étale locally on X there is a line bundle with p -curvature given by (the corresponding pull-back of) α . By the above this is equivalent to finding a splitting bundle for the Azumaya algebra $\mathcal{D}_{X,\alpha}$, which is always possible étale locally. It remains to show the exactness in the middle. Let α_1, α_2 be two closed forms on an étale open $U \rightarrow X$ such that $c_p(\alpha_1) = c_p(\alpha_2)$. Then by the discussion in Construction 2.47 line bundles

¹³Here an object is given by a vector bundle \mathcal{E} on $X^{\{p\}}$ together with an isomorphism $\gamma: \mathcal{D}_{X,\alpha} \simeq \text{End}_{\mathcal{O}_X^{\{p\}}}(\mathcal{E})$, and morphisms are isomorphisms of vector bundles that preserve this data.

with connection $(\mathcal{O}_U, d + \alpha_1)$ and $(\mathcal{O}_U, d + \alpha_2)$ are locally isomorphic, which means¹⁴ that (after some cover) $\alpha_1 = \alpha_2 + d \log f$ for some function f . \square

2.9. Restricted Poisson structures. We recall and try to motivate the definition of restricted Poisson structure due to [BK].

Recall that a Poisson bracket on a commutative ring A is a Lie bracket $\{-, -\}$ on A that is also a derivation in each variable. If the corresponding derivations are B -linear for some algebra B with a homomorphism $B \rightarrow A$ then we will say that the Poisson bracket is over B (or B -linear).

Construction 2.49 (Free Poisson algebra). Let V be a B -module. Let $T(V)$ be the tensor algebra over B and $L(V)$ the free Lie algebra over B on V correspondingly. Then $T(V)$ is the universal enveloping algebra of $L(V)$. Viewed this way, $T(V)$ acquires a PBW-filtration $F_*^{PBW} T(V)$, for which $\text{gr}_*^{PBW} T(V) \simeq \text{Sym}_B^* L(V)$. One can also consider the corresponding Rees algebra¹⁵ $Q(V) := \bigoplus_i F_i^{PBW} T(V) \cdot h^i$ corresponding to $F_*^{PBW} T(V)$; this is an h -torsion free $B[h]$ -algebra such that $Q(V)[h^{-1}] \simeq T(V)[h, h^{-1}]$ and $Q(V)/h \simeq \text{gr}_*^{PBW} T(V)$. Since $[F_i^{PBW} T(V), F_j^{PBW} T(V)] \subset F_{i+j-1}^{PBW} T(V)$ one has $[Q(V), Q(V)] \subset h \cdot Q(V)$ and there is a natural Poisson bracket on $\text{gr}_*^{PBW} T(V)$ defined by

$$h \cdot \{f, g\} := [\tilde{f}, \tilde{g}] \pmod{h^2}$$

where \tilde{f} and \tilde{g} are some lifts of $f, g \in \text{gr}_*^{PBW} T(V) \simeq Q(V)/h$ to $Q(V)$.

The resulting Poisson bracket on $\text{gr}_*^{PBW} T(V) \simeq \text{Sym}_B^* L(V)$ is identified with the Kostant-Kirillov bracket, namely the unique bracket that on linear functions $z, w \in L(V) \subset \text{Sym}_B^* L(V)$ is given by $\{z, w\} := [z, w] \in L(V)$. We call $\text{Sym}_B^* L(V)$ with the above bracket the *free Poisson algebra* on V and denote it $\text{Pois}(V)$.

This is justified by the universal property: namely, any B -linear map $f: V \rightarrow A$ to a Poisson algebra A extends to a Poisson algebra homomorphism $\text{Pois}(V) \rightarrow A$. Indeed, f extends to a Lie algebra map $f': L(V) \rightarrow A$ with respect to the Poisson bracket on A , and then we can extend f' to an algebra map $\text{Sym}_B^* L(V) \rightarrow A$. Since the Poisson bracket on $\text{Sym}_B^* L(V)$ is uniquely defined by its restriction to $L(V)$ we get that this map is automatically Poisson. In particular, for any Poisson algebra A there is a natural Poisson algebra map $\text{Pois}(A) \rightarrow A$.

Remark 2.50. The construction $V \mapsto \text{Pois}(V)$ defines a monad on B -modules. Indeed, we have a natural map $V \rightarrow \text{Pois}(V)$ and a natural map $\text{Pois}(\text{Pois}(V)) \rightarrow \text{Pois}(V)$ induced by the Poisson structure on $\text{Pois}(V)$ that endow $\text{Pois}(-)$ with a structure of a monad. Then, using the morphism $\text{Pois}(A) \rightarrow A$ one can define Poisson B -algebras simply as modules over $\text{Pois}(-)$.

The idea of a restricted Poisson structure on A is that together with a Poisson bracket one should include a restricted p -power operation $-^{[p]}: A \rightarrow A$ that would turn $(A, \{-, -\})$ in a restricted Lie algebra. However, the natural question is how $-^{[p]}$ should interact with the multiplication on A . This is what we will try to explain in the rest of this section. First, motivated by Construction 2.49 we can build the free restricted Poisson algebra:

¹⁴Indeed, if (\mathcal{O}_U, ∇) and (\mathcal{O}_U, ∇') are isomorphic, the isomorphism is given by multiplication by f and then $\nabla' = \nabla + d \log f$.

¹⁵In [BK] $Q(V)$ is called the algebra of *quantized polynomials* in V , hence the notation.

Construction 2.51 (Free restricted Poisson algebra: part I). Let V be a B -module and consider the free restricted Lie algebra $L_{\text{rest}}(V)$ on V . The underlying B -module of $L_{\text{rest}}(V)$ has a natural filtration $L_{\text{rest}}(V)^{\leq *}$ with $L_{\text{rest}}(V)^{\leq -1} = 0$, $L_{\text{rest}}(V)^{\leq 0} \simeq L(V)$, and $L_{\text{rest}}(V)^{\leq i+1}$ is inductively defined as $L_{\text{rest}}(V)^{\leq i} + (L_{\text{rest}}(V)^{\leq i})^{[p]}$, where by the Jacobson's formula the latter expression is a well-defined B -module. One has $\text{gr}_*(L_{\text{rest}}(V)) \simeq \bigoplus_{i \geq 0} L(V)^{(i)}$, where $L(V)^{(i)} := L(V) \otimes_{B, F_B^i} B$ is the i 'th Frobenius twist.

We can consider the corresponding universal enveloping algebra and its Rees algebra $Q_{\text{rest}}(V) := \bigoplus_i F_i^{PBW} U(L_{\text{rest}}(V))$ with respect to the PBW-filtration. One can then define the free restricted Poisson algebra $\text{Pois}_{\text{rest}}(V)$ as $Q_{\text{rest}}(V)/h \simeq \text{Sym } L_{\text{rest}}(V)$. As in Construction 2.49 this is naturally a Poisson algebra via the Lie bracket on $L_{\text{rest}}(V)$. However, we also need to endow $\text{Sym } L_{\text{rest}}(V)$ with a natural restricted structure. For this we need a little more preparation.

Let L be a restricted Lie algebra over B (e.g. $L_{\text{rest}}(V)$), and consider its universal enveloping algebra $U(L)$. Let us denote by $U_h(L) := \bigoplus_n F_n^{PBW} U(L) \cdot h^n$ its Rees algebra with respect to the PBW-filtration. L embeds in $U_h(L)$ as $h \cdot L \subset h \cdot F_1^{PBW} U(L) \subset U_h(L)$ and generates $U_h(L)$ over $B[h]$.

Consider the map $\theta: L \rightarrow U_h(L)$ which sends $x \in L$ to $x^p - h^{p-1}x^{[p]}$. Note that it is F_B -linear.

Lemma 2.52. *θ is additive and its image lands in the center $Z(U_h(L)) \subset U_h(L)$.*

Proof. We will denote by $[-, -]_L$ the Lie bracket in L to distinguish it from the one in $U_h(L)$. The additivity follows from the Jacobson's formula (see Lemma 2.13 and part(2) of Definition 2.16). For the statement about the center, it is enough to show that $[\theta(x), y] = 0$ for $y \in L$. Note that for $x, y \in L$ we have $[x, y] = h \cdot [x, y]_L$. Thus for $y \in L$ we have $[x^p, y] = \text{ad}(x)^p(y) = h^p \text{ad}_L(x)^p(y)$. On the other hand, $[x^{[p]}, y] = h \text{ad}_L(x)^p(y)$. This shows that $[\theta(x), y] = 0$. \square

The map from 2.52 gives a map of B -algebras

$$\theta: \text{Sym } L^{(1)} \rightarrow Z(U_h(L)).$$

Let $-^{(1)}: L \rightarrow L^{(1)}$ be the natural map sending x to $x \otimes 1$. Consider also the map $p: Q_{\text{rest}}(V) \twoheadrightarrow \text{Pois}_{\text{rest}}(V) \simeq Q_{\text{rest}}(V)/h$ given by reduction modulo h . Note that given $x \in \text{Pois}_{\text{rest}}(V)$ and any lift $\tilde{x} \in Q_{\text{rest}}(V)$ the element $(\tilde{x})^p \in Q_{\text{rest}}(V)$ is well-defined modulo h^p . Indeed, by Jacobson's formula

$$(\tilde{x} + hy)^p = \tilde{x}^p + \sum_{i=1}^{p-1} L_i(\tilde{x}, hy) \pmod{h^p},$$

but each $L_i(\tilde{x}, hy)$ is Lie polynomial of total Lie degree¹⁶ $p-1$ and (see the explicit description of L_i in Lemma 2.13) $L_i(\tilde{x}, hy) = h^i L_i(\tilde{x}, y)$. Thus $(\tilde{x} + hy)^p = \tilde{x}^p \pmod{h^p}$.

Then one can reconstruct the restricted structure on L in terms of θ , namely

$$h^{p-1}x^{[p]} = \tilde{x}^p - \theta(x^{(1)}) \pmod{h^p}.$$

Under the identification $h^{p-1}Q_{\text{rest}}(V)/h^p \simeq Q_{\text{rest}}(V)/h$ this defines $x^{[p]}$ uniquely. The idea now is to define the restricted structure on the whole of $\text{Sym } L_{\text{rest}}(V)$ in a similar manner using the map θ .

¹⁶Meaning that it is expressed as a sum of terms each involving a composition of $p-1$ commutators.

Lemma 2.53. (1) For any $x \in \text{Pois}_{\text{rest}}(V)$ and any lift $\tilde{x} \in Q_{\text{rest}}(V)$ the element $\tilde{x}^p - \theta(x^{(1)})$ is divisible by h^{p-1} .

(2) The map $-^{[p]}: \text{Pois}_{\text{rest}}(V) \rightarrow \text{Pois}_{\text{rest}}(V)$ defined by

$$h^{p-1}x^{[p]} = \tilde{x}^p - \theta(x^{(1)}) \pmod{h^p}$$

endows $\text{Pois}_{\text{rest}}(V) \simeq \text{Sym } L_{\text{rest}}(V)$ with a restricted Lie algebra structure that agrees with the one on $L_{\text{rest}}(V)$.

Proof. Consider the $B[h]/h^{p-1}$ -algebra given by $Q_{\text{rest}}(V)/h^{p-1}$. By the discussion above concerning the Jacobson's formula, $x \mapsto \tilde{x}^p$ produces a well-defined map $\text{Pois}_{\text{rest}}(V) \simeq Q_{\text{rest}}(V)/h \rightarrow Q_{\text{rest}}(V)/h^{p-1}$. By [BK, Lemma 1.3] this is in fact an algebra homomorphism. Moreover, it agrees with θ on L_{rest} and since $\text{Pois}_{\text{rest}}(V) := \text{Sym } L_{\text{rest}}(V)$ it agrees with θ on the whole $\text{Pois}_{\text{rest}}(V)$. We get that $\tilde{x}^p - \theta(x^{(1)}) = 0$ modulo h^{p-1} for any $x \in \text{Pois}_{\text{rest}}(V)$.

For (2) we need to check the defining properties of restricted Lie algebra (Definition 2.16). First of all, $-^{[p]}$ is Frobenius-linear since $x \rightarrow \tilde{x}^p$ and θ are. Note that since $\{x, y\} = h[\tilde{x}, \tilde{y}] \pmod{h^2}$ one has

$$\text{ad}_{[-, -]}(\tilde{x}^p)(\tilde{y}) = \text{ad}_{[-, -]}(\tilde{x})^p(\tilde{y}) = h^p \cdot \text{ad}_{\{-, -\}}(x)^p(y) \pmod{h^{p+1}}.$$

Since $\theta(x^{(1)})$ is central applying $[-, \tilde{y}]$ to the expression for $x^{[p]}$ we get that

$$h^p \text{ad}_{\{-, -\}}(x^{[p]})(y) = h^p \text{ad}_{\{-, -\}}(x)^p(y) \pmod{h^{p+1}}.$$

Finally, the formula for $(x + y)^{[p]}$ follows from the Jacobson formula. Namely, θ is a homomorphism, so we have $\theta(x^{(1)} + y^{(1)}) = \theta(x^{(1)}) + \theta(y^{(1)})$, and

$$h^{p-1} \cdot ((x + y)^{[p]} - x^{[p]} - y^{[p]}) = (\tilde{x} + \tilde{y})^p - \tilde{x}^p - \tilde{y}^p \pmod{h^p}$$

where the right hand side is given by $h^{p-1} \sum_i L_i(x, y)$ (2.13, here since each L_i is a Lie polynomial of degree $p - 1$ we have $L_i(\tilde{x}, \tilde{y}) = h^{p-1} L_i(x, y)$). \square

Construction 2.54 (Free restricted Poisson algebra: part II). Continuing Construction 2.51, via Lemma 2.53 we can now endow $\text{Pois}_{\text{rest}}(V)$ not only with a Poisson bracket, but also a p -power operation $-^{[p]}$ such that $(\text{Pois}_{\text{rest}}(V), \{-, -\}, -^{[p]})$ becomes a restricted Lie algebra. This allows to consider $\text{Pois}_{\text{rest}}(\text{Pois}_{\text{rest}}(V))$ and a homomorphism of algebras

$$\text{Pois}_{\text{rest}}(\text{Pois}_{\text{rest}}(V)) \rightarrow \text{Pois}_{\text{rest}}(V)$$

defined (via the identification $\text{Pois}_{\text{rest}}(\text{Pois}_{\text{rest}}(V)) \simeq \text{Sym } L_{\text{rest}}(\text{Pois}_{\text{rest}}(V))$) by sending $L_{\text{rest}}(\text{Pois}_{\text{rest}}(V))$ to $\text{Pois}_{\text{rest}}(V)$ using the structure of restricted Lie algebra, and then extending by multiplicativity. We also have an embedding $V \subset L_{\text{rest}}(V) \subset \text{Pois}_{\text{rest}}(V)$ which with the map above endows $\text{Pois}_{\text{rest}}$ with a structure of a monad on the category of B -modules.

Definition 2.55 (Restricted Poisson algebra). A *restricted Poisson algebra* A over B is a module over the monad given by $V \mapsto \text{Pois}_{\text{rest}}(V)$.

Remark 2.56. We note that there are a lot of structures underlying the restricted Poisson structure. Let A be a restricted Poisson B -algebra.

- There is a natural map of monads $\text{Pois}(-) \rightarrow \text{Pois}_{\text{rest}}(-)$ induced by the embedding $\text{Sym}^*(L(V)) \rightarrow \text{Sym}^*(L_{\text{rest}}(V))$, which in turn is induced by $L(V) \rightarrow L_{\text{rest}}(V)$ for each B -module V . Thus by Remark 2.50, A has a natural Poisson algebra structure.

- There is a natural map of monads $L_{\text{rest}}(V) \rightarrow \text{Pois}_{\text{rest}}(-)$ induced by the embedding $L_{\text{rest}}(V) \rightarrow \text{Sym}^*(L_{\text{rest}}(V))$ for each B -module V . This gives a restricted Lie algebra structure on A .
- There is a natural diagram of monads

$$\begin{array}{ccc} L(-) & \longrightarrow & \text{Pois}(-) \\ \downarrow & & \downarrow \\ L_{\text{rest}}(-) & \longrightarrow & \text{Pois}_{\text{rest}}(-) \end{array}$$

given by the above maps and embeddings $L(V) \rightarrow \text{Sym}^*(L(V))$ and $L(V) \rightarrow L_{\text{rest}}(V)$ for each B -module V correspondingly, which shows that the Lie algebra structure on A given by the above Poisson structure and restricted Lie structure on A agree.

Finally, let us describe more explicitly how $(-)^{[p]}$ in $\text{Pois}_{\text{rest}}(V)$ interacts with multiplication.

Construction 2.57 (Polynomial $P(x, y)$ and formula for $(xy)^{[p]}$). Consider $V \simeq Bx \oplus By$: a free B -module of rank 2 with basis given by x and y . Consider the (non-restricted) algebras $Q(V)$ and $\text{Pois}(V)$. Then by [BK, Equation 1.3] (which in turn follows from [BK, Lemma 1.3]) the expression $(xy)^p - x^p y^p$ is divisible by h^{p-1} . One can then define a Poisson polynomial $P(x, y) \in \text{Pois}(x, y) := \text{Pois}(V)$ by

$$h^{p-1}P(x, y) = (xy)^p - x^p y^p \pmod{h^p Q(V)}.$$

Then, for the restricted structures in $Q_{\text{rest}}(V)$ we have an equality

$$\begin{aligned} h^{p-1}(xy)^{[p]} &:= (xy)^p - \theta(xy) = x^p y^p + h^{p-1}P(x, y) - (x^p - h^{p-1}x^{[p]})(y^p - h^{p-1}y^{[p]}) = \\ &= h^{p-1}(x^p y^{[p]} + x^{[p]} y + P(x, y)) \pmod{h^p}, \end{aligned}$$

which shows that $(xy)^{[p]} = x^p y^{[p]} + x^{[p]} y + P(x, y) \in \text{Pois}_{\text{rest}}(V)$. Note that for any V this formula uniquely defines the restricted structure on $\text{Pois}_{\text{rest}}(V) \simeq \text{Sym}^*(L_{\text{rest}}(V))$ by induction on the degree.

This way for any restricted Poisson algebra A and any two elements $x, y \in A$ we have

$$(2.8) \quad (xy)^{[p]} = x^p y^{[p]} + x^{[p]} y + P(x, y),$$

where the value of $P(x, y) \in A$ is computed via the map $\text{Pois}(A) \rightarrow A$. In fact, since the above formula defines the restricted structure on a free restricted Poisson algebra uniquely, one can give an alternative definition of a restricted Poisson algebra as a Poisson B -algebra A with a restricted p -power operation $-^{[p]}$ such that $(A, \{-, -\}, -^{[p]})$ defines a restricted Lie algebra and that the relation (2.8) is satisfied for any $x, y \in A$. This is how it is defined in [BK] (namely, see [BK, Definition 1.8]).

Equation 2.8 is hard to check in practice in the original form essentially because the definition of $P(x, y)$ is complicated (and not very explicit). But, at least when $p > 2$ there is an equivalent condition that is easier to check:

Remark 2.58. Let $p > 2$. Let $x \in \text{Pois}(V)$ be some element. Then, for any lift $\tilde{x} \in Q(V)$ we have $(\tilde{x}^2)^p = \tilde{x}^p \cdot \tilde{x}^p$ and so $P(x, x) = 0$. From this and (2.8) we get that

$$(2.9) \quad (x^2)^{[p]} = 2x^{[p]}x^p.$$

Note that $xy = \frac{1}{4}((x+y)^2 - (x-y)^2)$. Thus, applying the restricted power, we get

$$(xy)^{[p]} = \frac{1}{4}((x+y)^2 - (x-y)^2)^{[p]}$$

which, using Jacobson's formula, gives an expression for $P(x, y)$ in terms of $-^{[p]}$ applied to squares and some Poisson polynomial in $x, y, x^{[p]}$ and $y^{[p]}$. This shows that checking Equation (2.8) reduces to checking that $-^{[p]}$ together with the Poisson bracket give a restricted structure in the Lie algebra structure and that Equation (2.8) holds for all x .

In particular, $(A, \{-, -\}, -^{[p]})$ defines a restricted Poisson algebra 2.8 if and only if

- $(A, \{-, -\})$ is a Poisson algebra;
- $(A, \{-, -\}, -^{[p]})$ defines a restricted Lie algebra;
- $(x^2)^{[p]} = 2x^{[p]}x^p$ for any $x \in A$.

This simplification of [BK, Definiton 1.8] was also previously observed in [BYZ].

2.10. Restricted symplectic geometry.

Definition 2.59. A *symplectic scheme* (X, ω) over S is a pseudo-smooth S -scheme X with a non-degenerate globally defined closed 2-form $\omega \in H^0(X, \Omega_X^2)$.

Since ω is non-degenerate it induces an identification $\mathcal{T}_X \xrightarrow{\sim} \Omega_X^1$ via $\xi \mapsto \iota_\xi \omega$, where ι_ξ is the contraction with ξ . This way ω also defines a section ω^{-1} of $\wedge_{\mathcal{O}_X}^2 \mathcal{T}_X$, which then gives a Poisson bracket on \mathcal{O}_X : $\{f, g\} := \langle \omega^{-1}, df \wedge dg \rangle$. The property that $d\omega = 0$ translates to the Jacobi identity for $\{-, -\}$: in particular $(\mathcal{O}_X, \{-, -\})$ defines a sheaf of Lie algebras over \mathcal{O}_S .

Having a Poisson bracket $\{-, -\}$, for any $f \in \mathcal{O}_X$ the bracket $\{f, -\}$ defines a derivation $H_f: \mathcal{O}_X \rightarrow \mathcal{O}_X$. This gives a map of sheaves $H_-: \mathcal{O}_X \rightarrow \mathcal{T}_X$ locally sending $f \mapsto H_f$.

The following definition is classical:

Definition 2.60. A vector field $\xi \in H^0(X, \mathcal{T}_X)$ is called *Hamiltonian* if $\xi = H_f$ for some $f \in H^0(X, \mathcal{O}_X)$.

Remark 2.61. Tracing through the definitions one can see that the equation $\xi = H_f$ is in fact equivalent to $\iota_\xi \omega = df$. Since ω is non-degenerate it follows that Hamiltonian vector fields generate \mathcal{T}_X over \mathcal{O}_X Zariski locally on X .

One of the key ideas in [BK] is that in char p there is a natural enhancement of the usual Poisson structure given by *restricted* Poisson structure, where the latter amounts to a sheafified version of Definition 2.55, as reformulated in Remark 2.57.

Definition 2.62. A *restricted Poisson structure* on a pseudo-smooth S -scheme X is given by

- (1) an \mathcal{O}_S -linear Poisson bracket $\{-, -\}$ on \mathcal{O}_X ;
- (2) a restricted p -th power operation $-^{[p]}: \mathcal{O}_X \rightarrow \mathcal{O}_X$ that turns $(\mathcal{O}_X, \{-, -\})$ into a restricted Lie algebra;
- (3) for any $f, g \in \mathcal{O}_X$ one has

$$(2.10) \quad (fg)^{[p]} = f^p g^{[p]} + f^{[p]} g^p + P(f, g),$$

where $P(x, y)$ is the Poisson polynomial from Remark 2.57.

By Remark 2.58, if $p > 2$, (2.10) is equivalent to the simpler relation

$$(f^2)^{[p]} = 2f^{[p]}f^p$$

for any $f \in \mathcal{O}_X$.

It is natural to ask when a Poisson structure extends to a restricted one, and what such extensions are given by. Note that for the restricted Poisson structure for any $f \in \mathcal{O}_X$ we have

$$H_{f^{[p]}} = H_f^{[p]} \in \mathcal{T}_X,$$

since vector fields act as H_f^p on \mathcal{O}_X . Thus Hamiltonian vector fields should be closed under the the p -th restricted power operation $-^{[p]}$. This imposes some restrictions on the underlying Poisson bracket. For example, in [BK, Theorem 1.11] Bezrukavnikov and Kaledin show that, given a *symplectic* S -scheme (X, ω) , Hamiltonian vector fields are closed under $-^{[p]}$ if and only if¹⁷ $C(\omega) = 0$.

More generally, in [BK, Theorem 1.12], for symplectic pseudo-smooth (X, ω) , they were able to fully describe all restricted Poisson structures compatible with ω . Below we include a different proof of their result. First, note that there is a well defined map of sheaves $\Omega_X^1/d(\mathcal{O}_X) \rightarrow \Omega_X^2$ induced by d .

Proposition 2.63. *Let $p > 2$. Let (X, ω) be a pseudo-smooth symplectic S -scheme. Then there is a natural bijection*

$$(2.11) \quad \left\{ \begin{array}{l} \text{global sections} \\ [\eta] \text{ of } \Omega_X^1/d(\mathcal{O}_X) \\ \text{s.t. } d[\eta] = \omega \end{array} \right\} \xleftrightarrow{1 \text{ to } 1} \left\{ \begin{array}{l} \text{restricted Poisson} \\ \text{structures on } X \\ \text{compatible with } \omega \end{array} \right\}.$$

Locally on X , given a lift of $[\eta] \in H^0(X, \Omega_X^1/d(\mathcal{O}_X))$ to a globally defined 1-form $\eta \in H^0(X, \Omega_X^1)$, the corresponding restricted p -th power operation $-^{[p]\eta} : \mathcal{O}_X \rightarrow \mathcal{O}_X$ is given by the formula

$$(2.12) \quad f^{[p]\eta} := H_f^{p-1}(\iota_{H_f}\eta) - \iota_{H_f^{[p]}\eta}.$$

Proof. First, we construct a map from the left to the right in somewhat inexplicit terms, which in the end will out to agree with (2.12). The idea is to realize \mathcal{O}_X as a Lie algebra of a certain group scheme over $X^{\{p\}}$ associated to η .

More precisely, for any $\eta \in H^0(X, \Omega_X^1)$ define a group sheaf \tilde{G}_η over $X^{\{p\}}$ as follows: for $T \rightarrow X^{\{p\}}$ put

$$\tilde{G}_\eta(T) := \{(\phi, f), \phi \in \text{Aut}_T(X \times_{X^{\{p\}}} T), f \in \mathcal{O}(X \times_{X^{\{p\}}} T), \text{ s.t. } \phi^*\eta - \eta = df\}.$$

Here the multiplication is given by $(\phi_1, f_1) \cdot (\phi_2, f_2) = (\phi_1\phi_2, \phi_2^*(f_1) + f_2)$. Since $X \rightarrow X^{\{p\}}$ is finite locally free and d is $\mathcal{O}_X^{\{p\}}$ -linear it is clear that \tilde{G}_η is represented by an affine group scheme over $X^{\{p\}}$.

The corresponding Lie algebra is given by the vector bundle

$$\text{Lie}(\tilde{G}_\eta) = \{(\theta, f), \theta \in \mathcal{T}_X, f \in \mathcal{O}_X, \text{ s.t. } L_\theta\eta = df\}$$

where L_θ denotes the Lie derivative. The Lie bracket on $\text{Lie}(\tilde{G}_\eta)$ is given by

$$[(\theta_1, f_1), (\theta_2, f_2)] = ([\theta_1, \theta_2], \theta_1(f_2) - \theta_2(f_1)).$$

¹⁷Here $C : F_{X^*}^{\{p\}}\Omega_{X,cl}^2 \rightarrow \Omega_X^{2,\{p\}}$ is the Cartier operator and the condition $C(\omega) = 0$ is equivalent to ω being exact locally in Zariski topology.

One has a natural embedding of Lie algebras

$$\mathrm{Lie}(\tilde{G}_\eta) \subset \mathcal{D}_{X, \leq 1}, \quad (\theta, f) \mapsto \theta + f,$$

and below we will occasionally consider elements of $\mathrm{Lie}(\tilde{G}_\eta)$ as differential operators of order ≤ 1 .

Let now η be such that $d\eta = \omega$. We define a map

$$\alpha: \mathcal{O}_X \rightarrow \mathrm{Lie}(\tilde{G}_\eta) \subset \mathcal{D}_{X, \leq 1}$$

by $\alpha(f) := H_f + f + \iota_{H_f}\eta$. Since

$$L_{H_f}\eta = (\iota_{H_f}d + d\iota_{H_f})\eta = \iota_{H_f}\omega + d(\iota_{H_f}\eta) = d(f + \iota_{H_f}\eta),$$

$\alpha(f)$ indeed lands in $\mathrm{Lie}(\tilde{G}_\eta)$.

Lemma 2.64. *The map $\alpha: (\mathcal{O}_X, \{-, -\}) \rightarrow \mathrm{Lie}(\tilde{G}_\eta)$ is an isomorphism of Lie algebras.*

Proof. We have

$$\begin{aligned} [\alpha(f), \alpha(g)] &= [H_f + f + \iota_{H_f}\eta, H_g + g + \iota_{H_g}\eta] = [H_f, H_g] + L_{H_f}(g + \iota_{H_g}\eta) - \\ &\quad - L_{H_g}(f + \iota_{H_f}\eta) = H_{\{f, g\}} + 2\{f, g\} + L_{H_f}\iota_{H_g}\eta - L_{H_g}\iota_{H_f}\eta. \end{aligned}$$

Since $L_\theta\iota_\xi = \iota_\xi L_\theta + \iota_{[\theta, \xi]}$ this further simplifies to

$$\begin{aligned} [\alpha(f), \alpha(g)] &= H_{\{f, g\}} + 2\{f, g\} + \iota_{H_g}L_{H_f}\eta + \iota_{[H_f, H_g]}\eta - L_{H_g}\iota_{H_f}\eta = \\ &= H_{\{f, g\}} + 2\{f, g\} + \iota_{H_g}\iota_{H_f}d\eta + \iota_{H_g}d\iota_{H_f}\eta + \iota_{[H_f, H_g]}\eta - \iota_{H_g}d\iota_{H_f}\eta = \\ &= H_{\{f, g\}} + 2\{f, g\} + \{g, f\} + \iota_{[H_f, H_g]}\eta = H_{\{f, g\}} + \{f, g\} + \iota_{[H_f, H_g]}\eta, \end{aligned}$$

which is exactly $\alpha(\{f, g\})$. Thus α is a map of Lie algebras.

It remains to see that α is an isomorphism. Let $\mathcal{T}_X^{\mathrm{Ham}} \subset \mathcal{T}_X$ denote the $(\mathcal{O}_X^{\{p\}})$ -linear Lie subalgebra of Hamiltonian vector fields. Note that if $(\theta, f) \in \mathrm{Lie}(\tilde{G}_\eta)$, then $\theta \in \mathcal{T}_X^{\mathrm{Ham}}$. Indeed,

$$\iota_\theta\omega = (L_\theta - d \circ \iota_\theta)(\eta) = d(f - \iota_\theta\eta),$$

so $\theta = H_{f - \iota_\theta\eta}$ by Remark 2.61. The map $\mathrm{Lie}(\tilde{G}_\eta) \rightarrow \mathcal{T}_X^{\mathrm{Ham}}$ is surjective with H_f being hit by $\alpha(f)$, and its kernel is given by pairs $(0, f)$ with $f \in \mathcal{O}_X$ with $df = 0$, which is $\mathcal{O}_X^{\{p\}} \subset \mathcal{O}_X$. We get a short exact sequence of Lie algebras

$$0 \rightarrow \mathcal{O}_X^{\{p\}} \rightarrow \mathrm{Lie}(\tilde{G}_\eta) \rightarrow \mathcal{T}_X^{\mathrm{Ham}} \rightarrow 0$$

(with the zero bracket on $\mathcal{O}_X^{\{p\}}$). There is a similar exact sequence for $(\mathcal{O}_X, \{-, -\})$, with the map $\mathcal{O}_X \rightarrow \mathcal{T}_X^{\mathrm{Ham}}$ given by $f \mapsto H_f$, and whose kernel is the Poisson center $\mathcal{O}_X^{\{p\}}$. The map α gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X^{\{p\}} & \xrightarrow{f \mapsto f} & \mathcal{O}_X & \xrightarrow{f \mapsto H_f} & \mathcal{T}_X^{\mathrm{Ham}} \longrightarrow 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X^{\{p\}} & \xrightarrow{f \mapsto (0, f)} & \mathrm{Lie}(\tilde{G}_\eta) & \xrightarrow{(\theta, f) \mapsto \theta} & \mathcal{T}_X^{\mathrm{Ham}} \longrightarrow 0 \end{array}$$

with left and right vertical maps being identity maps. Thus α is an isomorphism. \square

Recall that the Lie algebra of an affine group scheme carries a canonical structure of a restricted Lie algebra (see [Demazure-Gabriel](#)). It is defined as follows. Let $R := \mathcal{O}_X^{\{p\}}$ and put $R[\varepsilon]$ to be the algebra of dual numbers over R (so $\varepsilon^2 = 0$). The Lie algebra $\text{Lie}(\tilde{G}_\eta)$ (as a sheaf of R -modules) is given by definition by the kernel of the map $\tilde{G}_\eta(R[\varepsilon]) \rightarrow \tilde{G}_\eta(R)$ induced by reduction $R[\varepsilon] \rightarrow R$. Now, one can consider an auxiliary ring $R[\varepsilon_1, \dots, \varepsilon_p]$ (with all $\varepsilon_i^2 = 0$), and an R -subalgebra $S \subset R[\varepsilon_1, \dots, \varepsilon_p]$ generated by the sum $\sigma = \varepsilon_1 + \dots + \varepsilon_p$ and the product $\pi = \varepsilon_1 \dots \varepsilon_p$; it is easy to see that S is isomorphic to the subring of Σ_p -invariants in $R[\varepsilon_1, \dots, \varepsilon_p]$, and that $S \simeq R[\sigma, \pi]/(\sigma^p, \pi^2, \sigma\pi)$. Note that we have a natural projection $S \rightarrow R[\pi]/\pi^2$. For $x \in \text{Lie}(\tilde{G}_\eta)$ denote by $e^{\varepsilon x}$ the corresponding element of $\tilde{G}_\eta(R[\varepsilon])$. The elements $e^{\varepsilon_i x}$ in $\tilde{G}_\eta(R[\varepsilon_1, \dots, \varepsilon_p])$ commute with each other¹⁸, and so $e^{\varepsilon_1 x} \dots e^{\varepsilon_p x}$ gives an Σ_p -invariant element in $\tilde{G}_\eta(R[\varepsilon_1, \dots, \varepsilon_p])$. Since \tilde{G}_η is representable we have $\tilde{G}_\eta(R[\varepsilon_1, \dots, \varepsilon_p])^{\Sigma_p} \simeq \tilde{G}_\eta(S)$, thus $e^{\varepsilon_1 x} \dots e^{\varepsilon_p x}$ gives an element in $\tilde{G}_\eta(S)$ and we can take its image in $\tilde{G}_\eta(R[\pi]/\pi^2)$, which is equal to $e^{\pi y}$ for some $y \in \text{Lie}(\tilde{G}_\eta)$. Then one can show that the association

$$x \mapsto y =: x^{[p]}$$

defines a structure of restricted Lie algebra (see [par 7, 3.4. Demazure-Gabriel](#)) on $\text{Lie}(\tilde{G}_\eta)$. Let $(\theta, f) \in \text{Lie}(\tilde{G}_\eta)$. Then to compute $(\theta, f)^{[p]}$ via the above recipe we need to look at the image of

$$(1 + \varepsilon_1 \theta, \varepsilon_1 f) \cdot \dots \cdot (1 + \varepsilon_p \theta, \varepsilon_p f) \in \tilde{G}_\eta(R[\varepsilon_1, \dots, \varepsilon_p])$$

in $\tilde{G}_\eta(R[\pi]/\pi^2)$. It is not hard to see that it is given by a pair $(1 + \pi \theta^{[p]}, \pi \theta^{p-1}(f))$. In other words, we get that

$$(\theta, f)^{[p]} = (\theta^{[p]}, \theta^{p-1}(f)).$$

Via the identification $\alpha: \mathcal{O}_X \xrightarrow{\sim} \text{Lie}(\tilde{G}_\eta)$ we can now endow \mathcal{O}_X with the structure of restricted Lie algebra; more precisely, it is given by $f \mapsto \alpha^{-1}(\alpha(f)^{[p]})$. This agrees with [\(2.12\)](#): indeed, we have

$$\alpha(f)^{[p]} = (H_f, f + \iota_{H_f} \eta)^{[p]} = (H_f^{[p]}, H_f^{p-1}(f + \iota_{H_f} \eta)) = (H_f^{[p]}, H_f^{p-1}(\iota_{H_f} \eta))$$

and this way

$$f^{[p]\eta} := \alpha^{-1}(\alpha(f)^{[p]}) = H_f^{p-1}(\iota_{H_f} \eta) - \iota_{H_f^{[p]}} \eta.$$

Moreover, $f \mapsto f^{[p]\eta}$ defines a restricted Poisson structure:

$$\begin{aligned} (f^2)^{[p]\eta} &= H_{f^2}^{p-1}(\iota_{H_{f^2}} \eta) - \iota_{H_{f^2}^{[p]}} \eta = (2f)^{p-1} H_f^{p-1}(2f \cdot \iota_{H_f} \eta) - (2f)^p \iota_{H_f^{[p]}} \eta = \\ &= (2f)^p (H_f^{p-1}(\iota_{H_f} \eta) - \iota_{H_f^{[p]}} \eta) = 2f^p f^{[p]\eta}, \end{aligned}$$

where above we used that $H_{f^2} = 2f H_f$.

Note that the association $\eta \mapsto (-)^{[p]\eta}$ in fact only depends on the class $[\eta]$ in $\Omega_X^1/d(\mathcal{O}_X)$: indeed, if $\eta' = \eta + dg$, then

$$f^{[p]\eta'} - f^{[p]\eta} = H_f^{p-1}(\iota_{H_f} df) - \iota_{H_f^{[p]}} df = (\text{ad } f)^p(g) - (\text{ad } f)^p(g) = 0.$$

This way, having a section $[\eta] \in H^0(X, \Omega_X^1/d(\mathcal{O}_X))$ we can still associate a well-defined restricted Poisson structure $(-)^{[p][\eta]}$ by lifting $[\eta]$ to an actual 1-form η locally and gluing the resulting operations $(-)^{[p]\eta}$ globally.

¹⁸One has $e^{\varepsilon_i x} e^{\varepsilon_j x} e^{-\varepsilon_i x} e^{-\varepsilon_j x} = e^{\varepsilon_i \varepsilon_j [x, x]} = 1$.

It remains to show that the association $[\eta] \mapsto (-)^{[p][\eta]}$ is bijective. Note that both sides of (2.11), considered as Zariski sheaves on $X^{\{p\}}$, are torsors over $\mathcal{T}_X^{\{p\}}$. Indeed, the sheaf of sections $[\eta] \in \Omega_X^1/d(\mathcal{O}_X)$ with $d[\eta] = \omega$ is naturally a torsor over $\mathcal{H}_{X, \text{dR}}^1 \simeq \Omega_X^{1, \{p\}}$ and we can identify the latter with $\mathcal{T}_X^{\{p\}}$ by pulling back to $X^{\{p\}}$ the isomorphism $\mathcal{T}_X \xrightarrow{\sim} \Omega_X^1$ induced by ω (recall that it sends $\xi \mapsto \iota_\xi \omega$). For the restricted Poisson structures note that by (2.10) the difference of two p -th restricted power operations $(-)^{[p]_1}$ and $(-)^{[p]_2}$ defines a Frobenius-derivation, which moreover should take values in the Poisson center, since both $\{f^{[p]_1}, g\}$ and $\{f^{[p]_2}, g\}$ are equal to $H_f^{[p]}(g)$ (by the definition of restricted Lie algebra). It is also immediate to check that if one adds to a given $(-)^{[p]}$ such a derivation it still defines a restricted Lie algebra structure, and also a restricted Poisson structure. Moreover, such derivations are exactly identified with the sections of $\mathcal{T}_X^{\{p\}}$. Thus it remains to see that the association $[\eta] \mapsto (-)^{[p][\eta]}$ respects the torsor structure. Let α be a local section of Ω_X^1 with $d\alpha = 0$, then

$$f^{[p]\eta+\alpha} - f^{[p]\eta} = H_f^{p-1}(\iota_{H_f}\alpha) - \iota_{H_f^{[p]}}\alpha,$$

which by Remark 2.41 is the same as $-\iota_{H_f^{\{p\}}}\mathcal{C}(\alpha)$. This shows that if we act on $[\eta]$ by a vector field $\xi^{\{p\}} \in \mathcal{T}_X^{\{p\}}$ the corresponding p -th power operation will change by

$$-\iota_{H_f^{\{p\}}}\iota_{\xi^{\{p\}}}\omega^{\{p\}} = (\iota_\xi \iota_{H_f}\omega)^p = \xi(f)^p,$$

which is exactly what we need. \square

Given Proposition 2.63 the following definition now is logical.

Definition 2.65. A *restricted symplectic S -scheme* $(X, [\eta])$ is a pair of a pseudo-smooth S -scheme X and a section $[\eta] \in H^0(X, \Omega_X^1/d(\mathcal{O}_X))$ such that $\omega := d\eta \in H^0(X, \Omega_X^2)$ is non-degenerate.

Indeed, by Proposition 2.63 this data is equivalent to a symplectic structure (given by ω) together with a compatible restricted Poisson structure.

The following remark will be useful.

Remark 2.66 (No difference in doing differential geometry on X as a scheme over $X^{\{p\}}$ or S). Let X be an S -scheme and let $X \rightarrow X^{\{p\}}$ be the reduced Frobenius. Then we claim that providing X with the structure of a restricted symplectic scheme over S is the same as providing it with the same structure over the reduced twist $X^{\{p\}}$. Indeed, we have a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X^{\{p\}} \\ & \searrow & \downarrow \\ & & S \end{array}$$

which gives an exact sequence

$$F_X^{\{p\}*}\Omega_{X^{\{p\}}}^1 \xrightarrow{F_X^{\{p\}*}} \Omega_X^1 \rightarrow \Omega_{X/X^{\{p\}}}^1 \rightarrow 0.$$

Since the pull-back map $F_X^{\{p\}*}\Omega_{X^{\{p\}}}^1 \rightarrow \Omega_X^1$ is 0 we get that $\Omega_{X/X^{\{p\}}}^1 \simeq \Omega_X^1$, and so X is pseudo-smooth over $X^{\{p\}}$. We also have isomorphisms $\Omega_{X/X^{\{p\}}}^i \simeq \Omega_X^i$ and, more generally of the de Rham complex $\text{dR}_{X/X^{\{p\}}} \simeq \text{dR}_X$. Lastly, essentially by

the definition of reduced twist, $X^{\{p\}}$ for X considered as an $X^{\{p\}}$ -scheme or as an S -scheme coincide (because $\mathcal{O}_X^{\{p\}}$ is defined as the image of relative Frobenius). Thus symplectic and, more generally, restricted symplectic structures on X considered as an $X^{\{p\}}$ -scheme or an S -scheme are the same thing.

2.11. The group scheme G_0 and Darboux lemma. Another insight of Bezrukavnikov and Kaledin is that restricted Poisson structures on X can be interpreted in terms of “formal geometry” and related torsors. To recall it let us introduce one more piece of notation.

Construction 2.67 (Algebra A_0). Fix a number $d \in \mathbb{N}$ and let $S = \text{Spec } \mathbb{F}_p$. Let V be a vector space scheme of dimension d over \mathbb{F}_p (so $V \simeq \text{Spec } \mathbb{F}_p[x_1, \dots, x_d]$) and let $W = V \oplus V^\vee$ be the total space of the cotangent bundle to V : thus W is a symplectic vector space of dimension $2d$. Let y_i be the dual coordinates to x_i ; one has $W \simeq \text{Spec } \mathbb{F}_p[x_1, \dots, x_d, y_1, \dots, y_d]$.

Consider the ring of functions on the Frobenius neighborhood $F_W^{-1}(\{0\})$ of 0 in W (see Example 2.11). Explicitly, put¹⁹

$$A_0 := \mathbb{F}_p[x_i, y_j]_{1 \leq i, j \leq d} / x_i^p = y_j^p = 0.$$

We have $\Omega_{A_0}^1 \simeq (\oplus_{i=1}^d A_0 \cdot dx_i) \oplus (\oplus_{i=1}^d A_0 \cdot dy_i)$ (namely, since $dx_i^p = dy_j^p = 0$ the map $\Omega_W^1 \otimes_{\mathbb{F}_p[x_i, y_j]} A_0 \rightarrow \Omega_{A_0}^1$ is an isomorphism). In particular, A_0 is pseudo-smooth over \mathbb{F}_p . We endow $\text{Spec } A_0$ with a natural restricted symplectic structure given by the 1-form

$$\eta_{\text{can}} = \sum_{i=1}^d y_i dx_i \in \Omega_{A_0}^1.$$

Indeed, the 2-form $\omega := d\eta_{\text{can}} = \sum_i dy_i \wedge dx_i \in \Omega_{A_0}^2$ is easily seen to be non-degenerate.

Similarly, for any \mathbb{F}_p -algebra R we can consider an R -algebra

$$A_0(R) := A_0 \otimes_{\mathbb{F}_p} R.$$

We have $\Omega_{A_0(R)/R}^1 \simeq \Omega_{A_0}^1 \otimes_{\mathbb{F}_p} R$ and via this identification the pair

$$(\text{Spec } A_0(R), [\eta_{\text{can}} \otimes 1])$$

defines a restricted symplectic R -scheme (see Definition 2.65). Further we will denote $\eta_{\text{can}} \otimes 1$ by η_{can} for convenience.

Construction 2.68 (Group scheme G_0). Consider the functor

$$A_0: R \mapsto A_0(R)$$

on \mathbb{F}_p -algebras. Let $\underline{\text{Aut}}(A_0)$ be the functor

$$\underline{\text{Aut}}(A_0): R \mapsto \text{Aut}_R(A_0(R)) \simeq \text{Iso}_R(\text{Spec } A_0(R), \text{Spec } A_0(R))$$

that sends an \mathbb{F}_p -algebra R to the group of R -linear *algebra* automorphisms. Since $A_0(R)$ is a free module over R one sees that $\underline{\text{Aut}}(A_0)$ is represented by the affine group subscheme of $\text{GL}(A_0)$ given by those linear automorphisms that preserve multiplication on A_0 . Finally, let $G_0 \subset \underline{\text{Aut}}(A_0)$ be the functor

$$R \mapsto \text{Aut}_R^{\text{rest}}((\text{Spec } A_0(R), [\eta_{\text{can}}]))$$

that sends R to the group of R -algebra automorphisms of $A_0(R)$ that also preserve the restricted symplectic structure given by η_{can} . It can be seen to be represented by a

¹⁹This algebra is denoted by A in [BK, Section 3].

group subscheme of $\underline{\text{Aut}}(A_0)$: namely, one adds equations to $\phi \in \underline{\text{Aut}}(A_0)$ saying that $\mathbb{C}(\phi^* \eta_{\text{can}} - \eta_{\text{can}}) = 0$ (which is equivalent to $[\eta_{\text{can}}] \in \Omega_{A_0(R)}^1/d(A_0(R))$ being preserved under ϕ).

Remark 2.69. The group scheme $H := \underline{\text{Aut}}(A_0)$ is not particularly nice, for example it is not reduced. Indeed, let $I_0 = (x_1, \dots, x_d, y_1, \dots, y_d) \subset A_0$ be the maximal ideal. Note that for a reduced R any $\phi \in \text{Aut}_R(A_0(R))$ preserves $I_0(R) := I_0 \otimes R$: indeed $I_0(R)$ in this case coincides with nilradical of $A_0(R)$. If H were reduced this would also be true for the $\mathcal{O}(H)$ -point of H given by $H \xrightarrow{\text{id}} H$ and, since any R -point of H factors through this one, we would get that any $\phi \in \text{Aut}_R(A_0(R))$ for any R preserves $I_0(R)$. This, however, is not true: whenever $\alpha_p(R) := \{x \in R \mid x^p = 0\}$ is non-zero there is a natural automorphism of $A_0(R)$ sending x_i to $x_i + \varepsilon$ for some $\varepsilon \in \alpha_p(R)$ and this automorphism doesn't preserve $I_0(R)$. Similarly, G_0 is also not reduced: in fact η_{can} is also preserved by the above automorphism.

Construction 2.70 (Torsor of ‘‘Frobenius-frames’’). Let X be a pseudo-smooth scheme over a base scheme S such that $\text{rk } \Omega_{X/S}^1 = 2d$. Let A_0 be the algebra from Construction 2.67 for that particular d . Consider the reduced Frobenius map $F_X^{\{p\}}: X \rightarrow X^{\{p\}}$ (Construction 2.10).

There is a canonical $\underline{\text{Aut}}(A_0)$ -torsor \mathcal{M}_X on $X^{\{p\}}$ which is defined as follows. Namely, for a map $T \rightarrow X^{\{p\}}$ one puts

$$\mathcal{M}_X(T) := \text{Iso}_T(\text{Spec } A_0 \times T, X \times_{X^{\{p\}}} T).$$

This is a sheaf in flat topology on the category of schemes over $X^{\{p\}}$ endowed with a natural action of $\underline{\text{Aut}}(A_0)$. Moreover, flat locally on $X^{\{p\}}$ one has an isomorphism

$$\mathcal{M}_X|_U \simeq \underline{\text{Aut}}(A_0) \times U.$$

Indeed, let $U \rightarrow X$ be a Zariski cover that trivializes $\Omega_{X/S}^1$. Then U is still pseudo-smooth, so $F_U^{\{p\}}: U \rightarrow U^{\{p\}}$ is faithfully flat (as well as the composite map $U \rightarrow U^{\{p\}} \rightarrow X^{\{p\}}$) and (by Lemma 2.71 below) we have

$$X \times_{X^{\{p\}}} U \simeq U \times_{U^{\{p\}}} U \simeq \text{Spec } A_0 \times U.$$

This way the restriction $\mathcal{M}_X|_U$ is identified with the functor

$$(T \rightarrow U) \mapsto \text{Iso}_T(\text{Spec } A_0 \times T, \text{Spec } A_0 \times T)$$

which is exactly $X \times \underline{\text{Aut}}(A_0)$. Thus the action of $\underline{\text{Aut}}(A_0)$ on \mathcal{M}_X is effective and the latter indeed defines an $\underline{\text{Aut}}(A_0)$ -torsor over $X^{\{p\}}$. In [BK] it is called the *torsor of Frobenius frames on X* .

Lemma 2.71. *Let X be a pseudo-smooth scheme over a base scheme S . Then Zariski locally on X one has a natural isomorphism*

$$X \times_{X^{\{p\}}} X \simeq X \times \text{Spec } A_0.$$

Proof. When X is smooth over S , $F_X^{\{p\}}: X \rightarrow X^{\{p\}}$ is given by the relative Frobenius $F_{X/S}: X \rightarrow X^{(1)}$ and the statement is classical: Zariski locally one has an étale map $X \rightarrow \mathbb{A}_S^{2d}$ and in the case of affine space the isomorphism is easy to construct explicitly.

For a general pseudo-smooth scheme we use Proposition 2.35: namely, X Zariski locally is identified with the Frobenius neighborhood $(F_Y^{\{p\}})^{-1}(Z)$ in a smooth S -scheme Y of a closed subscheme $Z \hookrightarrow Y^{\{p\}}$. Following Remark 2.33 the reduced

Frobenius map $F_X^{\{p\}} : X \rightarrow X^{\{p\}} \simeq Z$ is the pull-back of $F_Y^{\{p\}} : Y \rightarrow Y^{\{p\}}$. This way we have

$$X \times_{X^{\{p\}}} X \simeq (Y \times_{Y^{\{p\}}} Y) \times_Y Z$$

and we can reduce to the smooth case. \square

Construction 2.72 (Restricted symplectic structures as reductions of \mathcal{M}_X to G_0). Another observation of [BK] is that restricted symplectic structures on X can be identified with reductions of the $\underline{\text{Aut}}(A_0)$ -torsor \mathcal{M}_X to the subgroup $G_0 \subset \underline{\text{Aut}}(A_0)$.

Let us recall this identification. By Remark 2.66, given a restricted symplectic S -scheme $(X, [\eta])$ one can equivalently consider it as a restricted symplectic scheme over $X^{\{p\}}$. One can define a functor $\mathcal{M}_{X, [\eta]}$ on schemes over $X^{\{p\}}$ as follows. First, note that for any T one gets a restricted symplectic T -scheme $(\text{Spec } A_0, [\eta_{\text{can}}]) \times T$. Second, given a map $T \rightarrow X^{\{p\}}$ the fiber product $(X, [\eta]) \times_{X^{\{p\}}} T$ is also a restricted symplectic T -scheme. Then one defines a functor

$$(T \rightarrow X^{\{p\}}) \mapsto \mathcal{M}_{X, [\eta]}(T) := \text{Iso}_T^{\text{rest}}((\text{Spec } A_0, \eta_{\text{can}}) \times T, (X, [\eta]) \times_{X^{\{p\}}} T)$$

where $\text{Iso}_T^{\text{rest}}$ denotes isomorphisms over T that preserve the restricted symplectic structure. This is a sheaf in flat topology on the category of schemes over $X^{\{p\}}$ with a natural action of G_0 . Moreover, picking an affine refinement of a flat cover $U \rightarrow X^{\{p\}}$ as in Construction 2.70 and using Remark 2.73 we get that $(X, [\eta]) \times_{X^{\{p\}}} U$ is in fact isomorphic to $(\text{Spec } A_0, \eta_{\text{can}}) \times U$. Thus, the restriction $\mathcal{M}_{X, [\eta]}|_U$ can be identified with the functor

$$(T \rightarrow U) \mapsto \text{Iso}_T^{\text{rest}}((\text{Spec } A_0, [\eta_{\text{can}}]) \times T, (\text{Spec } A_0, [\eta_{\text{can}}]) \times T)$$

or, in other words,

$$\mathcal{M}_{X, [\eta]}|_U \simeq G_0 \times U.$$

This way $\mathcal{M}_{X, [\eta]}$ defines a G_0 -torsor on $X^{\{p\}}$, and by construction one has a natural isomorphism

$$\alpha : \mathcal{M}_{X, [\eta]} \times^{G_0} \underline{\text{Aut}}(A_0) \simeq \mathcal{M}_X.$$

Conversely, having a G_0 -torsor \mathcal{M}' over $X^{\{p\}}$, such that

$$\mathcal{M}' \times^{G_0} \underline{\text{Aut}}(A_0) \simeq \mathcal{M}_X,$$

we can take a flat cover $U \rightarrow X$ as above such that $X \times_{X^{\{p\}}} U \simeq \text{Spec } A_0 \times U$ and descend the restricted symplectic U -scheme given by $(\text{Spec } A_0, [\eta_{\text{can}}]) \times U$ to $X^{\{p\}}$ using the descent data provided by \mathcal{M}' and α .

Remark 2.73. Let R be an \mathbb{F}_p -algebra. Then one can show that any two restricted R -linear Poisson structures on $\text{Spec } A_0(R)$ are isomorphic (see [BK, Proposition 3.4]).

Finally, let us fix some notation:

Definition 2.74. The G_0 -torsor $\mathcal{M}_{X, [\eta]}$ over $X^{\{p\}}$ is called the *torsor of Darboux frames* associated to $(X, [\eta])$.

Remark 2.75. The fact that there exists a faithfully flat cover $U \rightarrow X^{\{p\}}$ such that $X \times_{X^{\{p\}}} U \simeq (\text{Spec } A_0, [\eta_{\text{can}}]) \times U$ as restricted Poisson schemes, can be considered as an analogue of the Darboux lemma in the restricted symplectic setting. Indeed, we see that $(\text{Spec } A_0, [\eta_{\text{can}}])$ serves as a ‘‘local model’’ for any restricted symplectic structure.

3. SHEAVES OF ALGEBRAS $\mathcal{A}_{\mathbb{S}}$ AND $\mathcal{A}_{\mathbb{S}}^b$ AND THEIR CATEGORIES OF MODULES

For the rest of this section set $\mathbb{S} := \mathbb{P}^1$. For each dimension n we will define two quasi-coherent sheaves of algebras on \mathbb{S} . The first one is the \mathbb{P}^1 -version $\mathcal{A}_{\mathbb{S}}$ of the Rees construction of *reduced Weyl algebra* (on $2n$ generators), which encodes Hodge and conjugate filtration on the latter. The second is an auxiliary algebra $\mathcal{A}_{\mathbb{S}}^b$ which is obtained as another central reduction from Weyl algebra on $4n$ generators. Following [BV] we will construct an $\mathcal{O}_{\mathbb{S}}$ -linear action of G_0 on $\mathcal{A}_{\mathbb{S}}^b - \text{Mod}$, which will be used later to construct the canonical action of G_0 on the category $\mathcal{A}_{\mathbb{S}} - \text{Mod}$.

3.1. Rees construction over \mathbb{P}^1 . First let us remind the usual Rees construction and its properties.

Let \mathcal{B} be a quasicoherent sheaf of associative (but not necessarily commutative) algebras over a base scheme Y . Let $\mathcal{B}_{\leq *} = \dots \subset \mathcal{B}_{\leq n} \subset \mathcal{B}_{\leq n+1} \subset \dots$ be an increasing filtration on \mathcal{B} indexed by integers. We assume that the filtration is multiplicative ($\mathcal{B}_{\leq i} \cdot \mathcal{B}_{\leq j} \subset \mathcal{B}_{\leq i+j}$) and exhaustive (namely $\mathcal{B} = \text{colim}_n \mathcal{B}_{\leq n}$).

Construction 3.1 (Rees construction over \mathbb{A}^1). Let h be a formal variable. Let $\mathcal{B}_{\leq *}$ be as above.

Define the *Rees construction* $\mathcal{B}_{\mathbb{A}^1}$ of the filtered algebra $\mathcal{B}_{\leq *}$ as a subsheaf of algebras

$$\mathcal{B}_{\mathbb{A}^1} := \oplus_i \mathcal{B}_{\leq i} \cdot h^i \subset \mathcal{B}[h, h^{-1}] := \mathcal{B} \otimes_{\mathbb{Z}} \mathbb{Z}[h, h^{-1}].$$

Since $\mathcal{B}_{\leq i} \cdot \mathcal{B}_{\leq j} \subset \mathcal{B}_{\leq i+j}$ and $\mathcal{B}_{\leq i} \subset \mathcal{B}_{\leq i+1}$, the multiplication on $\mathcal{B}_{\mathbb{A}^1}$ is well-defined and $\mathcal{B}_{\mathbb{A}^1}$ defines a (quasicoherent) subsheaf of $\mathcal{O}_Y[h]$ -algebras in $\mathcal{B}[h, h^{-1}]$. One also has isomorphisms

$$(3.1) \quad \mathcal{B}_{\mathbb{A}^1}/h \simeq \text{gr}_* \mathcal{B} \quad \text{and} \quad \mathcal{B}_{\mathbb{A}^1}[h^{-1}] \simeq \mathcal{B}[h, h^{-1}].$$

Here for the second isomorphism we used the exhaustiveness of filtration.

Using a monoidal equivalence of categories of quasicoherent $\mathcal{O}_Y[h]$ -modules and quasicoherent sheaves on $\mathbb{A}^1 \times Y$ we can view the $\mathcal{O}_Y[h]$ -algebra $\mathcal{B}_{\mathbb{A}^1}$ as a quasicoherent sheaf of algebras $\mathcal{B}_{\mathbb{A}^1}$ over $\mathbb{A}^1 \times Y$. Formulas 3.1 then give isomorphisms

$$\mathcal{B}_{\mathbb{A}^1}|_{\{0\} \times Y} \simeq \text{gr}_* \mathcal{B} \quad \text{and} \quad \mathcal{B}_{\mathbb{A}^1}|_{\mathbb{G}_m \times Y} \simeq \mathcal{O}_{\mathbb{G}_m} \boxtimes \mathcal{B}.$$

Construction 3.2 (A variant over the center). Let $Z(\mathcal{B}) \subset \mathcal{B}$ be the center and define the filtration $Z(\mathcal{B})_{\leq *} := Z(\mathcal{B}) \cap \mathcal{B}_{\leq *}$. This filtration is also multiplicative and exhaustive and we can consider the corresponding Rees construction $Z(\mathcal{B})_{\mathbb{A}^1}$. Then $\mathcal{B}_{\mathbb{A}^1}$ is naturally a sheaf of algebras over $Z(\mathcal{B})_{\mathbb{A}^1}$ and as such also defines a quasi-coherent sheaf of algebras over the relative spectrum $\text{Spec}_{\mathbb{A}^1 \times Y}(Z(\mathcal{B})_{\mathbb{A}^1})$.

Construction 3.3 (Descent to $[\mathbb{A}^1/\mathbb{G}_m]$). Consider the standard action of \mathbb{G}_m on \mathbb{A}^1 (such that $t \circ h = th$ for $t \in \mathbb{G}_m$). It induces a \mathbb{G}_m -action on $Y \times \mathbb{A}^1$ and endows $\mathcal{O}_Y[h]$ with a \mathbb{Z} -grading such that \mathcal{O}_Y has grading 0 and the variable h has grading 1. Consider the corresponding quotient stack $[\mathbb{A}^1/\mathbb{G}_m] \times Y$. Using faithfully flat descent the abelian category of quasicoherent sheaves on $[\mathbb{A}^1/\mathbb{G}_m] \times Y$ can be identified with the category of quasicoherent sheaves of graded $\mathcal{O}_Y[h]$ -modules on Y .

The algebra $\mathcal{B}_{\mathbb{A}^1}$ comes with a natural grading $(\mathcal{B}_{\mathbb{A}^1})^i := \mathcal{B}_{\leq i} \cdot h^i$ that makes it into a graded $\mathcal{O}_Y[h]$ -algebra. This way $\mathcal{B}_{\mathbb{A}^1}$ defines a quasi-coherent sheaf of algebras $\text{x}\mathcal{B}_{[\mathbb{A}^1/\mathbb{G}_m]}$ on $[\mathbb{A}^1/\mathbb{G}_m] \times Y$.

The descent above also has a variant over the center. Namely, $Z(\mathcal{B})_{\mathbb{A}^1}$ is also a graded $\mathcal{O}_Y[h]$ -algebra, which endows $\text{Spec}_{\mathbb{A}^1 \times Y}(Z(\mathcal{B})_{\mathbb{A}^1})$ with a natural \mathbb{G}_m -action.

Moreover, $\mathcal{B}_{\mathbb{A}^1}$ is a graded $Z(\mathcal{B})_{\mathbb{A}^1}$ -algebra and as such defines a quasi-coherent sheaf of algebras on the quotient stack $[\mathrm{Spec}_{\mathbb{A}^1 \times Y}(Z(\mathcal{B})_{\mathbb{A}^1})/\mathbb{G}_m]$.

Now, having two multiplicative exhaustive filtrations on \mathcal{B} we can glue the two corresponding Rees constructions into a sheaf over $\mathbb{P}^1 \times Y$:

Construction 3.4 (Rees construction over \mathbb{P}^1). Let \mathcal{B} be a quasi-coherent sheaf of algebras over Y endowed with an exhaustive increasing filtration $\mathcal{B}_{\leq *}$ and an exhausting decreasing filtration $\mathcal{B}^{\geq *}$. We will denote such data by $(\mathcal{B}, \mathcal{B}_{\leq *}, \mathcal{B}^{\geq *})$.

Cover \mathbb{P}^1 by two charts given by affine lines $\mathbb{A}_+^1 \simeq \mathrm{Spec} \mathbb{Z}[h]$ and $\mathbb{A}_-^1 \simeq \mathrm{Spec} \mathbb{Z}[h^{-1}]$, with intersection $\mathbb{G}_m = \mathbb{P}^1 \setminus (\{0\} \sqcup \{\infty\}) \simeq \mathrm{Spec} \mathbb{Z}[h, h^{-1}]$. On these charts take Rees constructions²⁰ associated to the two filtrations:

$$\mathcal{B}_{\mathbb{A}_+^1} := \bigoplus_i \mathcal{B}_{\leq i} \cdot h^i \subset \mathcal{B}[h, h^{-1}] \supset \mathcal{B}_{\mathbb{A}_-^1} := \bigoplus_i \mathcal{B}^{\geq -i} \cdot h^{-i}.$$

$\mathcal{B}_{\mathbb{A}_+^1}$ and $\mathcal{B}_{\mathbb{A}_-^1}$ are the $\mathcal{O}_Y[h]$ and $\mathcal{O}_Y[h^{-1}]$ subalgebras of $\mathcal{B}[h, h^{-1}]$, and as such define quasicoherent sheaves of algebras on $\mathbb{A}_+^1 \times Y$ and $\mathbb{A}_-^1 \times Y$ correspondingly. Moreover, we have natural isomorphisms

$$\mathcal{B}_{\mathbb{A}_+^1}|_{\mathbb{G}_m \times Y} \simeq \mathcal{O}_{\mathbb{G}_m} \boxtimes \mathcal{B} \simeq \mathcal{B}_{\mathbb{A}_-^1}|_{\mathbb{G}_m \times Y}.$$

Using the composite isomorphism as the gluing data we obtain a quasi-coherent sheaf of algebras on $\mathbb{P}^1 \times Y$ which we will denote $\mathcal{B}_{\mathbb{P}^1}$ and call *Rees construction over \mathbb{P}^1* .

The above construction also localizes over the center of \mathcal{B} . Namely, $\mathcal{B}_{\mathbb{P}^1}$ naturally defines a sheaf of algebras on $\mathrm{Spec}_{\mathbb{P}^1 \times Y} Z(\mathcal{B})_{\mathbb{P}^1}$ which we will continue to denote $\mathcal{B}_{\mathbb{S}}$. Here $Z(\mathcal{B})$ denotes the center of \mathcal{B} .

Construction 3.5 (Descent to $[\mathbb{P}^1/\mathbb{G}_m]$). Analogously to Construction 3.3 algebras $\mathcal{B}_{\mathbb{A}_+^1}$ and $\mathcal{B}_{\mathbb{A}_-^1}$ are endowed with natural gradings $(\mathcal{B}_{\mathbb{A}_+^1})_i := \mathcal{B}_{\leq i} \cdot h^i$ and $(\mathcal{B}_{\mathbb{A}_-^1})_i := \mathcal{B}^{\geq i} \cdot h^i$. This way they define quasi-coherent sheaves of algebras on $[\mathbb{A}_+^1/\mathbb{G}_m] \times Y$ and $[\mathbb{A}_-^1/\mathbb{G}_m] \times Y$. Moreover, the restrictions of both $\mathcal{B}_{[\mathbb{A}_+^1/\mathbb{G}_m]}$ and $\mathcal{B}_{[\mathbb{A}_-^1/\mathbb{G}_m]}$ to $[\mathbb{G}_m/\mathbb{G}_m] \times Y \simeq Y$ are naturally identified with $\mathcal{B} = \mathrm{colim}_{i \rightarrow \infty} \mathcal{B}_{\leq i} = \mathrm{colim}_{i \rightarrow \infty} \mathcal{B}^{\geq -i}$. This way they glue to a quasi-coherent sheaf of algebras $\mathcal{B}_{[\mathbb{P}^1/\mathbb{G}_m]}$ on $[\mathbb{P}^1/\mathbb{G}_m] \times Y$.

This construction also localizes over the center of \mathcal{B} . Namely $\mathcal{B}_{[\mathbb{P}^1/\mathbb{G}_m]}$ naturally defines a quasi-coherent sheaf of algebras over the relative spectrum $\mathrm{Spec}_{[\mathbb{P}^1/\mathbb{G}_m] \times Y}(Z(\mathcal{B})_{[\mathbb{P}^1/\mathbb{G}_m]})$.

Example 3.6 (Split case). Assume \mathcal{B} is commutative (so $\mathcal{B} \simeq Z(\mathcal{B})$). Let \mathcal{B} be graded $\mathcal{B} \simeq \bigoplus_i \mathcal{B}_i$ and consider the split exhausting filtrations associated to it:

$$\mathcal{B}_{\leq n} := \bigoplus_{i \leq n} \mathcal{B}_i \text{ and } \mathcal{B}^{\geq n} := \bigoplus_{i \geq n} \mathcal{B}_i.$$

Then there is a natural isomorphism

$$\mathrm{Spec}_{\mathbb{P}^1 \times Y}(\mathcal{B}_{\mathbb{P}^1}) \simeq \mathrm{Spec}_Y(\mathcal{B}) \times \mathbb{P}^1.$$

Moreover, this isomorphism is \mathbb{G}_m -equivariant, where the \mathbb{G}_m -action on $\mathrm{Spec}_Y(\mathcal{B})$ is defined by the grading on \mathcal{B} . This way we also get an isomorphism

$$\mathrm{Spec}_{[\mathbb{P}^1/\mathbb{G}_m] \times Y}(\mathcal{B}_{[\mathbb{P}^1/\mathbb{G}_m]}) \simeq [(\mathrm{Spec}_Y(\mathcal{B}) \times \mathbb{P}^1)/\mathbb{G}_m].$$

To see these isomorphisms explicitly, note that

$$(\mathcal{B}[h])_n \simeq \bigoplus_{i+j=n} \mathcal{B}_i \cdot h^j \simeq \bigoplus_{i \leq n} \mathcal{B}_i \simeq \mathcal{B}_{\leq n}$$

²⁰Here, one can view $\mathcal{B}_{\mathbb{A}_-^1}$ as an instance of 3.1 for the increasing filtration given by $F_i \mathcal{B} := \mathcal{B}^{\geq -i}$, and where the coordinate on \mathbb{A}^1 is h^{-1} .

and, this way $\mathcal{B}[h] \simeq \mathcal{B}_{\mathbb{A}_+^1}$ as a graded \mathcal{O}_Y -algebra. Similarly,

$$(\mathcal{B}[h^{-1}])_n \simeq \oplus_{i-j=n} \mathcal{B}_i \cdot h^{-j} \simeq \oplus_{i \geq n} \mathcal{B}_i \simeq \mathcal{B}^{\geq n}$$

and so $\mathcal{B}[h^{-1}] \simeq \mathcal{B}_{\mathbb{A}_-^1}$. Moreover, under this identifications, the gluing data over $\mathbb{G}_m \times Y$ is just given by the isomorphism $(\mathcal{B}[h])[h^{-1}] \simeq (\mathcal{B}[h^{-1}])[h]$. This way we can identify $\mathcal{B}_{\mathbb{P}^1}$ with the pushforward $p_*\mathcal{O}$ for the projection $p: \text{Spec}_Y(\mathcal{B}) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, hence the statement above.

3.2. Twistor differential operators $\mathcal{D}_{X, \mathbb{P}^1}$. We now discuss a particular case of the above construction in the case of the sheaf of differential operators on a pseudo-smooth scheme. Let us clarify the setup.

Construction 3.7 (Hodge and conjugate filtrations on \mathcal{D}_X). Namely, let X be a pseudo-smooth S -scheme and let \mathcal{D}_X be the sheaf of differential operators that we have defined in Section 2.3. The pushforward $F_{X^*}^{\{p\}} \mathcal{D}_X$ defines a quasi-coherent sheaf of algebras over $X^{\{p\}}$. Let us discard $F_{X^*}^{\{p\}}$ from the notation and implicitly consider \mathcal{D}_X as a sheaf of algebras over $\mathcal{O}_X^{\{p\}}$. Recall that we have an isomorphism $\theta^{-1}: Z(\mathcal{D}_X) \simeq \text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}}$ (see Corollary 2.38). Let us temporarily discard the subscript given by $\mathcal{O}_X^{\{p\}}$ to lighten up the notations.

We endow \mathcal{D}_X with two filtrations:

- (Hodge filtration). Consider filtration $\text{Fil}_{\leq *}^H := \mathcal{D}_{X, \leq *}$ by the order of differential operator (see Remark 2.18). One has $\mathcal{D}_{X, \leq n} = 0$ if $n < 0$ and $\mathcal{D}_{X, \leq 0} \simeq \mathcal{O}_X \subset \mathcal{D}_X$:

$$\dots = 0 = 0 \subset \mathcal{O}_X \subset \mathcal{D}_{X, \leq 1} \subset \mathcal{D}_{X, \leq 2} \subset \dots$$

- (Conjugate filtration). This is a filtration $\text{Fil}_{\text{cnj}}^{\geq *}$ defined as follows. Namely one has $\text{Fil}_{\text{cnj}}^{\geq n} \simeq \mathcal{D}_X$ if $n \leq 0$:

$$\dots = \mathcal{D}_X = \mathcal{D}_X \subset \text{Fil}_{\text{cnj}}^{\geq 1} \subset \text{Fil}_{\text{cnj}}^{\geq 2} \dots$$

and $\text{Fil}_{\text{cnj}}^{\geq i}$ is defined as $\text{Sym}^{\lfloor \frac{i}{p} \rfloor} \mathcal{T}_X^{\{p\}} \cdot \mathcal{D}_X \subset \mathcal{D}_X$. In particular, it is a p -step filtration (one has $\text{Fil}_{\text{cnj}}^{\geq i} = \text{Fil}_{\text{cnj}}^{\geq i-1}$ unless $p|i$).

Remark 3.8. The conjugate filtration $\text{Fil}_{\text{cnj}}^{\geq *}$ is a reindexed variant of a filtration that appear in [OV].

Remark 3.9 (Induced filtrations on the center). Filtrations induced on $Z(\mathcal{D}_X)$ by the Hodge and conjugate filtrations on \mathcal{D}_X above are in fact split. Namely, we claim that the two filtrations $Z(\mathcal{D}_X) \cap \text{Fil}_{\leq *}^H$ and $Z(\mathcal{D}_X) \cap \text{Fil}_{\text{cnj}}^{\geq *}$ are obtained by construction in Example 3.6 from the single grading on $Z(\mathcal{D}_X)$ given as follows:

$$Z(\mathcal{D}_X)_n := \begin{cases} 0 & \text{if } p \nmid n \\ \text{Sym}^k \mathcal{T}_X^{\{p\}} & \text{if } n = p \cdot k \end{cases}$$

This is clear for the conjugate filtration: since $Z(\mathcal{D}_X) \cap \text{Fil}_{\leq -n}^{\text{cnj}}$ is given by $\text{Sym}^{\geq \lfloor \frac{n}{p} \rfloor} \mathcal{T}_X^{\{p\}}$ which is also a direct sum $\oplus_i Z(\mathcal{D}_X)_i$ with $i \geq n$. For the Hodge filtration this requires a slight argument.

More precisely, we need to show that $\text{Fil}_{\leq n}^H \cap Z(\mathcal{D}_X) \simeq \text{Sym}^{\leq \lfloor \frac{n}{p} \rfloor} \mathcal{T}_X^{\{p\}}$ (where the latter is isomorphic to $\oplus_i Z(\mathcal{D}_X)_i$ with $i \geq n$). Recall that the map $\theta: \text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}} \rightarrow$

$Z(\mathcal{D}_X)$ is induced by $v^{\{p\}} \mapsto \theta(v^{\{p\}}): v^p - v^{\{p\}}$ and so $\theta(\mathcal{T}_X) \subset \text{Fil}_{\leq p}^H$. Consequently, $\theta(\text{Sym}^{\leq k} \mathcal{T}_X) \subset \text{Fil}_{\leq pk}^H$ and to show the statement it is enough to show that the map $\text{gr}_{pk} \theta: \text{Sym}^k \mathcal{T}_X^{\{p\}} \rightarrow \text{gr}_{\leq pk}^H$ between the associated graded pieces is an embedding. We have $\theta(v^{\{p\}}) = v^p \in \text{gr}_p^H$ and this way we see that the map of $\mathcal{O}_X^{\{p\}}$ -algebras $\text{gr}_*(\theta): \text{Sym}^* \mathcal{T}_X^{\{p\}} \rightarrow \text{Sym}_{\mathcal{O}_X}^* \mathcal{T}_X$ is the map given by reduced Frobenius $F_{T^*X}^{\{p\}}: T^*X \rightarrow (T^*X)^{\{p\}}$ (as affine schemes over $\mathcal{O}_X^{\{p\}}$), which is an embedding essentially by definition.

Following the discussion in Example 3.6 we then get the following description of the Rees construction for $Z(\mathcal{D}_X)$. Namely, the grading on $Z(\mathcal{D}_X)$ that we considered above corresponds to the \mathbb{G}_m -action on $(T^*X)^{\{p\}} \simeq \text{Spec}_{X^{\{p\}}} Z(\mathcal{D}_X)$ where $t \in \mathbb{G}_m$ acts by rescaling the fibers of the vector bundle $(T^*X)^{\{p\}} \rightarrow X^{\{p\}}$ by t^p . We get a \mathbb{G}_m -equivariant isomorphism

$$\text{Spec}_{X^{\{p\}} \times \mathbb{S}} Z(\mathcal{D}_X)_{\mathbb{S}} \simeq (T^*X)^{\{p\}} \times \mathbb{S},$$

where the action on the right is diagonal.

It then also descends to an isomorphism

$$\text{Spec}_{X^{\{p\}} \times [\mathbb{S}/\mathbb{G}_m]} Z(\mathcal{D}_X)_{[\mathbb{S}/\mathbb{G}_m]} \simeq [((T^*X)^{\{p\}} \times \mathbb{S})/\mathbb{G}_m].$$

for the corresponding quotients by \mathbb{G}_m .

Definition 3.10 (Twistor differential operators). We define the sheaf $\mathcal{D}_{X,\mathbb{S}}$ of *twistor differential operators* on X as the Rees construction (3.4) over \mathbb{S} associated to $(\mathcal{D}_X, \text{Fil}_{\leq *}, \text{Fil}_{\leq *}, \text{cnj})$. By Remark 3.9 and discussion in Example 3.6, $\mathcal{D}_{X,\mathbb{S}}$ defines a \mathbb{G}_m -equivariant quasi-coherent sheaf of algebras over $(T^*X)^{\{p\}} \times \mathbb{S}$ (with \mathbb{G}_m -action as in Remark 3.9).

Remark 3.11. Let us describe $\mathcal{D}_{X,\mathbb{S}}$ more explicitly. Namely, the restriction $\mathcal{D}_{X,\mathbb{S} \setminus \{\infty\}}$ is the Rees algebra associated to PBW-filtration on \mathcal{D}_X and as such can be identified with the subalgebra in $\mathcal{D}_X[h]$ generated by \mathcal{O}_X and $h \cdot \mathcal{T}_X$. One can also describe it as a sheaf of algebras over $\mathcal{O}_X^{\{p\}}[h]$ generated by \mathcal{O}_X and \mathcal{T}_X with the relations given by $f_1 \cdot f_2 - f_2 \cdot f_1 = 0$, $v \cdot f - f \cdot v = h \cdot v(f)$ and $v_1 \cdot v_2 - v_2 \cdot v_1 = h \cdot [v_1, v_2]$ (here we identify \mathcal{T}_X with its image in $\mathcal{D}_{X,\leq 1} \cdot h$ in the Rees construction).

Following Remark 3.9 and Example 3.6 we have an isomorphism $Z(\mathcal{D}_X)_{\mathbb{S} \setminus \{\infty\}} \simeq Z(\mathcal{D}_X)[h] \simeq (\text{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}})[h]$. The corresponding map $Z(\mathcal{D}_X)_{\mathbb{S} \setminus \{\infty\}} \rightarrow \mathcal{D}_{X,\mathbb{S} \setminus \{\infty\}}$ sends $\mathcal{O}_X^{\{p\}}$ to \mathcal{O}_X via $F_X^{\{p\}}$, while $v^{\{p\}} \in \mathcal{T}_X^{\{p\}}$ maps to²¹ $\theta_h(v) = v^p - h^{p-1} \cdot v^{\{p\}}$.

For the description of the other chart note that since $\text{Fil}_{\leq 0}^{\text{cnj}} = \mathcal{D}_X$ we have a natural embedding $Z(\mathcal{D}_X)[h^{-1}] \subset Z(\mathcal{D}_X)_{\mathbb{S} \setminus \{0\}}$. Moreover, by the way how $\text{Fil}_{\leq *}, \text{cnj}$ is defined, $\mathcal{D}_{X,\mathbb{S} \setminus \{0\}}$ is identified with the tensor product $\mathcal{D}_X[h^{-1}] \otimes_{Z(\mathcal{D}_X)[h^{-1}]} Z(\mathcal{D}_X)_{\mathbb{S} \setminus \{0\}}$. Using Remark 3.9 and Example 3.6, one can further identify $\mathcal{D}_{X,\mathbb{S} \setminus \{0\}}$ with the pullback of $\mathcal{D}_X \boxtimes \mathcal{O}_{\mathbb{S} \setminus \{0\}}$ along the endomorphism of $\lambda_{h^{-p}}$ of $(T^*X)^{\{p\}} \times (\mathbb{S} \setminus \{0\})$ given by the fiberwise dilation by h^{-p} . In particular, over $\{\infty\} \in \mathbb{S} \setminus \{0\}$, $\lambda_{h^{-p}}$ is the zero map and thus the fiber $\mathcal{D}_{X,\{\infty\}} := \mathcal{D}_{X,\mathbb{S} \setminus \{0\}}/h^{-1}$ is identified with $\mathcal{D}_{X,0} \otimes_{\mathcal{O}_X^{\{p\}}} \text{Sym}^* \mathcal{T}_X^{\{p\}}$, where $\mathcal{D}_{X,0}$ is the central reduction from Remark 2.29. Recall that $\mathcal{D}_{X,0} \simeq \text{End}_{\mathcal{O}_X^{\{p\}}}(\mathcal{O}_X)$ is a

²¹To see why this is the formula: we have $\theta(v) \in \text{Fil}_{\leq p}^H$ and $\theta(v) \cdot h^p = v^p h^p - v^{\{p\}} h^p = (vh)^p - h^{p-1}(v^{\{p\}}h)$.

split Azumaya algebra; this way $\mathcal{D}_{X, \{\infty\}}$ is Morita equivalent to $\mathcal{O}_X \otimes_{\mathcal{O}_X^{\{p\}}} \mathrm{Sym}^* \mathcal{T}_X^{\{p\}} \simeq \mathrm{Sym}_{\mathcal{O}_X} \mathcal{T}_X$. In particular, the category of quasi-coherent sheaves of $\mathcal{D}_{X, \{\infty\}}$ -modules is equivalent to quasi-coherent sheaves over T^*X .

The two restrictions $\mathcal{D}_{X, \mathbb{S} \setminus \{\infty\}}$ and $\mathcal{D}_{X, \mathbb{S} \setminus \{0\}}$ are then glued along $\mathbb{S} \setminus (\{0\} \sqcup \{\infty\})$ by identifying the restrictions of both algebras with $\mathcal{D}_X[h, h^{-1}]$.

3.3. Central reduction of twistor differential operators. We extend Construction 2.46 to the algebra of twistor differential operators. Let $\alpha \in H^0(X^{\{p\}}, \Omega_X^{1, \{p\}})$ be a global section, $i_\alpha: X^{\{p\}} \hookrightarrow (T^*X)^{\{p\}}$ the graph of α . Denote by

$$\mathcal{J}_{\alpha, \mathbb{S}} \subset Z(\mathcal{D}_X)_{\mathbb{S}} \subset \mathcal{D}_{X, \mathbb{S}}$$

the sheaf of ideals determined by the closed embedding $i_\alpha \times \mathrm{Id}_{\mathbb{S}}: X^{\{p\}} \times \mathbb{S} \hookrightarrow (T^*X)^{\{p\}} \times \mathbb{S}$. Let $\mathcal{D}_{X, \frac{\mathbb{S}}{h^p}, \mathbb{S}}$ be the quotient $\mathcal{D}_{X, \mathbb{S}} / \mathcal{J}_{\alpha, \mathbb{S}} \mathcal{D}_{X, \mathbb{S}}$.

Recall that we have the natural map $-\{p\}: H^0(X, \Omega_X^1) \rightarrow H^0(X^{\{p\}}, \Omega_X^{1, \{p\}})$, $\alpha \mapsto \alpha^{\{p\}}$. It turns out that the association $\alpha \mapsto \mathcal{D}_{X, \frac{\alpha^{\{p\}}}{h^p}, \mathbb{S}}$ for $\alpha \in H^0(X, \Omega_X^1)$ only depends on the class $[\alpha] \in H_{Zar}^0(X, \mathrm{coker}(\mathcal{O}_X \xrightarrow{d} \Omega_{X/S}^1))$ in a canonical way, as the discussion below shows.

Construction 3.12. Let $\mu \in H^0(X, \Omega_{X/S}^1)$ be a closed 1-form on X . Define an automorphism $\phi_{\frac{\mu}{h}}$ of algebra $\mathcal{D}_{X, \mathbb{S} \setminus \{\infty\}}$ (viewed as the subalgebra of $\mathcal{D}_X[h]$ generated by \mathcal{O}_X and $h \cdot \mathcal{T}_X$) by setting $\phi_{\frac{\mu}{h}}(f) = f$ and $\phi_{\frac{\mu}{h}}(hv) = hv + \iota_v \mu$, for every function f and every vector field v . Informally, this automorphism is the conjugation by $e^{\frac{1}{h} \int \mu}$: though this expression does not make sense as a function on $X \times \mathbb{S}$, the conjugation automorphism is well defined. For $\mu_1, \mu_2 \in H^0(X, \Omega_{X/S}^1)$, we have that

$$(3.2) \quad \phi_{\frac{\mu_1}{h}} \circ \phi_{\frac{\mu_2}{h}} = \phi_{\frac{\mu_1 + \mu_2}{h}}.$$

The action of $\phi_{\frac{\mu}{h}}$ on the center $(\mathrm{Sym}_{\mathcal{O}_X^{\{p\}}}^* \mathcal{T}_X^{\{p\}})[h]$ of the algebra $\mathcal{D}_{X, \mathbb{S} \setminus \{\infty\}}$ is given by the Katz p -curvature formula (see Remark 2.45): $\phi_{\frac{\mu}{h}}$ acts trivially on $\mathcal{O}_X^{\{p\}}$ and, for $v^{\{p\}} \in \mathcal{T}_X^{\{p\}}$, one has that $\phi_{\frac{\mu}{h}}(h^p v^{\{p\}}) = h^p v^{\{p\}} + \iota_{v^{\{p\}}}(\mu^{\{p\}} - h^{p-1} C(\mu))$. In particular, if $C(\mu) = 0$ i.e., μ is exact, then $\phi_{\frac{\mu}{h}}$ extends to an automorphism of $\mathcal{D}_{X, \mathbb{S}}$ and acts on the spectrum of its centre $(T^*X)^{\{p\}} \times \mathbb{S}$ as the translation by $\mu^{\{p\}}$. Thus, for an exact 1-form μ on X and a section $\alpha \in H^0(X^{\{p\}}, \Omega_X^{1, \{p\}})$, one has that $\phi_{\frac{\mu}{h}}(\mathcal{J}_{\alpha, \mathbb{S}}) = \mathcal{J}_{\alpha + \mu^{\{p\}}, \mathbb{S}}$. It follows that $\phi_{\frac{\mu}{h}}$ descends to an isomorphism:

$$\phi_{\frac{\mu}{h}}: \mathcal{D}_{X, \frac{\mathbb{S}}{h^p}, \mathbb{S}} \xrightarrow{\sim} \mathcal{D}_{X, \frac{\alpha + \mu^{\{p\}}}{h^p}, \mathbb{S}}.$$

Construction 3.13 (Generalized central reduction $\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}$). A class

$$[\eta] \in H_{Zar}^0\left(X, \mathrm{coker}\left(\mathcal{O}_X \xrightarrow{d} \Omega_{X/S}^1\right)\right)$$

gives rise to a certain sheaf of algebras $\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}$ locally given by a central reduction of $\mathcal{D}_{X, \mathbb{S}}$. This is a sheaf of quasi-coherent $\mathcal{O}_X^{\{p\}}$ -algebras that comes together with the following structure. For any open subset U together with a 1-form $\eta \in \Omega_{X/S}^1(U)$ representing $[\eta]$, we have an isomorphism of $\mathcal{O}_U^{\{p\}}$ -algebras

$$\gamma_{U, \eta}: (\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}})|_U \xrightarrow{\sim} \mathcal{D}_{U, \frac{(\eta|_U)^{\{p\}}}{h^p}, \mathbb{S}}$$

such that, for any two 1-forms η_1, η_2 on U representing the class $[\eta]$, one has that $\gamma_{U, \eta_2} \circ \gamma_{U, \eta_1}^{-1} = \phi_{\frac{\eta_2 - \eta_1}{h}}$. Formula (3.2) shows that the algebra $\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}$ together with the collections of $\gamma_{U, \eta}$ exists and is unique up to a unique isomorphisms. Moreover, the construction is functorial, meaning that the sheaf algebras $\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}$ is equivariant with respect to the group $\text{Aut}_S(X, [\eta])$ of automorphisms of X over S preserving the class $[\eta]$. For future reference we display the induced action $\text{Aut}_S(X, [\eta])$ on the direct image $\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}$ along the projection $\text{pr}_{S \times \mathbb{S}}: X \times \mathbb{S} \rightarrow S \times \mathbb{S}$ by $\mathcal{O}_{S \times \mathbb{S}}$ -algebra automorphisms

$$(3.3) \quad \psi: \text{Aut}_{\mathbb{S}}(X, [\eta]) \rightarrow \text{Aut}(\text{pr}_{S \times \mathbb{S}}^* \mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}).$$

The algebra $\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}$ has further symmetries: simultaneous multiplication of h and $[\eta]$ by a scalar λ doesn't change the algebra. More precisely, endow \mathbb{S} with an action χ of \mathbb{G}_m by homotheties: $\chi(h) = zh$, where z is the coordinate on \mathbb{G}_m . Then pullback of $\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}$ with respect to the morphism $\mathbb{G}_m \times X \times \mathbb{S} \rightarrow X \times \mathbb{S}$ given by the \mathbb{G}_m -action on the last factor is naturally isomorphic to $\mathcal{D}_{\mathbb{G}_m \times X, ([z^{-1} \text{pr}_X^* \eta])/h, \mathbb{S}}$, where $\mathbb{G}_m \times X$ is considered as a scheme over $\mathbb{G}_m \times S$ and $\text{pr}_X: \mathbb{G}_m \times X \rightarrow X$ is the projection.

Example 3.14 (The sheaf $\mathcal{D}_{X, 0, \mathbb{S}}$). Let's consider a particular case of the above construction when we have $[\eta] = 0$. The corresponding sheaf of algebras $\mathcal{D}_{X, 0, \mathbb{S}}$ is simply the restriction of $\mathcal{D}_{X, \mathbb{S}}$ to the zero section $X^{\{p\}} \times \mathbb{S} \rightarrow (T^*X)^{\{p\}} \times \mathbb{S}$, or in terms of sheaves of algebras we have $\mathcal{D}_{X, 0, \mathbb{S}} \simeq \mathcal{D}_{X, \mathbb{S}} \otimes_{Z(\mathcal{D}_X)_{\mathbb{S}}} \mathcal{O}_X^{\{p\}}$.

Let us describe more explicitly the gluing data defining $\mathcal{D}_{X, 0, \mathbb{S}}$. Namely, $\mathcal{D}_{X, 0, \mathbb{S} \setminus \{\infty\}}$ is the quotient of $\mathcal{D}_{X, \mathbb{S} \setminus \{\infty\}}$ (see Remark 3.11) by the two-sided ideal generated by $v^p - h^{p-1}v^{[p]}$, $v \in \mathcal{T}_X$. On the other hand, the restriction $\mathcal{D}_{X, 0, \mathbb{S} \setminus \{0\}}$ is really simple, namely $\mathcal{D}_{X, 0, \mathbb{S} \setminus \{0\}} \simeq \mathcal{D}_{X, 0}[h^{-1}]$. Indeed, by Remark 3.11 $\mathcal{D}_{X, \mathbb{S} \setminus \{0\}} \simeq \mathcal{D}_X[h^{-1}] \otimes_{Z(\mathcal{D}_X)[h^{-1}]} Z(\mathcal{D}_X)_{\mathbb{S} \setminus \{0\}}$ from which we get that

$$\mathcal{D}_{X, 0, \mathbb{S} \setminus \{0\}} \simeq \mathcal{D}_X[h^{-1}] \otimes_{Z(\mathcal{D}_X)[h^{-1}]} \mathcal{O}_X^{\{p\}}[h^{-1}] \simeq \mathcal{D}_{X, 0}[h^{-1}].$$

Observe that for every class $[\eta]$ the restriction of $\mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}$ to $X \times \{\infty\} \hookrightarrow X \times \mathbb{S}$ is naturally isomorphic to that of $\mathcal{D}_{X, 0, \mathbb{S}}$. In particular, the former is a canonically split Azumaya algebra over $X \times \{\infty\}$.

3.4. Twistor reduced Weyl algebra $\mathcal{A}_{\mathbb{S}}$. We will be now interested in a very particular case of the Construction 3.13. Namely, let us fix a number $n \in \mathbb{N}$. Consider the algebra $C := \mathbb{F}_p[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$: this is the (ring of functions on the) Frobenius neighborhood of $\{0\}$ in an n -dimensional affine space over \mathbb{F}_p . Put $Y := \text{Spec } C$ considered as a scheme over \mathbb{F}_p . This is a pseudo-smooth \mathbb{F}_p -scheme (with $\Omega_C^1 \simeq \oplus_{i=1}^n C \cdot dx_i$) and one has $Y^{\{p\}} \simeq \text{Spec}(C^p) \simeq \text{Spec } \mathbb{F}_p$.

Definition 3.15. We define the *twistor reduced Weyl algebra* $\mathcal{A}_{\mathbb{S}}$ as $\mathcal{D}_{Y, 0, \mathbb{S}}$. This is a quasi-coherent sheaf of algebras on \mathbb{S} .

Remark 3.16. Note that by (the proof of) Lemma 2.71 any pseudo-smooth X considered as a scheme over $X^{\{p\}}$ flat locally looks like C with $n = \dim X$. This way $\mathcal{A}_{\mathbb{S}}$ can serve as a model example of quantization (in this context of T^*X).

Let us describe $\mathcal{A}_{\mathbb{S}}$ in terms of its restrictions to $\mathbb{S} \setminus \{\infty\}$ and $\mathbb{S} \setminus \{0\}$ and the gluing data. First, let us describe the algebra $D_Y := \Gamma(Y, \mathcal{D}_Y)$. We have $\mathcal{T}_Y \simeq \oplus_{i=1}^n C \cdot \partial_i$

with $\partial_i(x_j) = \delta_{ij}$. Thus

$$D_Y \simeq \mathbb{F}_p \langle x_i, y_j \rangle_{1 \leq i, j \leq n} / ([x_i, x_j] = [y_i, y_j] = x_i^p = 0, [y_j, x_i] = \delta_{ij})$$

with y_i corresponding to ∂_i . One sees easily that $\partial_i^{[p]} = 0$ and so the center $Z(D_Y)$ is described as

$$Z(D_Y) \simeq \mathbb{F}_p[y_1^p, \dots, y_n^p] \subset D_Y.$$

Hodge filtration on D_Y is given by the total degree in y_i 's. Following Remark 3.11 the corresponding Rees algebra $D_{Y, \mathbb{S} \setminus \{\infty\}}$ is the $\mathbb{F}_p[h]$ -algebra given by

$$D_{Y, \mathbb{S} \setminus \{\infty\}} \simeq \mathbb{F}_p[h] \langle x_i, y_j \rangle_{1 \leq i, j \leq n} / ([x_i, x_j] = [y_i, y_j] = x_i^p = 0, [y_j, x_i] = h \cdot \delta_{ij})$$

The center $Z(D_{Y, \mathbb{S} \setminus \{\infty\}})$ is identified with

$$Z(D_{Y, \mathbb{S} \setminus \{\infty\}}) \simeq \mathbb{F}_p[h, y_1^p, \dots, y_n^p] \subset D_{Y, \mathbb{S} \setminus \{\infty\}}.$$

We have $D_{Y, 0, \mathbb{S} \setminus \{\infty\}} := D_{Y, \mathbb{S} \setminus \{\infty\}} \otimes_{Z(D_{Y, \mathbb{S} \setminus \{\infty\}})} \mathbb{F}_p[h]$ and so

$$D_{Y, 0, \mathbb{S} \setminus \{\infty\}} \simeq \mathbb{F}_p[h] \langle x_i, y_j \rangle_{1 \leq i, j \leq n} / ([x_i, x_j] = [y_i, y_j] = y_j^p = x_i^p = 0, [y_j, x_i] = h \cdot \delta_{ij}).$$

Finally, $D_{Y, 0} := D_Y \otimes_{Z(D_Y)} \mathcal{O}(Y)^{\{p\}}$ is described as

$$D_{Y, 0} := \mathbb{F}_p \langle x_i, y_j \rangle_{1 \leq i, j \leq n} / ([x_i, x_j] = [y_i, y_j] = x_i^p = y_j^p = 0, [y_j, x_i] = \delta_{ij}).$$

Following Remark 2.29 the natural action of $D_{Y, 0}$ on C (where x_i acts by multiplication and y_i by ∂_i) produces an isomorphism

$$D_{Y, 0} \simeq \text{End}_{\mathbb{F}_p}(C) \simeq \text{Mat}_{p^n}(\mathbb{F}_p).$$

Remark 3.17 (Explicit description of $\mathcal{A}_{\mathbb{S}}$). Now, following Example 3.14 we can explicitly describe $\mathcal{A}_{\mathbb{S}}$. Let $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}$ and $\mathcal{A}_{\mathbb{S} \setminus \{0\}}$ be the restriction of $\mathcal{A}_{\mathbb{S}}$ to the corresponding charts. We have $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}} \simeq \mathcal{D}_{Y, 0, \mathbb{S} \setminus \{\infty\}}$, and so the $\mathbb{F}_p[h]$ -algebra $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}} := \Gamma(\mathbb{S} \setminus \{\infty\}, \mathcal{A}_{\mathbb{S} \setminus \{\infty\}})$ is described as

$$\mathcal{A}_{\mathbb{S} \setminus \{\infty\}} \simeq \mathbb{F}_p[h] \langle x_i, y_j \rangle_{1 \leq i, j \leq n} / ([x_i, x_j] = [y_i, y_j] = y_j^p = x_i^p = 0, [y_j, x_i] = h \cdot \delta_{ij}).$$

We note that $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}$ is a free $\mathbb{F}_p[h]$ -module of rank p^{2n} (by ordered monomials $x^I y^J$ with the degrees less or equal than $p-1$ in each variable).

On the other hand, $\mathcal{A}_{\mathbb{S} \setminus \{0\}} \simeq \mathcal{D}_{Y, 0, \mathbb{S} \setminus \{0\}} \simeq \mathcal{D}_{Y, 0}[h^{-1}]$, and so by the above the $\mathbb{F}_p[h^{-1}]$ -algebra $\mathcal{A}_{\mathbb{S} \setminus \{0\}} := \Gamma(\mathbb{S} \setminus \{0\}, \mathcal{A}_{\mathbb{S} \setminus \{0\}})$ is given by

$$\mathcal{A}_{\mathbb{S} \setminus \{0\}} \simeq \mathbb{F}_p[h^{-1}] \langle x'_i, y'_j \rangle_{1 \leq i, j \leq n} / ([x'_i, x'_j] = [y'_i, y'_j] = x_i'^p = y_j'^p = 0, [y'_j, x'_i] = \delta_{ij}).$$

Similarly, this is a free $\mathbb{F}_p[h^{-1}]$ -module of rank p^{2n} . It follows that $\mathcal{A}_{\mathbb{S}}$ is locally free as an $\mathcal{O}_{\mathbb{S}}$ -module of rank p^{2n} . The identification $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}[h^{-1}] \simeq \mathcal{A}_{\mathbb{S} \setminus \{0\}}[h]$ is given by $x_i = x'_i$ and $y_i = h \cdot y'_i$ (since y_i and y'_i in fact corresponds to $h \cdot \partial_i$ and ∂_i).

Remark 3.18. Since $D_{Y, 0} \simeq \text{Mat}_{p^n}(\mathbb{F}_p)$ we get that $\mathcal{A}_{\mathbb{S} \setminus \{0\}} \simeq \text{Mat}_{p^n}(\mathbb{F}_p)[h^{-1}]$. Also, $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}$ embeds into $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}[h^{-1}] \simeq \mathcal{A}_{\mathbb{S} \setminus \{0\}}[h] \simeq \text{Mat}_{p^n}(\mathbb{F}_p)[h, h^{-1}]$, and so $\mathcal{A}_{\mathbb{S}}$ is an order in the matrix algebra of rank p^n over \mathbb{S} .

Construction 3.19 (A module $\mathcal{V}_{\mathbb{S}}$ over $\mathcal{A}_{\mathbb{S}}$). We define a sheaf $\mathcal{V}_{\mathbb{S}}$ of $\mathcal{A}_{\mathbb{S}}$ -modules as follows. As an $\mathcal{O}_{\mathbb{S}}$ -module $\mathcal{V}_{\mathbb{S}}$ is free:

$$\mathcal{V}_{\mathbb{S}} = \mathcal{O}_{\mathbb{S}} \otimes \mathbb{F}_p[x_1, \dots, x_n] / (x_i^p, 1 \leq i \leq n).$$

The action of $x_i \in \Gamma(\mathbb{S}, \mathcal{A}_{\mathbb{S}})$ on $\mathcal{V}_{\mathbb{S}}$ is given by the multiplication on the right factor; the action of $y_i \in \Gamma(\mathbb{S} \setminus \{\infty\}, \mathcal{A}_{\mathbb{S}})$ is given by the formula $y_i(f) = h \otimes \frac{\partial f}{\partial x_i}$, for $f \in \mathbb{F}_p[x_1, \dots, x_n] / (x_i^p, 1 \leq i \leq n)$.

3.5. **Algebra A_0 .** Let $i_0: \{0\} \simeq \text{Spec } \mathbb{F}_p \rightarrow \mathbb{S}$ be an embedding of 0.

Definition 3.20. We define the \mathbb{F}_p -algebra A_0 to be $i_0^* \mathcal{A}_{\mathbb{S}}$. Equivalently, $A_0 \simeq \mathcal{A}_{\mathbb{S} \setminus \{\infty\}}/h$ and so (using Remark 3.17)

$$A_0 \simeq \mathbb{F}_p[x_i, y_j]_{1 \leq i, j \leq n} / (x_i^p = y_j^p = 0).$$

Let A_h be the h -adic completion of the $k[[h]]$ -algebra $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}$. Since $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}$ is finitely-generated as a $k[[h]]$ -module one has $A_h \simeq \mathcal{A}_{\mathbb{S} \setminus \{\infty\}} \otimes_{k[[h]]} k[[h]]$.

For an \mathbb{F}_p -algebra R we denote by $A_0(R)$ the R -algebra $A_0 \otimes_{\mathbb{F}_p} R$. Note that $\text{Spec } A_0(R)$ is pseudo-smooth over R for any R .

Construction 3.21. We denote by $\text{ev}_0: A_0 \rightarrow \mathbb{F}_p$ the ‘‘evaluation at 0’’-map that sends all x_i and y_j to 0. We let $I_0 := \text{Ker}(\text{ev}_0) \subset A_0$ be the ideal given by its kernel. Obviously, $I_0 = (x_1, \dots, x_n, y_1, \dots, y_n)$. For any R we put $I_0(R) := I_0 \otimes_{\mathbb{F}_p} R \subset A_0(R)$. Note that $I_0(R) \subset \alpha_p(A_0(R))$.

Let us now apply some of the results of Section 2 to $S := \text{Spec } R$ and $X := \text{Spec } A_0(R)$. Following Example 2.11(2) we have $A_0(R)^{\{p\}} \simeq R \subset A_0(R)$; this way the reduced Frobenius map $F_X^{\{p\}}: X \rightarrow X^{\{p\}}$ is identified with the projection $X \rightarrow S$. On the other hand, since $x_i^p = y_j^p = 0$ the reduced twist map $W_{A_0(R)}^{\{p\}}: A_0(R) \rightarrow R$ induced by $a \mapsto a^p \in R \simeq A_0(R)^{\{p\}}$ is identified with the composition $A_0(R) \xrightarrow{\text{ev}_0} R \xrightarrow{(-)^p} R$, $a \mapsto \text{ev}_0(a)^p$.

Consider the $A_0(R)$ -module $\Omega_{A_0(R)}^1 := \Omega_{A_0(R)/R}^1$ of R -relative differential 1-forms. $\Omega_{A_0(R)}^1$ is a free $A_0(R)$ -module of rank $2n$ spanned by dx_i and dy_i for $i = 1, \dots, n$. Consider also the R -module $\Omega_{A_0(R)}^{1, \{p\}} := \Omega_{A_0(R)}^1 \otimes_{A_0(R), \text{ev}_0^p} R$. We let $\Omega_{A_0(R)}^{1, cl} \subset \Omega_{A_0(R)}^1$ be the R -module of closed 1-forms. We have the de Rham differential $A_0(R) \xrightarrow{d} \Omega_{A_0(R)}^{1, cl}$ and the Cartier operation $\mathbb{C}: \Omega_{A_0(R)}^{1, cl} \rightarrow \Omega_{A_0(R)}^{1, \{p\}}$.

Lemma 3.22. *There is a short exact sequence of R -modules*

$$0 \rightarrow A_0(R)/R \xrightarrow{d} \Omega_{A_0(R)}^{1, cl} \xrightarrow{\mathbb{C}} \Omega_{A_0(R)}^{1, \{p\}} \rightarrow 0$$

Proof. This follows by applying the global sections functor to the Cartier short exact sequence (2.3) on the affine R -scheme $X = \text{Spec } A_0(R)$. \square

Remark 3.23. Under an isomorphism

$$\Omega_{A_0(R)}^{1, \{p\}} \simeq (\oplus_{i=1}^n R \cdot dx_i) \oplus (\oplus_{i=1}^n R \cdot dy_i),$$

the Cartier operation has the following description. Namely, given $\alpha \in \Omega_{X/R, cl}^1$ we can write it as a linear combination of monomials in x_i, y_j times dx_k, dy_l with coefficients in R and look at the coefficients $a_i \in R$ and $b_j \in R'$ in front of summands of the form $x_i^{p-1} dx_i$ and $y_j^{p-1} dy_j$. Then $\mathbb{C}(\alpha) = \sum_i a_i \cdot dx_i + \sum_j b_j \cdot dy_j \in \Omega_{A_0(R)}^{1, \{p\}}$.

We can also consider maps $d \log: A_0(R)^\times \rightarrow \Omega_{A_0(R)}^{1, cl}$ and the map $-^{\{p\}}: \Omega_{A_0(R)}^{1, cl} \rightarrow \Omega_{A_0(R)}^{1, \{p\}}$ sending α to $\alpha \otimes 1 \in \Omega_{A_0(R)}^{1, \{p\}}$.

Lemma 3.24. *There is a short exact sequence*

$$(3.4) \quad 0 \rightarrow A_0(R)^\times / R^\times \xrightarrow{d \log} \Omega_{A_0(R)}^{1, cl} \xrightarrow{\alpha \mapsto \mathbb{C}(\alpha) - \alpha^{\{p\}}} \Omega_{A_0(R)}^{1, \{p\}} \rightarrow 0.$$

Proof. Let $X := \text{Spec } A_0(R)$. By Example 2.11, $X^{[p]} \simeq S := \text{Spec } R$. Recall Milne's exact sequence (Proposition 2.48). We claim that $H_{\text{et}}^1(S, F_{X^*}^{[p]} \mathcal{O}_X^\times / \mathcal{O}_S^\times) = 0$. Indeed, by Remark 3.25 below, $F_{X^*}^{[p]} \mathcal{O}_X^\times$ splits as $\mathcal{O}_S^\times \times (1 + I_0)^\times$ and $H_{\text{et}}^1(S, (1 + I_0)^\times) = 0$. The other terms in Milne's exact sequence are coherent sheaves, and so also don't have higher cohomology. Thus, applying global sections to 2.48 we exactly get (3.4) above. \square

Remark 3.25. Recall the notation $I_0(R) := \text{Ker}(\text{ev}_0)(R)$. Consider the subgroup $(1 + I_0(R))^\times \subset A_0(R)^\times$: it is well-defined since $I_0(R)$ lies in the nilradical of $A_0(R)$, and identifies with the kernel of the map $A_0(R)^\times \rightarrow R^\times$ induced by ev_0 . Together with the map $R^\times \rightarrow A_0(R)^\times$ this gives a splitting $A_0(R)^\times \simeq R^\times \times (1 + I_0(R))^\times$ functorially in R . Moreover, $(1 + I_0(R))^\times$ has a (functorial in R) finite filtration by

$$(1 + I_0(R))^\times \supset (1 + I_0(R)^2)^\times \supset (1 + I_0(R)^3)^\times + \dots$$

with $(1 + I_0(R)^n)/(1 + I_0(R)^{n+1}) \simeq I_0(R)^n/I_0(R)^{n+1}$, which is a free R -module spanned by monomials in x_i 's and y_j 's of degree n .

This shows that the étale sheaf $(1 + I_0)^\times: R \mapsto (1 + I_0(R))^\times$ on the big site of affine \mathbb{F}_p -schemes has a finite filtration with the associated graded given by a coherent sheaf. In particular, for any specific $\text{Spec } R$ the restriction of this sheaf to the small étale site $(\text{Spec } R)_{\text{et}}$ doesn't have higher cohomology: $H_{\text{et}}^{>0}(\text{Spec } R, (1 + I_0)^\times) = 0$.

Construction 3.26 (Restricted structure on the Frobenius neighborhood). We endow $X = \text{Spec } A_0(R)$ with the structure of a restricted symplectic R -scheme as in Construction 2.67. Namely, we consider $(X, [\eta_{\text{can}}])$ with

$$\eta_{\text{can}} := \sum_{i=1}^n y_i dx_i \in \Omega_{A_0(R)}^1.$$

The corresponding symplectic form is given by $\omega := d\eta_{\text{can}} = \sum_i dy_i \wedge dx_i \in \Omega_{A_0(R)}^2$.

For the lemma that will follow shortly we will need the following generalization of Definition 2.60.

Definition 3.27. Let X be a symplectic S -scheme. A derivation $\xi \in H^0(X, \mathcal{T}_X)$ is called *log-Hamiltonian* if $\xi = f^{-1} \cdot H_f$ for some $f \in H^0(X, \mathcal{O}_X^\times)$.

Remark 3.28. Similarly to the Hamiltonian case, the condition $\xi = f^{-1} \cdot H_f$ is equivalent to $\iota_\xi \omega = d \log f$.

Given a vector field $\xi \in H^0(X, \mathcal{T}_X)$ we denote by $L_\xi: \Omega_X^i \rightarrow \Omega_X^i$ the Lie derivative along ξ . We recall the Cartan formula: $L_\xi = \iota_\xi \circ d + d \circ \iota_\xi$ in terms of the de Rham differential and contraction. A vector field ξ is called *Poisson* if $L_\xi \omega = 0$.

Lemma 3.29. (1) A Poisson vector field $\xi \in \mathcal{T}_{A_0(R)}$ is Hamiltonian if and only if for all $f \in A_0(R)$ we have

$$\xi(f^{[p]}) = H_f^{p-1}(\xi(f)).$$

(2) A Poisson derivation $\xi \in \mathcal{T}_{A_0(R)}$ is log-Hamiltonian if and only if for all $f \in A$ we have

$$(3.5) \quad \xi(f^{[p]}) = H_f^{p-1}(\xi(f)) + \xi(f)^p.$$

Proof. The proof essentially follows [BK, Lemma 2.5]. By Cartan formula and the assumption $L_\xi\omega = 0 \Rightarrow d(\iota_\xi\omega) + 0 = 0$ we get that $\beta := \iota_\xi\omega \in \Omega_{A_0(R)}^1$ is closed. By Remark 2.61 and Lemma 3.22 ξ is Hamiltonian if and only if $C(\beta) = 0$. Similarly, by Lemma 3.24 ξ is log-Hamiltonian if and only if $C(\beta) = \beta^{\{p\}}$. Since ω is non-degenerate, Hamiltonian vector fields generate $\mathcal{T}_{A_0(R)}$. Then vector fields of the form $H_f^{\{p\}}$ generate $\mathcal{T}_{A_0(R)}^{\{p\}}$, and so it is enough to understand the pairing of $C(\beta) \in \Omega_{A_0(R)}^{1,\{p\}}$ with (the twists of) Hamiltonian vector fields. By Remark 2.41 $\iota_{H_f^{\{p\}}}(C(\beta)) = \iota_{H_f^{\{p\}}}\beta - H_f^{p-1}(\iota_{H_f}\beta)$. Substituting $\beta = \iota_\xi\omega$ we get $\iota_{H_f^{\{p\}}}(C(\beta)) = \xi(f^{\{p\}}) - H_f^{p-1}(\xi(f)) \in R \subset A_0(R)$. Thus, $C(\beta) = 0$ if and only if $\xi(f^{\{p\}}) - H_f^{p-1}(\xi(f)) = 0$ for all $f \in A_0(R)$. This gives Part 1. Part 2 is obtained similarly, by noting that $\iota_{H_f^{\{p\}}}(\beta^{\{p\}}) = (\iota_{H_f}\beta)^{\{p\}} = \xi(f)^p$. \square

3.6. h -separable algebras. Let $\text{Sch}_{\mathbb{S}}$ be the category of schemes over \mathbb{S} . To construct a group action on the category of $\mathcal{A}_{\mathbb{S}}$ -modules it will be crucial to have some understanding of the algebra automorphisms of $\mathcal{A}_{\mathbb{S}}$. This will become easier if we restrict to a certain subcategory in $\text{Sch}_{\mathbb{S}}$, where automorphisms of $\mathcal{A}_{\mathbb{S}}$ are closely related to G_0 . The goal of this section is to define this subcategory.

Construction 3.30. Let $R \in \text{CAlg}_{\mathbb{F}_p[h]/}$. For an element $b \in R/h$ let $\tilde{b} \in R$ be a lift. Then the operation $b \mapsto \tilde{b}^p$ gives a well-defined map $R/h \rightarrow R/h^p$. Indeed, given another lift $\tilde{b}' = \tilde{b} + hc$ we have $(\tilde{b}')^p = \tilde{b}^p + h^p c^p$.

Consider $\alpha_p(R/h) = \{a \in R/h \mid a^p = 0\} \subset R/h$. Then restricting the above map to $\alpha_p(R/h)$ gives a map $\delta: \alpha_p(R/h) \rightarrow hR/h^p R$.

Definition 3.31. An $\mathbb{F}_p[h]$ -algebra R is called h -separable if

- (1) it is h -torsion free (i.e., multiplication by h is injective on R);
- (2) the map $\delta_R: \alpha_p(R/h) \rightarrow hR/h^p R$ is equal to 0.

Remark 3.32. It is not hard to see that the latter condition is equivalent to the following: for any $a \in R$ one has

$$a^p \in hR \iff a^p \in h^p R.$$

We denote by $\text{CAlg}_{\mathbb{F}_p[h]/}^{h\text{-sep}}$ the category of h -separable $\mathbb{F}_p[h]$ -algebras.

Example 3.33. (1) Let C be an \mathbb{F}_p -algebra. Then $R := C \otimes_{\mathbb{F}_p} \mathbb{F}_p[h]$ is h -separable.

(2) Any $\mathbb{F}_p[h, h^{-1}]$ -algebra R is h -separable. Indeed, it is obviously h -torsion free and the second condition is vacuous since $R/h = 0$.

(3) $R := \mathbb{F}_p[h^{1/p}]$ is not h -separable. Indeed, $R/h \simeq \mathbb{F}_p[x]/x^p$ where x is the class of $h^{1/p}$, and the map δ_R sends x to $h \in hR/h^p R$.

(4) More generally, let K be any extension of $\mathbb{F}_p(h)$ ramified at 0. Let $R \subset K$ be the integral closure of $\mathbb{F}_p[h]$ inside K . Then R is not h -separable.

Note that $\alpha_p(R/h)$ has a natural structure of R -module such that δ_R is a Frobenius-linear map of R -modules. Let $\delta_R^\ell: \alpha_p(R/h) \otimes_{R, F_R} R \rightarrow hR/h^p R$ be its linearization. Obviously, $\delta_R = 0 \iff \delta_R^\ell = 0$.

Even though it might not yet be clear what is the motivation behind the notion h -separability, let us show that it is local in étale topology:

Lemma 3.34. *Let R be an $\mathbb{F}_p[h]$ -algebra and let $R \rightarrow R'$ be an étale cover. Then R is h -separable if and only if R' is.*

Proof. R' is faithfully flat over R , and so R is h -torsion free if and only if so is R' . It remains to relate δ_R and $\delta_{R'}$. Since $R \rightarrow R'$ is étale, the relative Frobenius $F_{R'/R}$ is an isomorphism. We have a factorization

$$\begin{array}{ccccc} R & \xrightarrow{\text{id}_R} & R & \xrightarrow{F_R} & R \\ \downarrow & & \downarrow & & \downarrow \\ R' & \xrightarrow{F_{R'/R}} & R' & \xrightarrow{W_{R'/R}} & R' \\ & \searrow & & \nearrow & \\ & & & & F_{R'} \end{array}$$

of $F_{R'}$ as the composition $W_{R'/R} \circ F_{R'/R}$ (recall that $W_{R'/R}$ by definition is the pull-back of F_R to R'). Identifying $\alpha_p(B)$ with the kernel of F_B and the diagram above, it is not hard to see that the linearized map $\delta_{R'}^\ell: \alpha_p(R'/h) \otimes_{R', F_{R'}} R' \rightarrow hR'/h^p R'$ is identified with $F_{R'/R}^{-1}$ applied to the base change

$$\delta_R^\ell \otimes R': \alpha_p(R/h) \otimes_{R, F_R} R \otimes_R R' \rightarrow hR/h^p R \otimes_R R'.$$

Thus, $\delta_{R'}^\ell = 0 \Leftrightarrow \delta_R = 0$. □

Remark 3.35. Let R be an h -separable $k[h]$ -algebra. Then for any $a \in R/h$ the image $\tilde{a}^p \in R/h^p$ only depends on the p -th power $a^p \in R/h$. Indeed, if $a, b \in R/h$ are such that $a^p = b^p$, then $a - b \in \alpha_p(R/h)$ and $\tilde{a}^p - \tilde{b}^p = \delta_R(a - b) = 0$, so $\tilde{a}^p = \tilde{b}^p$. In other words, the map $(\tilde{})^p: R/h \rightarrow R/h^p$ factors through the image of the absolute Frobenius on R/h . This gives a well-defined map

$$s: (R/h)^p \simeq (R/h)/\alpha_p(R/h) \rightarrow R/h^p.$$

Definition 3.36. An \mathbb{S} -scheme $X \rightarrow \mathbb{S}$ is called *h -separable* if its base change to $\mathbb{S} \setminus \{\infty\}$ has a Zariski cover by $\{U_i := \text{Spec } R_i\}$ where each R_i is an h -separable $\mathbb{F}_p[h]$ -algebra.

Example 3.37. (1) If the map $X \rightarrow \mathbb{S}$ factors through $\mathbb{S} \setminus \{0\}$ then X is h -separable. Indeed, in this case $X_{\mathbb{S} \setminus \{0\}}$ can be covered by spectra of $\mathbb{F}[h, h^{-1}]$ -algebras that are h -separable.

(2) Let R be an h -separable $k[h]$ -algebra. Then the composition $\text{Spec } R \rightarrow \mathbb{S} \setminus \{\infty\} \rightarrow \mathbb{S}$ defines an h -separable \mathbb{S} -scheme.

We consider the big site $\text{Sch}_{/\mathbb{S}}^{h\text{-sep}} \subset \text{Sch}_{/\mathbb{S}}$ of all h -separable schemes over \mathbb{S} endowed with the étale topology²².

Remark 3.38. Any \mathbb{S} -scheme that lives over $\mathbb{S} \setminus \{0\}$ is h -separable, and so we have a natural embedding $\text{Sch}_{/\mathbb{S} \setminus \{0\}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$. We also have a fully faithful embedding $(\text{CAlg}_{\mathbb{F}_p[h]}^{h\text{-sep}})^{\text{op}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ with the image given by those separable h -schemes that live over $\mathbb{S} \setminus \{\infty\}$ and are affine. Moreover, any $(X \rightarrow \mathbb{S}) \in \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ has a Zariski cover $U_i \rightarrow X$ such that each $U_i \rightarrow \mathbb{S}$ belongs to one of the two subcategories.

²²In fact for all the computations we will only need to use Zariski topology, but we just wanted to point out that étale topology also make sense in this setup.

The functor $\mathcal{A}_{\mathbb{S}}: (f: X \rightarrow \mathbb{S}) \mapsto \Gamma(X, f^*\mathcal{A}_{\mathbb{S}})$ defines a sheaf on $\text{Aff}_{/\mathbb{S}}^{h\text{-sep}}$. Its restriction to $(\text{CAlg}_{\mathbb{F}_p[h]})^{\text{op}} \subset \text{Aff}_{/\mathbb{S}}^{h\text{-sep}}$ is given by

$$R \mapsto A_{\mathbb{S}\setminus\{\infty\}}(R) := A_{\mathbb{S}\setminus\{\infty\}} \otimes_{k[h]} R,$$

(see Remark 3.17 to recall the definition of $A_{\mathbb{S}\setminus\{\infty\}}$). Explicitly,

$$A_{\mathbb{S}\setminus\{\infty\}}(R) \simeq R\langle x_i, y_j \rangle_{1 \leq i, j \leq n} / ([x_i, x_j] = [y_i, y_j] = y_j^p = x_i^p = 0, [y_j, x_i] = h \cdot \delta_{ij}).$$

Note that the underlying R -module of $A_{\mathbb{S}\setminus\{\infty\}}(R)$ is free of rank p^{2n} .

Taking the fiber $(X \rightarrow \mathbb{S}) \mapsto X_{\{0\}}$ over $\{0\} \in \mathbb{S}$ defines a map of sites $i_0^{-1}: \text{Sch}_{/\mathbb{S}}^{h\text{-sep}} \rightarrow \text{Sch}_{/\mathbb{F}_p}$. On $(\text{CAlg}_{\mathbb{F}_p[h]})^{\text{op}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ this functor is given by sending an $\mathbb{F}_p[h]$ -algebra R to $\text{Spec } R/h$. Consider the sheaf A_0 on $\text{Sch}_{/\mathbb{F}_p}$ sending $X \mapsto \Gamma(X, A_0 \otimes_{\mathbb{F}_p} \mathcal{O}_X)$ and the corresponding pushforward $i_{0*}A_0$. Its restriction to $\text{Sch}_{/\mathbb{S}\setminus\{0\}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ is 0, while on $(\text{CAlg}_{\mathbb{F}_p[h]})^{\text{op}} \subset \text{Aff}_{/\mathbb{S}}^{h\text{-sep}}$ it is given by $R \mapsto A_0(R/h)$.

We have a natural map $\mathcal{A}_{\mathbb{S}} \rightarrow i_{0*}A_0$ (induced by the isomorphism $A_0 \simeq i_0^*\mathcal{A}_{\mathbb{S}}$ as a quasi-coherent sheaf on \mathbb{S}). Its restriction to $\text{Sch}_{/\mathbb{S}\setminus\{0\}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ is the zero map $\mathcal{A}_{\mathbb{S}} \rightarrow 0$, while on $(\text{CAlg}_{\mathbb{F}_p[h]})^{\text{op}} \subset \text{Aff}_{/\mathbb{S}}^{h\text{-sep}}$ it is given by reduction modulo h

$$A_{\mathbb{S}\setminus\{\infty\}}(R) \rightarrow A_{\mathbb{S}\setminus\{\infty\}}(R)/h \simeq A_0(R/h).$$

3.7. Automorphisms of $\mathcal{A}_{\mathbb{S}}$. We let $\underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})$ be the sheaf on big fpqc site of \mathbb{S} which sends $f: X \rightarrow \mathbb{S}$ to algebra automorphisms of $f^*\mathcal{A}_{\mathbb{S}}$. Given the explicit description of $\mathcal{A}_{\mathbb{S}}$ in Remark 3.17 one sees that $\underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})$ is represented by a finite type group scheme over \mathbb{S} . Since by definition $A_0 \simeq i_0^*\mathcal{A}_{\mathbb{S}}$ (as quasi-coherent sheaves) we get a natural “reduction mod h ”-map $i_0^*: \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}) \rightarrow i_{0*}(\underline{\text{Aut}}(A_0))$.

Recall the group subscheme $G_0 \subset \underline{\text{Aut}}(A_0)$ of restricted Poisson automorphisms of $(\text{Spec } A_0, [\eta_{\text{can}}])$ (Construction 2.68). The first goal of this section is to show that if we restrict to $\text{Sch}_{/\mathbb{S}}^{h\text{-sep}} \subset \text{Sch}_{/\mathbb{S}}$ the map $i_0^*: \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}) \rightarrow i_{0*}(\underline{\text{Aut}}(A_0))$ above factors through $i_{0*}G_0$.

The key is to reinterpret the restricted Poisson structure on $A_0(R/h)$ in terms of $A_{\mathbb{S}\setminus\{\infty\}}(R)$. Note that (since $A_0 := i_0^*\mathcal{A}_{\mathbb{S}}$) we have $A_0(R/h) \simeq A_{\mathbb{S}\setminus\{\infty\}}(R)/h$.

Remark 3.39 (Poisson structure). Let R be h -torsion free. Then $A_{\mathbb{S}\setminus\{\infty\}}(R)$ is h -torsion free as well, and we can identify $h^i A_{\mathbb{S}\setminus\{\infty\}}(R)/h^{i+1} A_{\mathbb{S}\setminus\{\infty\}}(R) \simeq A_0(R/h)$ (via dividing by h^i) for any $i \geq 0$. For an element $f \in A_0(R/h)$ let $\tilde{f} \in A_{\mathbb{S}\setminus\{\infty\}}(R)$ denote a lift under the projection $A_{\mathbb{S}\setminus\{\infty\}}(R) \twoheadrightarrow A_0(R/h)$. One can define a Poisson bracket $\{-, -\}$ on $A_0(R/h)$ by the equation $h\{f, g\} = [\tilde{f}, \tilde{g}] \pmod{h^2}$. One sees immediately from the relations defining $A_{\mathbb{S}\setminus\{\infty\}}(R)$ that $\{x_i, x_j\} = \{y_i, y_j\} = 0$, while $\{y_j, x_i\} = \delta_{ij}$ and so this Poisson structure agrees with the one defined by ω (see Construction 3.26).

The restricted p -th power operation $-^{[p]}$ (corresponding to the restricted Poisson structure on $A_0(R)$) is more subtle to identify, and this is where h -separability of R is important.

Construction 3.40 (Restricted structure). Note that for any $f \in A_0(R/h)$ and any lift $\tilde{f} \in A_{\mathbb{S}\setminus\{\infty\}}(R)$ the expression \tilde{f}^p is well-defined mod h^p . Indeed, by the

Jacobson's formula (see Section 2.2) we have

$$(\tilde{f} + h\tilde{g})^p = \tilde{f}^p + h^p\tilde{g}^p + \sum_{i=1}^{p-1} h^i L_i(\tilde{f}, \tilde{g}),$$

where each $L_i(\tilde{f}, \tilde{g})$ is a Lie polynomial of total degree $p-1$. Since the commutator $[A_{\mathbb{S}\setminus\{\infty\}}(R), A_{\mathbb{S}\setminus\{\infty\}}(R)] \subset hA_{\mathbb{S}\setminus\{\infty\}}(R)$ we get that $h^i L_i(\tilde{f}, \tilde{g}) \in h^{p-1+i}A_{\mathbb{S}\setminus\{\infty\}}(R)$, and, consequently, that $(\tilde{f} + h\tilde{g})^p = \tilde{f}^p \pmod{h^p}$.

Recall the map $\mathfrak{s}: (R/h)^p \rightarrow R/h^p$ from Remark 3.35. We will denote by \mathfrak{s} its composition with the embedding $R/h^p \hookrightarrow A_{\mathbb{S}\setminus\{\infty\}}(R)/h^p$. The p -restricted structure then appears as follows, namely for any $f \in A_0(R/h)$ and a lift $\tilde{f} \in A_{\mathbb{S}\setminus\{\infty\}}(R)$ one can show as in Lemma 2.53 that the difference $\tilde{f}^p - \mathfrak{s}(\text{ev}_0(f)^p)$ is divisible by h^{p-1} and define

$$h^{p-1} \cdot f^{[p]} := \tilde{f}^p - \mathfrak{s}(\text{ev}_0(f)^p) \pmod{h^p}.$$

Here we implicitly identified $h^{p-1}A_{\mathbb{S}\setminus\{\infty\}}(R)/h^p A_{\mathbb{S}\setminus\{\infty\}}(R) \simeq A_0(R/h)$ and considered $\mathfrak{s}(\text{ev}_0(f)^p)$ as an element of $A_{\mathbb{S}\setminus\{\infty\}}(R)/h^p$. By the same argument as in Lemma 2.53, one sees that this defines a restricted Poisson structure on $A_0(R/h)$.

Remark 3.41. It is important to note that the map $\mathfrak{s}(\text{ev}_0(-)^p): A_0(R/h) \rightarrow R/h^p$ given by $f \mapsto \mathfrak{s}(\text{ev}_0(f)^p) \in R/h^p$ is preserved²³ by any R/h -linear automorphism of $A_0(R/h)$. Indeed, $\text{Aut}_{R/h}(A_0(R/h))$ acts trivially on $A_0(R/h)^{\{p\}} \simeq R$, and $\text{ev}_0(f)^p$ is identified with the reduced Frobenius map $F_{A_0(R/h)}^{\{p\}}: A_0(R/h) \rightarrow A_0(R/h)^{\{p\}}$ which commutes with the action by automorphisms.

Lemma 3.42. *The restricted structures on $A_0(R/h)$ from Constructions 3.40 and 3.26 coincide.*

Proof. Once we know that both constructions give restricted Poisson structures it is enough to check that they act in the same way on the algebra generators x_i and y_i . We claim that in both cases $x_i^{[p]} = y_i^{[p]} = 0$. Indeed, for 3.40 this follows from $\text{ev}_0(x_i) = \text{ev}_0(y_j) = 0$. For 3.26, note that $H_{y_j} = \partial_{x_j}$ and $H_{x_i} = -\partial_{y_j}$. In both cases $H_f^{[p]} = 0$, and so we need to check that

$$H_{x_i}^{p-1}(\iota_{H_{x_i}} \eta) = H_{y_i}^{p-1}(\iota_{H_{y_i}} \eta) = 0.$$

We leave this as an exercise to the reader. \square

Lemma 3.43. *Let R be an h -separable \mathbb{F}_p -algebra. If ϕ is an R -linear algebra automorphism of $A_{\mathbb{S}\setminus\{\infty\}}(R)$, then the reduction of ϕ modulo h preserves the restricted Poisson structure on $A_0(R/h)$. In other words, $\phi_0 \in G_0(R/h)$.*

Proof. By Remark 3.39

$$h \cdot \{\phi_0(f), \phi_0(g)\} = [\phi(\tilde{f}), \phi(\tilde{g})] = \phi([\tilde{f}, \tilde{g}]) = h \cdot \phi_0(\{f, g\}) \pmod{h^2}$$

thus ϕ_0 preserves Poisson structure. Similarly,

$$h^{p-1}(\phi_0(f))^{[p]} = (\widetilde{\phi_0(f)})^p - \mathfrak{s}(\text{ev}_0(\phi_0(f))^p) = \phi(\tilde{f}^p - \mathfrak{s}(\text{ev}_0(f)^p)) = h^{p-1}\phi_0(f^{[p]}) \pmod{h^p},$$

so ϕ_0 also preserves the restricted structure. \square

Corollary 3.44. *The map $i_0^*: \underline{\text{Aut}}(A_{\mathbb{S}}) \rightarrow i_{0*}\underline{\text{Aut}}(A_0)$ of sheaves on $\text{Sch}_{\mathbb{S}}^{h\text{-sep}}$ factors through $i_{0*}G_0$.*

²³Meaning for any $\phi \in \text{Aut}_{R/h}(A_0(R/h))$ and $f \in A_0(R/h)$ we have $\mathfrak{s}(\text{ev}_0(f)^p) = \mathfrak{s}(\text{ev}_0(\phi(f))^p)$.

Proof. Indeed, by Remark 3.38 it is enough to check on the subcategories $\text{Sch}_{/\mathbb{S}\setminus\{0\}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ and $(\text{CAlg}_{\mathbb{F}_p[h]})^{\text{op}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$. The restriction of both $i_{0*}\underline{\text{Aut}}(A_0)$ and $i_{0*}G_0$ to the first subcategory is trivial and so the statement is trivial as well. For $R \in (\text{CAlg}_{\mathbb{F}_p[h]})^{\text{op}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ this is exactly the statement of Lemma 3.43. \square

Let $r: \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}) \rightarrow i_{0*}G_0$ denote the resulting map of sheaves on $\text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$. We are now going to describe the kernel of r . Note that $\mathcal{A}_{\mathbb{S}}^{\times}$ acts on $\mathcal{A}_{\mathbb{S}}$ by conjugation and that its image lands in this kernel. Once again, it is enough to see this separately for objects over $\mathbb{S}\setminus\{0\}$ or for spectra of h -separable $k[h]$ -algebras. The restriction of $\text{Ker}(r)$ to $\mathbb{S}\setminus\{0\}$ is the whole $\underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})$, so there is nothing to prove. For a $k[h]$ -separable algebra R the action of $\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)^{\times}$ on $\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)$ by conjugation is trivial modulo h since $[\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R), \mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)] \subset h \cdot \mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)$. Thus we have a natural map

$$\text{Ad}: \mathcal{A}_{\mathbb{S}}^{\times} \rightarrow \text{Ker}(r) \subset \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})$$

of Zariski sheaves on $\text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$. The map of algebras $\mathcal{O}_{\mathbb{S}} \rightarrow \mathcal{A}_{\mathbb{S}}$ induces a homomorphism $\mathbb{G}_m \rightarrow \mathcal{A}_{\mathbb{S}}^{\times}$ which factors through the invertible elements in the center $Z(\mathcal{A}_{\mathbb{S}})^{\times}$.

Proposition 3.45. *The map $\text{Ad}: \mathcal{A}_{\mathbb{S}}^{\times} \rightarrow \text{Ker}(r)$ is a surjection of Zariski sheaves on $\text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$. The kernel of $\text{Ad}(-)$ is canonically identified with $\mathbb{G}_m \subset \mathcal{A}_{\mathbb{S}}^{\times}$.*

Proof. Again, it is enough to prove the statement separately for the restrictions to $\text{Sch}_{/\mathbb{S}\setminus\{0\}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ and $(\text{CAlg}_{\mathbb{F}_p[h]})^{\text{op}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$.

For $\text{Sch}_{/\mathbb{S}\setminus\{0\}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ we have $\text{Ker}(r) \simeq \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}\setminus\{0\}})$ and we want to show that there is a short exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow \mathcal{A}_{\mathbb{S}\setminus\{0\}}^{\times} \rightarrow \text{Aut}(\mathcal{A}_{\mathbb{S}\setminus\{0\}}) \rightarrow 0.$$

By Remark 3.17 we can identify $\mathcal{A}_{\mathbb{S}\setminus\{0\}}$ with the matrix algebra Mat_{p^n} , and the sequence above turns into

$$0 \rightarrow \mathbb{G}_m \rightarrow \text{GL}_{p^n} \rightarrow \text{PGL}_{p^n} \rightarrow 0,$$

which is exact in Zariski topology.

The case of $(\text{CAlg}_{\mathbb{F}_p[h]})^{\text{op}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ is more involved. Let us first identify the kernel of Ad . By definition, it consists of the invertible elements $Z(\mathcal{A}_{\mathbb{S}})^{\times}$ of the center. Every element $a \in \mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)$ can be uniquely written as $\sum_{I,J} r_{I,J} x^I y^J$ with $r_{I,J} \in R$ (with $I = (i_1, \dots, i_n)$, $J = (j_1, \dots, j_n)$ and $0 \leq i_k, j_l \leq p-1$). Since R is h -torsion free, from the relations defining $\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)$ it is clear that if $r_{I,J} \neq 0$ for some non-zero I or J then there is x_i , or y_j such that one of the commutators $[a, x_i]$ or $[a, y_j]$ is non-zero. Thus $Z(\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R))^{\times} \simeq R^{\times}$ and $\text{Ker}(\text{Ad}(-)) \simeq \mathbb{G}_m$.

It remains to show that automorphisms of $\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)$ that are trivial modulo h are given by conjugation, at least after a Zariski localization $R \rightarrow R'$. We will first prove this after inverting h , then for h -completion of R , and then will obtain the result for R using Beauville-Laszlo gluing.

If $h \in R$ is invertible then $\text{Spec } R$ lies over $\text{Sch}_{/\mathbb{S}\setminus\{0\}}$, and we are done by the case $\text{Sch}_{/\mathbb{S}\setminus\{0\}} \subset \text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$.

If R is h -complete, then $\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)$ is also h -complete (since it is a free module of finite rank over R). We will approximate $\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)$ by $\mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)/h^n$ for all $n \geq 0$. As a base of induction let us show that any $\phi \in \text{Ker}(r)$ is equal to an inner automorphism modulo h^2 . In other words, we would like to find $x \in \mathcal{A}_{\mathbb{S}\setminus\{\infty\}}(R)^{\times}$ such that

$\phi = \text{Ad}(x) \pmod{h^2}$. Note that the map $\phi - \text{id} \pmod{h^2}: A_{\mathbb{S}\setminus\{\infty\}} \rightarrow A_{\mathbb{S}\setminus\{\infty\}}/h^2$ factors through $h \cdot A_{\mathbb{S}\setminus\{\infty\}}(R)$ and gives a well defined map $\xi_\phi: A_0(R/h) \simeq A_{\mathbb{S}\setminus\{\infty\}}/h \rightarrow hA_{\mathbb{S}\setminus\{\infty\}}/h^2A_{\mathbb{S}\setminus\{\infty\}} \simeq A_0(R/h)$, which is an R/h -linear derivation²⁴ of $A_0(R/h)$. For $\phi = \text{Ad}(x)$ with $x = x_0 \pmod{h}$ we have

$$h \cdot \xi_{\text{Ad}(x)}(a) = xax^{-1} - a = [x, a] \cdot x^{-1} = h \cdot \frac{1}{x_0} \{x_0, a\} \pmod{h^2},$$

and so one sees that $\xi_{\text{Ad}(x)} = \frac{1}{x_0} H_{x_0}$. Thus it is enough to show that for $\phi \in \text{Ker}(\mathfrak{r})$ the corresponding vector field ξ_ϕ is log-Hamiltonian.

Looking at the action on commutator modulo h^2 , one sees that ξ_ϕ is Poisson (namely, $L_{\xi_\phi}\omega = 0$). Thus, by Lemma 3.29(2) we need to show that ξ_ϕ satisfies equation (3.5). Moreover, it is enough to check (3.5) on $f \in I_0(R/h) = \{f \in A_0(R/h) \mid \text{ev}_0(f) = 0\}$: indeed, the constants R/h and $I_0(R/h)$ span all $A_0(R/h)$. By 3.40, for $f \in I_0(R/h)$ and any lift $\tilde{f} \in A_{\mathbb{S}\setminus\{\infty\}}(R)$ we have

$$\tilde{f}^p = h^{p-1} \cdot f^{[p]} \pmod{h^p}.$$

We then have

$$h^p \cdot \xi_\phi(f^{[p]}) = h^{p-1} \cdot (\phi(f^{[p]} - f^{[p]}) = (\tilde{f} + h \cdot \widetilde{\xi_\phi(f)})^p - \tilde{f}^p \pmod{h^{p+1}}.$$

Using Jacobson's formula (2.13) the right hand side is identified with $h^p(\xi_\phi(f)^p + L_1(\tilde{f}, \widetilde{\xi_\phi(f)})) \pmod{h^{p+1}}$. From this, using that $L_1(x, y) = \text{ad}(x)^{p-1}(y)$ (2.14) and identifying $A_0(R/h) \simeq h^p A_{\mathbb{S}\setminus\{\infty\}}(R)/h^{p+1} A_{\mathbb{S}\setminus\{\infty\}}(R)$, we get that

$$\xi_\phi(f^{[p]}) = \xi_\phi(f)^p + H_f^{p-1}(\xi_\phi(f)),$$

as desired.

For $n \geq 2$ one shows analogously that if $\phi: A_{\mathbb{S}\setminus\{\infty\}}(R) \rightarrow A_{\mathbb{S}\setminus\{\infty\}}(R)$ is an automorphism which is identity modulo h^n then there exists an element $x \in A_{\mathbb{S}\setminus\{\infty\}}(R)$ such that $\phi = \text{Ad}(1 + h^{n-1}x)$ modulo h^{n+1} . Namely, such ϕ also defines a derivation ξ'_ϕ of $A_0(R/h)$ by considering the map $\phi - \text{id} \pmod{h^{n+1}}: A_{\mathbb{S}\setminus\{\infty\}}(R) \rightarrow A_{\mathbb{S}\setminus\{\infty\}}(R)/h^{n+1}$ which factors through $A_h/h \simeq A_0$ and whose image lies in $h^n A_{\mathbb{S}\setminus\{\infty\}}(R)/h^{n+1} A_{\mathbb{S}\setminus\{\infty\}}(R) \simeq A_0$. As in the case of $n = 1$, looking at the action on the commutator one sees that ξ'_ϕ is Poisson. However, we now have

$$h^n \cdot \xi_{\text{Ad}(1+h^{n-1}x)}(a) = (1 + h^{n-1}x)a(1 - h^{n-1}x) = h^n \cdot [x_0, a] \pmod{h^{n+1}};$$

in other words, $\xi_{\text{Ad}(1+h^{n-1}x)} = H_{x_0}$. Also, given ϕ that is trivial modulo h^n with $n \geq 2$

$$h^{p+n-1} \cdot \xi_\phi(f^{[p]}) = h^{p-1} \cdot (\phi(f^{[p]} - f^{[p]}) = (\tilde{f} + h^n \cdot \widetilde{\xi_\phi(f)})^p - \tilde{f}^p \pmod{h^{p+n}},$$

from which (using Jacobson's formula again) we get

$$\xi_\phi(f^{[p]}) = H_f^{p-1}(\xi_\phi(f)),$$

so by Lemma 3.29(1) ξ_ϕ is Hamiltonian and ϕ agrees with $\text{Ad}(1 + h^{n-1}x)$ for some x .

This way, starting with an automorphism $\phi \in \text{Ker}(\mathfrak{r})$, we can construct a sequence of elements $x_1, x_2, x_3, \dots \in A_{\mathbb{S}\setminus\{\infty\}}(R)$ such that

$$\phi = \dots \circ \text{Ad}(1 + h^2 x_3) \circ \text{Ad}(1 + h x_2) \circ \text{Ad}(x_1) = \text{Ad}(\dots (1 + h^2 x_3)(1 + h x_2)x_1),$$

²⁴Indeed,

$$h \cdot \xi_\phi(xy) = \phi(\tilde{x} \cdot \tilde{y}) - \tilde{x} \cdot \tilde{y} = \phi(\tilde{x}) \cdot \phi(\tilde{y}) - \tilde{x} \cdot \tilde{y} = h \cdot (\xi_\phi(x) \cdot y + x \cdot \xi_\phi(y)) \pmod{h^2}.$$

where the infinite product on the right hand side is well-defined since R is h -complete.

Let now R be an arbitrary h -separable $k[h]$ -algebra. We are trying to show that the map of (Zariski) sheaves $\mathcal{A}_{\mathbb{S}}^{\times} \rightarrow \text{Ker}(r)$ on $\text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$ is surjective. It is enough to do so on stalks, and so we can assume R is local (and, in particular, that $\text{Pic}(R) = 0$). Since $A_{\mathbb{S}\setminus\{\infty\}}(R)$ is a free R -module of finite rank, by Beauville-Laszlo theorem [BL], an R -linear automorphism of $A_{\mathbb{S}\setminus\{\infty\}}(R)$ is uniquely encoded by a pair of automorphisms $\phi_1 \in \text{Aut}_{R_h^\wedge}(A_{\mathbb{S}\setminus\{\infty\}}(R_h^\wedge))$, $\phi_2 \in \text{Aut}_{R[h^{-1}]}(A_{\mathbb{S}\setminus\{\infty\}}(R[h^{-1}]))$ (where R_h^\wedge is the h -adic completion of R) such that their images in $\text{Aut}_{R_h^\wedge[h^{-1}]}(A_{\mathbb{S}\setminus\{\infty\}}(R_h^\wedge[h^{-1}]))$ agree. By the discussion above we can find elements $a_1 \in A_{\mathbb{S}\setminus\{\infty\}}(R_h^\wedge)$, $a_2 \in A_{\mathbb{S}\setminus\{\infty\}}(R[h^{-1}])$ such that $\phi_1 = \text{Ad}(a_1)$ and $\phi_2 = \text{Ad}(a_2)$; moreover, if we denote by a_1°, a_2° the images of a_1 and a_2 in $A_{\mathbb{S}\setminus\{\infty\}}(R_h^\wedge[h^{-1}])$ then we have $\text{Ad}(a_1^\circ) = \text{Ad}(a_2^\circ)$, so $a_1^\circ = s \cdot a_2^\circ$ for some central unit $s \in (R_h^\wedge[h^{-1}])^\times$. Now, again by Beauville-Laszlo, one has a natural embedding of the double quotient²⁵ $(R_h^\wedge)^\times \backslash (R_h^\wedge[h^{-1}])^\times / R[h^{-1}]^\times$ to $\text{Pic}(R)$. Since $\text{Pic}(R) = 0$ by our assumption, it means that we can decompose $s \in (R_h^\wedge[h^{-1}])^\times$ as $s_1^{-1}s_2$ with $s_1 \in (R_h^\wedge)^\times$ and $s_2 \in (R[h^{-1}])^\times$. Replacing a_1 and a_2 by s_1a_1 and s_2a_2 we now get $a_1^\circ = a_2^\circ$, while still $\phi_1 = \text{Ad}(a_1)$ and $\phi_2 = \text{Ad}(a_2)$. Using Beauville-Laszlo one more time the data of $a_1 \in A_{\mathbb{S}\setminus\{\infty\}}(R_h^\wedge)$, $a_2 \in A_{\mathbb{S}\setminus\{\infty\}}(R[h^{-1}])$ with $a_1^\circ = a_2^\circ$ gives an $a \in A_{\mathbb{S}\setminus\{\infty\}}(R)^\times$. Moreover, the data of $\text{Ad}(a_1) \in \text{Aut}_{R_h^\wedge}(A_{\mathbb{S}\setminus\{\infty\}}(R_h^\wedge))$, $\text{Ad}(a_2) \in \text{Aut}_{R[h^{-1}]}(A_{\mathbb{S}\setminus\{\infty\}}(R[h^{-1}]))$ is the same as for the ϕ . Thus, $\phi = \text{Ad}(a)$ and we are done. \square

Remark 3.46. From Proposition 3.45 we get an exact sequence of sheaves on $\text{Sch}_{/\mathbb{S}}^{h\text{-sep}}$:

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathcal{A}_{\mathbb{S}}^{\times} \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}) \xrightarrow{r} G_0.$$

In fact, if we restrict to even smaller subcategory of locally constant schemes (see the discussion just above Lemma 6.8) one can also show that the morphism r becomes surjective, and so the sequence above becomes a 4-term exact sequence.

Further we will need the following corollary:

Corollary 3.47. *Let T be an \mathbb{F}_p -scheme and let $\pi: T \times \mathbb{S} \rightarrow \mathbb{S}$ be the projection. Let R be the stalk of $\mathcal{O}_{T \times \mathbb{S}}$ at a point $x \in T \times \mathbb{S}$ and consider the corresponding map $\text{Spec } R \rightarrow \mathbb{S}$. Then every R -linear automorphism of $\mathcal{A}_{\mathbb{S}}(R)$ that lies in $\text{Ker}(r)(R)$ is inner.*

Proof. Note that $\text{Spec } R \rightarrow \mathbb{S}$ is h -separable. Indeed, either $\pi(x) = \{\infty\}$ and then $\text{Spec } R$ lies over $\mathbb{S}\setminus\{0\}$ or $\pi(x) \in \mathbb{S}\setminus\{\infty\}$ and then R is a localization of a $k[h]$ -algebra of the form Example 3.33(1) (and thus is also h -separable). \square

3.8. The algebra $\mathcal{A}_{\mathbb{S}}^{\flat}$ and the G_0 -action on $\mathcal{A}_{\mathbb{S}}^{\flat}\text{-Mod}$. We implicitly fix a number $n \in \mathbb{N}$. As in Construction 2.67 we endow $X := \text{Spec } A_0$ with a restricted symplectic structure given by the 1-form $\eta_{\text{can}} = \sum_i y_i dx_i$.

Definition 3.48. The sheaf of algebras $\mathcal{A}_{\mathbb{S}}^{\flat}$ is defined as $\mathcal{D}_{\text{Spec } A_0, \frac{[\eta_{\text{can}}]}{h}, \mathbb{S}}$.

²⁵Which corresponds to the gluing data for a line bundle on $\text{Spec } R$ given by trivial line bundles on $\text{Spec } R_h^\wedge$ and $\text{Spec } R[h^{-1}]$ and the gluing $R_h^\wedge[h^{-1}] \simeq R_h^\wedge[h^{-1}]$ for the restrictions to $\text{Spec } R_h^\wedge[h^{-1}]$ given by $s \in R_h^\wedge[h^{-1}]$.

Note that A_0 is a particular example of the algebra C considered in Section 3.4 where we take the number of parameters to be $2n$. We have $A_0^{\{p\}} \simeq \mathbb{F}_p$ and since $y_i^p = 0 \in A_0$ it follows that

$$(\eta_{\text{can}})^{\{p\}} = \sum_i y_i^p dx_i = 0.$$

Since $\mathcal{D}_{\text{Spec } A_0, \frac{[\eta]}{h}, \mathbb{S}}$ only depends on the pull-back $[\eta]^{\{p\}}$ we get that in fact

$$(3.6) \quad \mathcal{D}_{\text{Spec } A_0, \frac{[\eta_{\text{can}}]}{h}, \mathbb{S}} \simeq \mathcal{D}_{\text{Spec } A_0, 0, \mathbb{S}}.$$

In particular, the discussion in Section 3.4 applies to $\mathcal{D}_{\text{Spec } A_0, \frac{[\eta_{\text{can}}]}{h}, \mathbb{S}}$ as well.

Remark 3.49. The above isomorphism $\mathcal{A}_{\mathbb{S}}^b \simeq \mathcal{D}_{\text{Spec } A_0, 0, \mathbb{S}}$ determines a splitting of the Azumaya algebra $\mathcal{A}_{\mathbb{S}/\{0\}}^b$ (see Example 3.14).

The reason to interpret $\mathcal{D}_{\text{Spec } A_0, 0, \mathbb{S}}$ as $\mathcal{D}_{\text{Spec } A_0, \frac{[\eta_{\text{can}}]}{h}, \mathbb{S}}$ is that in the latter presentation it acquires a natural action of G_0 .

Construction 3.50 (G_0 -action on $\mathcal{A}_{\mathbb{S}}^b$). This is a variant of the construction in [BV, Section 3.1]. Let S be a scheme over \mathbb{F}_p . Denote by $\text{pr}_{\mathbb{S}}: S \times \mathbb{S} \rightarrow \mathbb{S}$ the projection. Applying Construction 3.13 (with $X = S \times \text{Spec } A_0$ and $[\eta]$ being the pullback of $[\eta_{\text{can}}]$ to X) we have a homomorphism

$$G_0(S) \rightarrow \text{Aut}(\text{pr}_{\mathbb{S}}^* \mathcal{A}_{\mathbb{S}}^b),$$

where the group displayed on the right-hand side consists of all $\mathcal{O}_{S \times \mathbb{S}}$ -linear algebra automorphisms of $\text{pr}_{\mathbb{S}}^* \mathcal{A}_{\mathbb{S}}^b$. This defines a homomorphism of group schemes over \mathbb{S} :

$$\psi: G_0 \times \mathbb{S} \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}^b).$$

Let $(X, [\eta])$ be a restricted symplectic S -scheme. Recall that $[\eta]$ defines the G_0 -torsor of Darboux frames $\mathcal{M}_{X, [\eta]} \rightarrow X^{\{p\}}$ (see Construction 2.72). Let $\text{pr}_{\mathbb{S}}: \mathcal{M}_{X, [\eta]} \times \mathbb{S} \rightarrow \mathbb{S}$ be the projection. The G_0 -action on $\mathcal{A}_{\mathbb{S}}^b$ endows $\text{pr}_{\mathbb{S}}^* \mathcal{A}_{\mathbb{S}}^b$ with the structure of G_0 -equivariant sheaf of algebras on $\mathcal{M}_{X, [\eta]} \times \mathbb{S}$ (where G_0 acts on $\mathcal{M}_{X, [\eta]}$). Denote the corresponding sheaf on the quotient $X^{\{p\}} \times \mathbb{S}$ by

$$\mathcal{M}_{X, [\eta]} \times^{G_0} \mathcal{A}_{\mathbb{S}}^b.$$

reference? is this in paper with Katia?

Proposition 3.51. *Let $(X, [\eta])$ be a restricted symplectic S -scheme. Then one has a natural isomorphism*

$$\mathcal{M}_{X, [\eta]} \times^{G_0} \mathcal{A}_{\mathbb{S}}^b \simeq \mathcal{D}_{X, \frac{[\eta]}{h}, \mathbb{S}}.$$

4. QUASI-COHERENT SHEAVES OF ABELIAN CATEGORIES

4.1. Sheaves of categories. For a (classical) additive category \mathcal{C} , let $\text{Center}(\mathcal{C})$ denote the *center* of \mathcal{C} , that is the ring of endomorphisms of the identity functor $\text{Id}: \mathcal{C} \rightarrow \mathcal{C}$. Given a commutative ring R , an R -linear structure on a presentable additive category \mathcal{C} is given by a ring homomorphism $R \rightarrow \text{Center}(\mathcal{C})$. Note that by [SAG, Remark D.1.3.4] this is equivalent to providing a continuous action of the monoidal category $R\text{-Mod}$ on \mathcal{C} . In particular, for any $M \in R\text{-Mod}$ we have a natural endofunctor of \mathcal{C} given by tensor product with M : $X \mapsto X \otimes_R M$. Let $\text{LinCat}_R^{\text{Add}}$ denote

the $(2, 1)$ -category of (classical) R -linear presentable additive categories (considered as an $(\infty, 1)$ -category).

Following [HA, Definition 1.3.5.1], by a *Grothendieck abelian category* we will mean a presentable abelian category \mathcal{C} for which the collection of monomorphisms is closed under filtered colimits. In particular, \mathcal{C} is cocomplete. We will denote by $\text{LinCat}_R^{\text{Ab}} \subset \text{LinCat}_R^{\text{Add}}$ the full subcategory spanned by those R -linear additive categories whose underlying category is equivalent to a Grothendieck abelian category. We will call an object of $\text{LinCat}_R^{\text{Ab}}$ an *abelian R -linear category* (or abelian categories over R), see [SAG, Definition D.1.4.1.(3)]. Note that morphisms in $\text{LinCat}_R^{\text{Ab}}$ are given by continuous additive functors, and they are not assumed to be exact.

Given a homomorphism $R \rightarrow R'$ there is a natural *base change* functor $\text{LinCat}_R^{\text{Ab}} \rightarrow \text{LinCat}_{R'}^{\text{Ab}}$, $\mathcal{C} \mapsto \mathcal{C} \otimes_R R'$ which is left adjoint to the forgetful functor $\text{LinCat}_{R'}^{\text{Ab}} \rightarrow \text{LinCat}_R^{\text{Ab}}$ ([SAG, Construction D.2.4.1]). The category $\mathcal{C} \otimes_R R' \in \text{LinCat}_{R'}^{\text{Ab}}$ can be explicitly described as the R' -linear category formed by pairs (M, α) , where M is an object of \mathcal{C} and $\alpha: R' \rightarrow \text{End}(M)$ is a homomorphism of R -algebras. The unit of adjunction $\mathcal{C} \rightarrow \mathcal{C} \otimes_R R'$ is given by the functor that sends an object $X \in \mathcal{C}$ to a pair given by $X \otimes_R R'$ with the natural action of R' on the right. Note that if R' is flat over R this functor is exact.

A key useful result that we are going to consistently use below is a form of flat descent for R -linear abelian categories. More precisely, by [SAG, Corollary D.6.8.4] the functor

$$\text{LinCat}_-^{\text{Ab}}: \text{CAlg}_{\mathbb{Z}} \rightarrow \widehat{\text{Cat}}_{\infty}$$

sending a commutative ring R to $\text{LinCat}_R^{\text{Ab}}$ forms an fpqc sheaf.

Construction 4.1. Let S be a base affine scheme and consider the category $\text{PStk}/_S$ of prestacks over S , by which we mean accessible functors $\mathcal{X}: \text{Aff}_S^{\text{op}} \rightarrow \text{Spc}$ to the category of spaces. Given a prestack²⁶ $\mathcal{X}: \text{Aff}_S^{\text{op}} \rightarrow \text{Spc}$ we define the category $\text{LinCat}_{\mathcal{X}}^{\text{Ab}}$ as the value on \mathcal{X} of the right Kan extension of functor $\text{LinCat}_-^{\text{Ab}}$; more explicitly

$$\text{LinCat}_{\mathcal{X}}^{\text{Ab}} := \lim_{\text{Spec } R \rightarrow \mathcal{X}} \text{LinCat}_R^{\text{Ab}}.$$

Note that, as a limit of $(2, 1)$ -categories, $\text{LinCat}_{\mathcal{X}}^{\text{Ab}}$ is still a $(2, 1)$ -category. Given a map of prestacks $f: \mathcal{X} \rightarrow \mathcal{Y}$ we will denote by

$$f^*: \text{LinCat}_{\mathcal{Y}}^{\text{Ab}} \rightarrow \text{LinCat}_{\mathcal{X}}^{\text{Ab}}$$

the natural pull-back functor. The association $\mathcal{X} \mapsto \text{LinCat}_{\mathcal{X}}^{\text{Ab}}$ sends colimits to limits. Since $\text{LinCat}_-^{\text{Ab}}$ was an fpqc sheaf, it also factors through the fpqc-sheafification, namely the pull-back

$$\iota^*: \text{LinCat}_{L_{\text{fpqc}}(\mathcal{X})}^{\text{Ab}} \rightarrow \text{LinCat}_{\mathcal{X}}^{\text{Ab}},$$

for the natural map $\iota: \mathcal{X} \rightarrow L_{\text{fpqc}}(\mathcal{X})$ to the fpqc sheafification is an equivalence.

Definition 4.2. Let $\mathcal{X}: \text{Aff}_S^{\text{op}} \rightarrow \text{Spc}$ be a prestack.

- (1) We will call an object \mathcal{C} of $\text{LinCat}_{\mathcal{X}}^{\text{Ab}}$ a *quasi-coherent sheaf of abelian categories on \mathcal{X}* . It amounts to the following piece of data:
 - (i) For every affine scheme $Z \rightarrow \mathcal{X}$ an abelian category $\mathcal{C}(Z)$ over $\mathcal{O}(Z)$.

²⁶By Spc here we mean the ∞ -category of spaces, also known as ∞ -groupoids.

(ii) For every morphism $u: Z' \rightarrow Z$ of affine schemes over \mathcal{X} , an equivalence

$$u^*: \mathfrak{C}(Z) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Z') \xrightarrow{\sim} \mathfrak{C}(Z'),$$

of categories over $\mathcal{O}(Z')$.

(iii) For every composable pair of morphisms u_1, u_2 of affine schemes over \mathcal{X} , an isomorphism $(u_1 \circ u_2)^* \xrightarrow{\sim} u_2^* \circ u_1^*$, such that the natural compatibility axiom for 3-fold compositions holds.

(2) For a quasi-coherent sheaf of categories $\mathfrak{C} \in \text{LinCat}_{\mathcal{X}}^{\text{Ab}}$ we define its category of global sections to be the additive category given by the limit

$$\mathfrak{C}(\mathcal{X}) := \lim_{Z \rightarrow \mathcal{X}} \mathfrak{C}(Z)$$

over affine schemes mapping to \mathcal{X} . We warn the reader that in general $\mathfrak{C}(\mathcal{X})$ does *not* need to be an abelian category (since functors in the limit above are typically not exact). However, it is abelian if \mathcal{X} is represented by an algebraic stack Remark 4.5(2).

Example 4.3. (1) Let \mathcal{X} be representable by an affine scheme $\text{Spec } R$. Then the category of affine schemes over \mathcal{X} has a final object given by the identity map $\text{Spec } R \rightarrow \mathcal{X}$, and so

$$\text{LinCat}_{\mathcal{X}}^{\text{Ab}} \simeq \text{LinCat}_R^{\text{Ab}}.$$

In other words, a quasi-coherent sheaf of abelian categories over $\text{Spec } R$ is just an R -linear abelian category.

(2) Let $\mathcal{X}: \text{Aff}_S^{\text{op}} \rightarrow \text{Spc}$ be a prestack. Let the *naive* category of quasi-coherent sheaves on \mathcal{X} to be the limit

$$\text{QCoh}^{\text{naive}}(\mathcal{X}) := \lim_{Z \rightarrow \mathcal{X}} \mathcal{O}(Z)\text{-Mod}$$

over affine schemes over \mathcal{X} ; this is an additive monoidal 1-category. Let $A \in \text{QCoh}^{\text{naive}}(\mathcal{X})$ be an algebra object; for any map $f: Z \rightarrow \mathcal{X}$ it defines an algebra $A(Z)$ over $\mathcal{O}(Z)$ and this association is functorial under pull-back. Then the association to $f: Z \rightarrow \mathcal{X}$ the abelian $\mathcal{O}(Z)$ -linear category $A(Z)\text{-Mod}$ defines an object $A\text{-Mod}$ of $\lim_{Z \rightarrow \mathcal{X}} \text{LinCat}_{\mathcal{O}(Z)}^{\text{Ab}}$, or, in other words, a quasi-coherent sheaf of abelian categories over \mathcal{X} .

As a standard consequence of fpqc descent for $\text{LinCat}_-^{\text{Ab}}$ one can use Čech resolutions to compute $\text{LinCat}_{\mathcal{X}}^{\text{Ab}}$:

Remark 4.4. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be an fpqc surjection of prestacks and let $\mathcal{X}^{(\bullet)}$ denote the corresponding Čech nerve. Then by [ref in HTT](#) the natural map $\text{colim}_{[\bullet] \in \Delta^{\text{op}}} \mathcal{X}^{(\bullet)} \rightarrow \mathcal{Y}$ becomes an isomorphism after applying the fpqc sheafification $L_{\text{fpqc}}(-)$. Applying $\text{LinCat}_-^{\text{Ab}}$, we get an equivalence of categories

$$\text{LinCat}_{\mathcal{Y}}^{\text{Ab}} \xrightarrow{\sim} \lim_{[\bullet] \in \Delta} \text{LinCat}_{\mathcal{X}^{(\bullet)}}^{\text{Ab}}.$$

Since all terms in the limit are $(2, 1)$ -categories, it can in fact be replaced by a similar limit over $\Delta_{\leq 3} \subset \Delta$.

Let us apply this to the situation when \mathcal{X} is an algebraic stack.

Remark 4.5 (Sheaves of categories on algebraic stacks). Let $\mathcal{X}: \text{Aff}_S^{\text{op}} \rightarrow \text{Grpd}$ be a classical²⁷ algebraic stack. Assume for simplicity that \mathcal{X} is a quasi-compact Artin stack with affine diagonal. Then:

- (1) There is a smooth cover $U = \text{Spec } R \rightarrow \mathcal{X}$ by an affine scheme, and all terms $U^{(\bullet)} = \text{Spec } R^{(\bullet)}$ of the corresponding Čech nerve are also affine. By Remark 4.4 we get an equivalence of $(2, 1)$ -categories

$$(4.1) \quad \text{LinCat}_{\mathcal{X}}^{\text{Ab}} \xrightarrow{\sim} \lim_{[\bullet] \in \Delta_{\leq 3}} \text{LinCat}_{R^{(\bullet)}}^{\text{Ab}}.$$

This way an object $\mathfrak{C} \in \text{LinCat}_{\mathcal{X}}^{\text{Ab}}$ amounts to an explicit piece of data similar to the one in Construction 4.1: an R -linear abelian category \mathfrak{C} , together with an equivalence of categories $u: \mathfrak{C} \otimes_R R^{(1)} \simeq \mathfrak{C} \otimes_R R^{(1)}$ for the two different maps $R = R^{(0)} \rightarrow R^{(1)}$, with a choice of an isomorphism of functors $\alpha: \delta_2^*(u) \xrightarrow{\sim} \delta_1^*(u) \circ \delta_0^*(u)$ for the pull-back to $R^{(2)}$, whose various pull-backs to $R^{(3)}$ are strictly compatible.

- (2) Let $\mathfrak{C} \in \text{LinCat}_{\mathcal{X}}^{\text{Ab}}$. By (4.1) we also have analogous formula for the global sections of \mathfrak{C} (as an additive category):

$$\mathfrak{C}(\mathcal{X}) \xrightarrow{\sim} \lim_{[\bullet] \in \Delta_{\leq 3}} \mathfrak{C}(U^{(\bullet)}).$$

Note however that all functors in the limit are exact (since the covering $U \rightarrow \mathcal{X}$ was flat). It follows that $\mathfrak{C}(\mathcal{X})$ is an abelian²⁸ category.

- (3)²⁹ Assume the base affine scheme S is Noetherian and the atlas U is of finite type over S . Then $\mathcal{X}: \text{Aff}_S^{\text{op}} \rightarrow \text{Grpd}$ is left Kan extended from the subcategory $\text{Aff}_S^{\text{ft}} \subset \text{Aff}_S$, namely

$$\mathcal{X} \xrightarrow{\sim} \text{colim}_{Z \rightarrow \mathcal{X}} Z$$

where $Z \in \text{Aff}_S^{\text{ft}}$. It follows that in the limit formula for $\text{LinCat}_{\mathcal{X}}^{\text{Ab}}$ as the right Kan extension one can restrict to finite type affine S -schemes:

$$\text{LinCat}_{\mathcal{X}}^{\text{Ab}} \simeq \lim_{\substack{Z \in \text{Aff}_S^{\text{ft}} \\ Z \rightarrow \mathcal{X}}} \text{LinCat}_{\mathcal{O}(Z)}^{\text{Ab}}.$$

Remark 4.6 (Sheaves of categories on formal prestacks). By an h -adic formal prestack we will mean a functor $\widehat{T}: \text{Aff}_{\mathbb{Z}[h]}^{\text{op}} \rightarrow \text{Spc}$ such that $T(Z) = \emptyset$ unless h is nilpotent on $\mathcal{O}(Z)$. A datum of such \widehat{T} is given by a collection of prestacks T_m over $S_m = \text{Spec } \mathbb{Z}[[h]]/h^{m+1}$, $m \geq 0$, together with isomorphisms $T_{m+1} \times_{S_{m+1}} S_m \xrightarrow{\sim} T_m$. The prestack \widehat{T} is reconstructed from $\{T_m\}_{m \in \mathbb{N}}$ as $\text{colim}_m T_m$ in prestacks over $\text{Spec } \mathbb{Z}[[h]]$. We get an equivalence of categories

$$\text{LinCat}_{\widehat{T}}^{\text{Ab}} \xrightarrow{\sim} \lim_m \text{LinCat}_{T_m}^{\text{Ab}}.$$

In particular, a quasi-coherent sheaf of categories over \widehat{T} is given by a collection $\{\mathfrak{C}_m, i_m^* \mathfrak{C}_{m+1} \xrightarrow{\sim} \mathfrak{C}_m, m \geq 0\}$, where \mathfrak{C}_m is a quasi-coherent sheaf of categories over T_m and $i_m^* \mathfrak{C}_{m+1}$ is the pullback of \mathfrak{C}_{m+1} to T_m .

²⁷We consider it as a prestack via the nerve functor $N: \text{Grpd} \rightarrow \text{Spc}$ whose essential image is given by 1-truncated spaces.

²⁸Note that abelian categories are closed under limits with respect to exact functors in $\text{LinCat}_R^{\text{Add}}$ [SAG, Remark D.1.6.5].

²⁹Do we need this?

4.2. Groups acting on a category. For an affine group scheme H over S acting on a fpqc-stack X and a quasi-coherent sheaf of categories \mathfrak{C} over X , an H -equivariant structure on \mathfrak{C} consists of a quasi-coherent sheaf of categories $\mathfrak{C}_{X/H}$ over the quotient stack X/H together with an equivalence of quasi-coherent sheaves of categories between the pullback of $\mathfrak{C}_{X/H}$ to X and \mathfrak{C} .

In particular, if H acts trivially on X we shall refer to the above data as an action of H on \mathfrak{C} . Note that, indeed, for every affine scheme $Z \rightarrow X$, the group $H(Z)$ acts on the abelian category $\mathfrak{C}(Z)$ (via the natural action of $H(Z)$ on the point $Z \rightarrow X \rightarrow X/H \simeq X \times BH$). For $h \in H(Z)$ we will denote by ϕ_h the corresponding autoequivalence of $\mathfrak{C}(Z)$.

The following construction will play an important role in what follows.

Construction 4.7 (A version of the Grothendieck construction). Let A be an algebra in $\mathrm{QCoh}^{\mathrm{naive}}(X)$ whose underlying quasi-coherent sheaf is locally free of finite type³⁰ and let \underline{A}^* the relatively affine group scheme³¹ of its invertible elements. Let

$$(4.2) \quad 1 \rightarrow \underline{A}^* \rightarrow \widehat{H}_X \rightarrow H \times_S X \rightarrow 1$$

be an extension of group schemes over X together with an action

$$(4.3) \quad \alpha: \widehat{H}_X \rightarrow \underline{\mathrm{Aut}}(A)$$

of \widehat{H}_X on A by algebra automorphisms such that

- (T) the action of \widehat{H}_X on \underline{A}^* induced by (4.3) is equal to the adjoint action arising from exact sequence (4.2).

Since $\underline{A}^* \subset \underline{A}$ is dense, it follows that the restriction of α to the subgroup $\underline{A}^* \subset \widehat{H}_X$ is equal to Ad . Extension (4.2) gives rise to an action of the group scheme H on $A\text{-Mod}$. Indeed, (4.3) defines an action of \widehat{H}_X on $A\text{-Mod}$ and we claim that the latter factors canonically through H . Indeed, for an affine scheme over X , $u: Z \rightarrow X$, and an element $a \in \underline{A}^*(Z) \subset \widehat{H}(Z)$, the corresponding endofunctor ϕ_a of $A\text{-Mod}(Z)$ is induced by the inner automorphism Ad_a of $u^*A(Z)$. Consequently, the multiplication by a induces an isomorphism between the identity endofunctor and ϕ_a . The above construction determines descent data for $A\text{-Mod}$ along the morphism $X \rightarrow X/H = X \times BH$, where BH is the classifying stack of H . We denote the corresponding quasi-coherent sheaf of categories over X/H by $A\text{-Mod}^H$ *Maybe $A\text{-Mod}^H$ is a better notation?*

Remark 4.8 (Explicit local description of $A\text{-Mod}^H$). Let Z be an fpqc-stack and let $f: Z \rightarrow X \times BH$ be a morphism. Let us give an explicit description of the category of global sections $A\text{-Mod}^H(Z)$. Let $u: \mathcal{M} \rightarrow Z$ be the H -torsor corresponding to f . The group \widehat{H}_X acts on \mathcal{M} via the projection $\widehat{H}_X \rightarrow H \times_S X$. Action (4.3) of \widehat{H}_X on A makes the pullback $A_{\mathcal{M}}$ of A to \mathcal{M} a \widehat{H}_X -equivariant sheaf of algebras on \mathcal{M} . Then $A\text{-Mod}^H(Z)$ is the category of \widehat{H}_X -equivariant (naive) quasi-coherent³² $A_{\mathcal{M}}$ -modules \mathcal{F} on \mathcal{M} , such that the action of \underline{A}^* on \mathcal{F} induced by the $A_{\mathcal{M}}$ -module structure coincides with the action of the subgroup $\underline{A}^* \hookrightarrow \widehat{H}_X$.

³⁰Meaning that for any affine scheme $f: Z \rightarrow X$ the pull-back f^*A is locally free of finite type over Z .

³¹That is, a group stack $\underline{A}^* \rightarrow X$ over X that is representable in affine schemes.

³²So, objects of the category of $A_{\mathcal{M}}$ -modules in $\mathrm{QCoh}^{\mathrm{naive}}(\mathcal{M}/\widehat{H}_X)$ (for the latter see Example 4.3(2)).

In the case $Z = X \times BH$ and f is the identity map we get the following explicit description of the category of global sections of $A\text{-Mod}^H$; namely an object in $A\text{-Mod}^H(X \times BG)$ is given by a quasi-coherent sheaf $M \in \text{QCoh}^{\text{naive}}(X)$ together with an action of A and an action $\beta: \widehat{H}_X \rightarrow \underline{\text{Aut}}_X(M)$ such that

- (1) $\beta_h(am) = \alpha_h(a)\beta_h(m)$ for $a \in A$, $m \in M$, $h \in \widehat{H}_X$;
- (2) $\beta_a(m) = am$ for $a \in A^* \subset \widehat{H}_X$ and $m \in M$.

Assume that, for $f: Z \rightarrow X \times BH$ as above, we are given a \widehat{H}_X -torsor $\widehat{\mathcal{M}} \rightarrow Z$ with $\widehat{\mathcal{M}}/\underline{A}^* = \mathcal{M}$. Using the action of \widehat{H}_X on A define a sheaf \mathcal{O}_Z -algebra $O = \widehat{\mathcal{M}} \times^{\widehat{H}_X} A_Z$. Applying the construction from Example 4.3(2) we consider the quasi-coherent sheaf categories $O\text{-Mod}$ over Z . By definition, the category $O\text{-Mod}(Z)$ is equivalent to the category of \widehat{H}_X -equivariant quasi-coherent modules over $A_{\widehat{\mathcal{M}}}$. An object $\mathcal{F} \in A\text{-Mod}_{X/H}(Z)$ determines via the pullback along $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$ an object of $O\text{-Mod}(Z)$. Repeating this construction for every affine scheme over Z we get a functor between quasi-coherent sheaves of categories over Z :

$$(4.4) \quad f^*(A\text{-Mod}_{X/H}) \rightarrow O\text{-Mod}.$$

Proposition 4.9. *Morphism (4.4) is an equivalence. In particular, it induces an equivalence of categories $A\text{-Mod}_{X/H}(Z) \xrightarrow{\sim} O\text{-Mod}(Z)$.*

Proof. Thanks to the faithfully flat descent for quasi-coherent sheaves of categories it suffices to exhibit a fpqc cover of $Z' \rightarrow Z$ such that functor (4.4) induces an equivalence of categories $A\text{-Mod}_{X/H}(Z') \xrightarrow{\sim} O\text{-Mod}(Z')$. We take $u: Z' = \widehat{\mathcal{M}} \rightarrow Z$. Then both categories are identified with the category of quasi-coherent $A_{\widehat{\mathcal{M}}}$ -modules and morphism (4.4) evaluated at $\widehat{\mathcal{M}}$ is isomorphic to the identity functor. \square

Example 4.10. Assume that the projection $\widehat{H}_X \rightarrow H \times_S X$ has a section $s: H \times_S X \rightarrow \widehat{H}_X$. Then the action of H on $A\text{-Mod}$ lifts to an action of H on A . Moreover, the section defines a lift $\widehat{\mathcal{M}} \rightarrow Z$ of $\mathcal{M} \rightarrow Z$ such that the corresponding sheaf of algebras is given by the formula $O = \mathcal{M} \times^H A$. Note that when the image of the section s commutes with $\underline{A}^* \subset \widehat{H}_X$ the action of H on A is automatically trivial (because $\underline{A}^* \subset \underline{A}$ is dense).

Remark 4.11. Let ϕ be an action of H on the category $A\text{-Mod}$. For a scheme Z over X and a point $h \in H(Z)$ denote by ϕ_h the corresponding autoequivalence of the category of A_Z -modules. Assume that, for every $Z \rightarrow X$ and $h \in H(Z)$, fpqc locally on Z the autoequivalence ϕ_h preserves the isomorphism class of the free A_Z -module A_Z . Then ϕ arises from data (4.2) and (4.3) for an appropriate \widehat{H}_X . Namely, one sets $\widehat{H}_X(Z)$ to be the group of pairs (h, α_h) , where $h \in H(Z)$ and $\alpha_h: \phi_h(A_Z) \xrightarrow{\sim} A_Z$ is an isomorphism of A_Z -modules. If X, H are schemes of finite type over a field then it is enough to check the above condition on ϕ for schemes Z which are also of finite type.

Remark 4.12 (Tensor product of actions). In the context of Construction 4.2 let A_1, A_2 be two quasi-coherent sheaves of algebras that are locally free of finite type over X . For $i = 1, 2$ let

$$1 \rightarrow \underline{A}_i^* \rightarrow \widehat{H}_{i,X} \rightarrow H_X \rightarrow 1$$

and $\alpha_i: \widehat{H}_{i,X} \rightarrow \underline{\text{Aut}}(A_i)$ define an action of H on $A_i\text{-Mod}$. Then we have a natural H -action on $(A_1 \otimes_{\mathcal{O}_X} A_2)\text{-Mod}$. In terms of short exact sequences it is explicitly given

by the extension

$$1 \rightarrow \underline{(A_1 \otimes A_2)^*} \rightarrow \widehat{H}_{1 \otimes 2, X} \rightarrow H_X \rightarrow 1$$

where $\widehat{H}_{1 \otimes 2, X}$ is defined as the quotient of $(H_{1, X} \times_{H_X} H_{2, X}) \times \underline{(A_1 \otimes A_2)^*}$ by the subgroup $\underline{A_1^*} \times \underline{A_2^*}$ embedded antidiagonally. The map $\alpha_{1 \otimes 2}: \widehat{H}_{1 \otimes 2, X} \rightarrow \underline{\text{Aut}}(A_1 \otimes A_2)$ is defined as follows: the components $H_{i, X}$ act separately on the A_i , while $\underline{(A_1 \otimes A_2)^*}$ acts by conjugation.

5. ACTION OF G_0 ON $\mathcal{A}_{\mathbb{S}}\text{-Mod}$.

5.1. Main Theorem. The action of G_0 on A_0 defines its action on $A_0\text{-Mod}$. The goal of this subsection is to extend the latter to an action of G_0 on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$.

Theorem 1. *Let H be an affine group scheme over \mathbb{F}_p equipped with a homomorphism $v: H \rightarrow G_0$. There exists a unique (up to a unique isomorphism) action of the group scheme H on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ equipped with a trivialization at $h = \infty$ such that the induced action of H on the center of $A_0\text{-Mod}$ (identified with A_0) equals the composition $H \xrightarrow{v} G_0 \hookrightarrow \underline{\text{Aut}}(A_0)$.*

Proof. Let's start with some preliminary remarks.

a) Set $H_{\mathbb{S}} = H \times \mathbb{S}$. We claim that giving an action of H on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ amounts specifying an extension

$$(5.1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \underline{\mathcal{A}_{\mathbb{S}}^*} & \longrightarrow & \widehat{H}_{\mathbb{S}} & \longrightarrow & H_{\mathbb{S}} \longrightarrow 1 \\ & & & \searrow \text{Ad} & \downarrow \alpha & & \\ & & & & \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}) & & \end{array}$$

satisfying property (T) in Construction 4.7. Indeed, using Remark 4.11 it suffices to check that, for any affine scheme Z finite type over \mathbb{S} , every autoequivalence ϕ of the category $\mathcal{A}_{\mathbb{S}}(Z)\text{-Mod}$ over $\mathcal{O}(Z)$ *fpqc* locally on Z preserves the isomorphism class of the free $\mathcal{A}_{\mathbb{S}}(Z)$ -module $\mathcal{A}_{\mathbb{S}}(Z)$. We shall see that this is true even Zariski locally on Z . Choose a point $z \in Z$ and let $Z_{(z)} := \text{Spec } \mathcal{O}_{Z, z}$ be the spectrum of the local ring. If the image of z in \mathbb{S} is not 0 then $\mathcal{A}_{\mathbb{S}}(Z)$ is an Azumaya algebra in some Zariski neighborhood U of z . Hence, the restriction of ϕ to U is given by the tensor product with a line bundle \mathcal{L} . Replacing U with a smaller neighbourhood over which \mathcal{L} is trivial, we get that ϕ restricted to that neighbourhood is isomorphic to the identity functor. If the image of z in \mathbb{S} is 0, then $\mathcal{A}_{\mathbb{S}}(Z_{(z)})$ is a local ring. Hence, by Kaplansky's theorem, every projective $\mathcal{A}_{\mathbb{S}}(Z_{(z)})$ -module is free. In particular, this can be applied to a projective module $\phi(\mathcal{A}_{\mathbb{S}}(Z_{(z)}))$. Using that the ring of endomorphisms of $\phi(\mathcal{A}_{\mathbb{S}}(Z_{(z)}))$ is a local ring (namely, $\mathcal{A}_{\mathbb{S}}(Z_{(z)})^{op}$) we conclude that $\phi(\mathcal{A}_{\mathbb{S}}(Z_{(z)}))$ is indecomposable, and therefore isomorphic to $\mathcal{A}_{\mathbb{S}}(Z_{(z)})$. This isomorphism then also extends to a Zariski neighborhood of z .

- b) By Lemma 5.5, $\widehat{H}_{\mathbb{S}}$ viewed as a $\underline{\mathcal{A}_{\mathbb{S}}^*}$ -torsor over $H_{\mathbb{S}}$ is locally trivial for the Zariski topology on $H_{\mathbb{S}}$.
- c) Under the identification in part a) a trivialization of the action at $h = \infty$ amounts to the choice of a section $H = H \times \infty \rightarrow \widehat{H}_{\mathbb{S}} \times_{\mathbb{S}} \infty$ whose image commutes with the subgroup $\underline{\mathcal{A}_{\mathbb{S}}^*} \times_{\mathbb{S}} \infty \subset \widehat{H}_{\mathbb{S}} \times_{\mathbb{S}} \infty$ (see Example 4.10). The action of $H = H \times 0$ on the center of $A_0\text{-Mod}$ is induced by $\alpha_0: \widehat{H}_{\mathbb{S}} \times_{\mathbb{S}} 0 \rightarrow \underline{\text{Aut}}(A_0)$. The uniqueness

part of Theorem 1 asserts that, for any two extensions (5.1), such that α_0 equals the composition

$$\widehat{H}_{\mathbb{S}} \times_{\mathbb{S}} 0 \rightarrow H_{\mathbb{S}} \times_{\mathbb{S}} 0 \xrightarrow{v} G_0 \subset \underline{\text{Aut}}(A_0),$$

together with chosen sections at $h = \infty$, there exists a unique isomorphism connecting the two that carries one section to the other.

We will first prove the uniqueness assertion in the theorem. Let T be an affine scheme over \mathbb{F}_p , $\pi: T \times \mathbb{S} \rightarrow \mathbb{S}$ the projection. The tensor product with a line bundle defines a functor

$$(5.2) \quad \text{Pic}(T \times \mathbb{S}) \rightarrow \text{Aut}(\pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod})$$

from the Picard groupoid of line bundles over $T \times \mathbb{S}$ to the groupoid of autoequivalences³³ of the quasi-coherent sheaf of categories $\pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod}$.

Lemma 5.1. *Functor (5.2) is fully faithful and its essential image consists of those autoequivalences $\mathfrak{H} \subset \text{Aut}(\pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod})$ that act identically on the center of $(\pi^* \mathcal{A}_{\mathbb{S}})_{|T \times 0}\text{-Mod}$ (identified with $A_0 \otimes \mathcal{O}(T)$).*

Proof. The only part that requires a proof is the essential surjectivity of the functor $\text{Pic}(T \times \mathbb{S}) \rightarrow \mathfrak{H}$. Let ϕ be an autoequivalence in \mathfrak{H} . We claim that ϕ , locally for the Zariski topology on $T \times \mathbb{S}$, is isomorphic to Id . Indeed, as explained in remarks (a) and (b), every autoequivalence of $\pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod}$ is, Zariski locally on $T \times \mathbb{S}$, induced by an automorphism of $\pi^* \mathcal{A}_{\mathbb{S}}$. Since $\phi \in \mathfrak{H}$, the latter automorphism is equal to Id modulo h . Now the claim follows from Lemma 3.47. Assigning to every open subset $U \subset T \times \mathbb{S}$ the set of isomorphisms $\phi|_U \simeq \text{Id}$ we define a torsor (for the Zariski topology) under the sheaf of central elements in $(\pi^* \mathcal{A}_{\mathbb{S}})^*$. The latter sheaf is equal to $\mathcal{O}_{\mathbb{S} \times T}^*$. This defines a line bundle L_{ϕ} over $T \times \mathbb{S}$ such that ϕ is the tensor product with L_{ϕ} . \square

Next, consider the groupoid \mathfrak{H}' formed by pairs (ϕ, α) , where $\phi \in \mathfrak{H}$ and $\alpha: \phi|_{T \times \infty} \xrightarrow{\sim} \text{Id}$. The equivalence $\text{Pic}(T \times \mathbb{S}) \xrightarrow{\sim} \mathfrak{H}$ induces an equivalence between \mathfrak{H}' and the groupoid of line bundles over $T \times \mathbb{S}$ equipped with a trivialization at $T \times \infty$. Since the Picard stack of $\mathbb{S} = \mathbb{P}_{\mathbb{F}_p}^1$ is isomorphic to $\mathbb{Z} \times B\mathbb{G}_m$ it follows that \mathfrak{H}' is discrete and its π_0 is the group $H^0(T, \mathbb{Z})$.

Returning to the proof of the uniqueness assertion recall that giving an action ψ of H on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ is equivalent to giving descent data along the morphism $\mathbb{S} \rightarrow \mathbb{S} \times BH$. In particular, any action gives rise to an autoequivalence

$$\phi: \pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod} \xrightarrow{\sim} \pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod},$$

where $\pi: H \times \mathbb{S} \rightarrow \mathbb{S}$ is the projection. If ϕ_1, ϕ_2 are equipped with trivializations at $h = \infty$ and induce the same action on the center of $\mathcal{A}_0\text{-Mod}$ then $\phi_1 \circ \phi_2^{-1}$ is an object of \mathfrak{H}' corresponding to some element $[\phi_1 \circ \phi_2^{-1}] \in H^0(H, \mathbb{Z})$. Using the cocycle *isomorphisms* related to ϕ_i it follows that $[\phi_1 \circ \phi_2^{-1}]$ is, in fact, a group homomorphism $H \rightarrow \mathbb{Z}$. The latter must be trivial because H is assumed to be affine and, in particular, quasi-compact. We conclude that there is a unique isomorphism $\phi_1 \xrightarrow{\sim} \phi_2$ that commutes with trivializations at $H \times \infty$. The discreteness of \mathfrak{H}' (applied to $T = H \times H$) implies that $\phi_1 \xrightarrow{\sim} \phi_2$ commutes with the cocycle *isomorphisms*. This completes the proof of the uniqueness assertion of the Theorem.

³³The composition of autoequivalences makes $\text{Aut}(\pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod})$ into a group object in the category of groupoids. Functor (5.2) is a homomorphism of groups in groupoids.

Remark 5.2. The argument above proves a more general assertion: if T is any scheme over \mathbb{F}_p and $\text{pr}_{\mathbb{S}}: \mathbb{S} \times T \rightarrow \mathbb{S}$ is the projection, then there exists at most one (up to a unique isomorphism) action of H on $\text{pr}_{\mathbb{S}}^* \mathcal{A}_{\mathbb{S}}\text{-Mod}$ equipped with a trivialization over $\infty \times T$ such that the induced action of H on the center of $(\text{pr}_{\mathbb{S}}^* \mathcal{A}_{\mathbb{S}})_{0 \times T}\text{-Mod}$ is given by v .

Let us prove the existence. Without loss of generality we may assume that $H = G_0$, $v = \text{Id}$. We shall use the action $\psi: G_0 \times \mathbb{S} \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}^b)$ from Construction 3.50 and the following result borrowed from [BV]. Let A_h (resp. A_h^b) be the h -adic completion of $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}$ (resp. $\mathcal{A}_{\mathbb{S} \setminus \{\infty\}}^b$). Set $\mathcal{B}_{\mathbb{S}} = \mathcal{A}_{\mathbb{S}} \otimes_{\mathcal{O}_{\mathbb{S}}} \mathcal{A}_{\mathbb{S}}^{b, \text{op}}$, and let $\underline{\text{Aut}}(B_h)$ be the group scheme over \mathbb{F}_p of $\mathbb{F}_p[[h]]$ -linear automorphisms of the h -adic completion of $\mathcal{B}_{\mathbb{S} \setminus \infty}$. That is, for a \mathbb{F}_p -algebra R , $\underline{\text{Aut}}(B_h)(R)$ is the group of $R[[h]]$ -algebra automorphisms of tensor product $B_h \hat{\otimes}_{\mathbb{F}_p[[h]]} R[[h]]$. Define a homomorphism $\Gamma_{\psi_0}: G_0 \hookrightarrow \underline{\text{Aut}}(B_0) = \underline{\text{Aut}}(A_0 \otimes A_0^b)$ by the formula $\Gamma_{\psi_0}(g) = g \otimes \psi_0(g)$.

Lemma 5.3 ([BV, Corollary 3.5]). *There exists a unique (up to a unique isomorphism) triple $(\widehat{G}^{\sharp}, \alpha, i)$ displayed in the digram*

$$(5.3) \quad \begin{array}{ccccccc} & & & B_h(h^{-1})^* & & & \\ & & & \uparrow i & & & \\ 1 & \longrightarrow & B_h^* & \longrightarrow & \widehat{G}^{\sharp} & \longrightarrow & G_0 \longrightarrow 1 \\ & & \searrow \text{Ad} & & \downarrow \alpha & & \downarrow \Gamma_{\psi_0} \\ & & & & \underline{\text{Aut}}(B_h) & \longrightarrow & \underline{\text{Aut}}(B_0) \end{array}$$

where the middle sequence is exact, the north east arrow is the natural inclusion, i is a monomorphism, and $\alpha(g) = \text{Ad}_{i(g)}$. In addition, if W is an irreducible representation of $B_h(h^{-1})$, $B_h(h^{-1}) \xrightarrow{\sim} \text{End}_{\mathbb{F}_p((h))}(W)$, there exists a $\mathbb{F}_p[[h]]$ -lattice $\Lambda \subset W$, invariant under the B_h -action on W and under the action of \widehat{G}^{\sharp} :

$$i: \widehat{G}^{\sharp} \hookrightarrow \underline{\text{Aut}}_{\mathbb{F}_p[[h]]}(\Lambda) \subset \underline{\text{Aut}}_{\mathbb{F}_p((h))}(W) = B_h(h^{-1})^*.$$

The plan of our construction is the following. Since $\mathcal{A}_{\mathbb{S}}$ and $\mathcal{A}_{\mathbb{S}}^b$ restricted to $\mathbb{S} \setminus \{0\}$ are split Azumaya algebras (see Remarks 3.18 and 3.49) their categories of modules are equivalent:

$$\mathcal{A}_{\mathbb{S} \setminus \{0\}}\text{-Mod} \xrightarrow{\sim} \mathcal{O}_{\mathbb{S} \setminus \{0\}}\text{-Mod} \xrightarrow{\sim} \mathcal{A}_{\mathbb{S} \setminus \{0\}}^b\text{-Mod}.$$

Thus $\psi|_{\mathbb{S} \setminus \{0\}}$ defines an action of G_0 on $\mathcal{A}_{\mathbb{S} \setminus \{0\}}\text{-Mod}$. On the other hand, if $\widehat{\mathbb{S}}$ denotes the formal completion of \mathbb{S} at $\{0\}$ the diagram (5.3) yields an action of G_0 on $\mathcal{B}_{\widehat{\mathbb{S}}}\text{-Mod}$ whose restriction to the punctured disk is trivialized. Together with $\psi|_{\widehat{\mathbb{S}}}$ the latter defines an action G_0 on $\mathcal{A}_{\widehat{\mathbb{S}}}\text{-Mod}$ equivalent to the above over the punctured disk. We use the Beauville-Laszlo theorem to glue the two pieces together. Let us provide the details.

First, we define an action of the group G_0 on $\mathcal{A}_{\widehat{\mathbb{S}}}\text{-Mod}$. Recall from Remark 4.6 that the latter amounts to specifying an action of G_0 on $\mathcal{A}_{\mathbb{S}_m}\text{-Mod}$, for every $m \geq 0$, together with the restriction isomorphisms.

Let us denote by $L^+ \mathbb{G}_m$ the group valued presheaf on affine k -schemes sending $\text{Spec } R$ to $R[[h]]^*$. Recall the following 4-term exact sequence

$$(5.4) \quad 1 \rightarrow L^+ \mathbb{G}_m \rightarrow \underline{A}_h^* \xrightarrow{\text{Ad}} \underline{\text{Aut}}(A_h) \rightarrow G_0 \rightarrow 1,$$

where the map $L^+G_m \rightarrow \underline{A}_h^*$ is induced by the homomorphism $k[[h]] \rightarrow A_h$, and the map $\underline{\text{Aut}}(A_h) \rightarrow G_0$ is given by reduction modulo h . Here, the surjectivity of the latter map follows from [BK, Lemma 3.10], while the exactness at other terms follows from Remark 3.46 (see also [BK, Remark 3.12])

Let $\underline{\text{Aut}}(A_h) \rightarrow \underline{\text{Aut}}(A_h^b)$ be the projection to G_0 followed by ψ and let $\Gamma_\psi: \underline{\text{Aut}}(A_h) \hookrightarrow \underline{\text{Aut}}(A_h) \times \underline{\text{Aut}}(A_h^b) \subset \underline{\text{Aut}}(B_h)$ be its graph. Set $\widehat{G} = \alpha^{-1}(\text{Im } \Gamma_\psi) \subset \widehat{G}^\sharp$. Note that $\alpha: \widehat{G} \rightarrow \text{Im } \Gamma_\psi$ is surjective: indeed, the image of Γ_ψ in $\underline{\text{Aut}}(B_0)$ is contained in $\Gamma_{\psi_0}(G_0)$, and the surjectivity follows from the exactness³⁴ in the middle of the sequence

$$\underline{B}_h^* \xrightarrow{\text{Ad}} \underline{\text{Aut}}(B_h) \rightarrow \underline{\text{Aut}}(B_0)$$

in (5.3). We derive from (5.3) the following commutative diagram:

$$(5.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \underline{A}_h^* & \xrightarrow{a \mapsto a \otimes 1} & \widehat{G} & \longrightarrow & G_0 \longrightarrow 1 \\ & & & \searrow \text{Ad} & \downarrow & & \\ & & & & \underline{\text{Aut}}(A_h) = G, & & \end{array}$$

where the sequence on top is exact by (5.4).

For a non-negative integer m , denote by $(\underline{A}_h^*)^{\geq m}$ the subgroup of \underline{A}_h^* of elements equal to 1 modulo h^{m+1} . Then the above diagram yields a compatible family of extensions

$$(5.6) \quad 1 \rightarrow \underline{A}_h^*/(\underline{A}_h^*)^{\geq m} \rightarrow \widehat{G}/(\underline{A}_h^*)^{\geq m} \rightarrow G_0 \rightarrow 1.$$

Observe that the group scheme $\underline{A}_h^*/(\underline{A}_h^*)^{\geq m}$ over $\text{Spec } \mathbb{F}_p$ is obtained from the group scheme $\underline{\mathcal{A}}_{\mathbb{S}_m}^*$ over \mathbb{S}_m by applying the Weil restriction of scalars functor, that is $\underline{A}_h^*/(\underline{A}_h^*)^{\geq m} = \underline{\text{Mor}}_{\mathbb{S}_m}(\mathbb{S}_m, \underline{\mathcal{A}}_{\mathbb{S}_m}^*)$. Therefore, by adjunction, sequence (5.6) gives rise to an extension of group schemes over \mathbb{S}_m

$$(5.7) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \underline{\mathcal{A}}_{\mathbb{S}_m}^* & \longrightarrow & \widehat{G}_{\mathbb{S}_m} & \longrightarrow & G_0 \times \mathbb{S}_m \longrightarrow 1 \\ & & & \searrow \text{Ad} & \downarrow & & \\ & & & & \underline{\text{Aut}}(\underline{\mathcal{A}}_{\mathbb{S}_m}). & & \end{array}$$

Explicitly, extension (5.7) is obtained by pulling back (5.6) to \mathbb{S}_m and then taking the pushout with respect to the homomorphism $\underline{\text{Mor}}_{\mathbb{S}_m}(\mathbb{S}_m, \underline{\mathcal{A}}_{\mathbb{S}_m}^*) \times \mathbb{S}_m \rightarrow \underline{\mathcal{A}}_{\mathbb{S}_m}^*$. Using Construction 4.2, diagram (5.7) yields the promised action of $G_0 \times \mathbb{S}_m$ on $\underline{\mathcal{A}}_{\mathbb{S}_m}$ -Mod. It is clear from the construction that the actions are compatible as m varies. Following Construction 4.2 and Remark 4.6, we denote by $\underline{\mathcal{A}}_{\mathbb{S}}\text{-Mod}^{G_0}$ the corresponding quasi-coherent sheaf of categories over $\widehat{\mathbb{S}}/G_0$.

Homomorphism ψ defines an action of G_0 on $\underline{\mathcal{A}}_{\mathbb{S}}^b\text{-Mod}$. Denote by $\underline{\mathcal{A}}_{\mathbb{S}}^b\text{-Mod}^{G_0}$ the corresponding quasi-coherent sheaf of categories over \mathbb{S}/G_0 .

Lemma 5.4. *There exist*

(i) *A pair $\Psi_*: \underline{\mathcal{A}}_{\mathbb{S}}\text{-Mod} \leftrightarrow \underline{\mathcal{A}}_{\mathbb{S}}^b\text{-Mod}$: Ψ^* of adjoint functors of the form*

$$(5.8) \quad \Psi_*(N) = \text{Hom}_{\underline{\mathcal{A}}_{\mathbb{S}}}(\Lambda_{\mathbb{S}}, N), \quad \Psi^*(N') = \Lambda_{\mathbb{S}} \otimes_{\underline{\mathcal{A}}_{\mathbb{S}}^b} N',$$

³⁴Recall that algebras $A_h := A_{h,n}$ and $A_h^b := A_{h,n}^b$ implicitly depend on the dimension parameter n , and that by (3.6) there is an isomorphism of algebras $A_h^b \simeq A_{h,2n}$. Consequently, we have an isomorphism $B_h \simeq A_{h,3n}$ and so the exactness follows from (5.4).

where $\Lambda_{\mathbb{S}}$ is a locally finitely generated $\mathcal{A}_{\mathbb{S}} \otimes_{\mathcal{O}_{\mathbb{S}}} \mathcal{A}_{\mathbb{S}}^{b,op}$ -module,

- (ii) A pair $(\Psi_*)_{\widehat{\mathbb{S}}/G_0}: \mathcal{A}_{\widehat{\mathbb{S}}}\text{-Mod}^{G_0} \leftrightarrow \mathcal{A}_{\mathbb{S}}^b\text{-Mod}^{G_0}: (\Psi^*)_{\widehat{\mathbb{S}}/G_0}$ of adjoint functors together with an isomorphism between its pullback to $\widehat{\mathbb{S}}$ and the pair $((\Psi_*)_{\widehat{\mathbb{S}}}, (\Psi^*)_{\widehat{\mathbb{S}}})$,
- (iii) An integer $N > 0$, such that the kernel and cokernel of the adjunction morphisms $\text{Id} \rightarrow \Psi_* \circ \Psi^*$, $\Psi^* \circ \Psi_* \rightarrow \text{Id}$ are killed by h^N (in particular, the adjunction morphisms are isomorphisms when restricted to $\mathbb{S} \setminus 0$).

Proof. The group G_0 acts on $\mathcal{A}_{\widehat{\mathbb{S}}}\text{-Mod}$ and on $\mathcal{A}_{\mathbb{S}}^b\text{-Mod}$. This yields an action of G_0 on $\mathcal{B}_{\widehat{\mathbb{S}}}\text{-Mod}$ and every global section of $\mathcal{B}_{\widehat{\mathbb{S}}}\text{-Mod}^{G_0}$ gives rise to a functor $\mathcal{A}_{\mathbb{S}}^b\text{-Mod}^{G_0} \rightarrow \mathcal{A}_{\widehat{\mathbb{S}}}\text{-Mod}^{G_0}$. In terms of group extensions this G_0 -action is given by diagram (5.3); indeed, following Remark 4.12 this reduces to identifying \widehat{G}^\sharp with the quotient of $\widehat{G} \times \underline{B}_h^*$ by \underline{A}_h^* embedded antidiagonally. Here, (5.3) produces a compatible sequence of diagrams of group schemes over \mathbb{F}_p

$$\begin{array}{ccccccc} 1 & \longrightarrow & \underline{B}_h^*/(\underline{B}_h^*)^{\geq m} & \longrightarrow & \widehat{G}^\sharp/(\underline{B}_h^*)^{\geq m} & \longrightarrow & G_0 \longrightarrow 1 \\ & & & & \downarrow & & \\ & & & & \underline{\text{Aut}}(\underline{B}_h/h^m), & & \end{array}$$

$\swarrow \text{Ad}$

for $m \geq 1$, which then gives an action of G_0 on $\mathcal{B}_{\widehat{\mathbb{S}}}\text{-Mod}$ (as in the passage from (5.6) to (5.7) above).

Moreover, the homomorphism $\iota: \widehat{G}^\sharp \rightarrow \underline{\text{Aut}}_{\mathbb{F}_p[[h]]}(\Lambda)$ from (5.3) restricts to homomorphisms $\iota_m: \widehat{G}^\sharp/(\underline{B}_h^*)^{\geq m} \rightarrow \underline{\text{Aut}}_{\mathbb{F}_p[[h]]/h^m}(\Lambda/h^m)$ for $m \geq 1$, compatibly with B_h/h^m -module structure on $\Lambda_m := \Lambda/h^m$ (meaning that the two corresponding actions of $\underline{B}_h^*/(\underline{B}_h^*)^{\geq m}$ on $\underline{\Lambda}_m$ agree). By Remark 4.8 we can consider each Λ_m as an object of $(\mathcal{B}_{\widehat{\mathbb{S}}/G_0}\text{-Mod})(\mathbb{S}_m/G_0)$, which then glue into an object

$$\Lambda_{\widehat{\mathbb{S}}/G_0} \in (\mathcal{B}_{\widehat{\mathbb{S}}}\text{-Mod}_{\widehat{\mathbb{S}}/G_0})(\widehat{\mathbb{S}}/G_0) \simeq \lim_{m \geq 0} (\mathcal{B}_{\widehat{\mathbb{S}}/G_0}\text{-Mod})(\mathbb{S}_m/G_0)$$

of the global sections over $\widehat{\mathbb{S}}/G_0$. Let us denote by $\Lambda_{\widehat{\mathbb{S}}}$ the pull-back of $\Lambda_{\widehat{\mathbb{S}}/G_0}$ via $\widehat{\mathbb{S}} \rightarrow \widehat{\mathbb{S}}/G_0$. Merely as a $\mathcal{B}_{\widehat{\mathbb{S}}}$ -module, $\Lambda_{\widehat{\mathbb{S}}}$ can be extended to a $\mathcal{B}_{\mathbb{S}}$ -module $\Lambda_{\mathbb{S}}$, locally free over $\mathcal{O}_{\mathbb{S}}$. Indeed, we have a Morita equivalence $\mathcal{B}_{\mathbb{S} \setminus \{0\}}\text{-Mod} \xrightarrow{\sim} \mathcal{O}_{\mathbb{S} \setminus \{0\}}\text{-Mod}$. Using this equivalence, $\Lambda_{\mathbb{S}}$ determines a 1-dimensional space over $\mathbb{F}_p((h))$. To construct $\Lambda_{\mathbb{S}}$, choose a free rank one $\mathbb{F}_p[[h^{-1}]$ -submodule of the latter and use the Beauville-Laszlo theorem [BL]. Finally, we define the desired pair of adjoint functors by formulae (5.8). The adjunction morphisms $\text{Id} \rightarrow \Psi_* \circ \Psi^*$, $\Psi^* \circ \Psi_* \rightarrow \text{Id}$ are induced by morphisms of bimodules that are finitely generated as $\mathcal{O}_{\mathbb{S}}$ -modules and also are isomorphisms away from $0 \in \mathbb{S}$. Thus the cones of these morphisms are killed by some power of h . \square

We return to the construction of the action of G_0 on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$. Let $\pi: G_0 \times \mathbb{S} \rightarrow \mathbb{S}$ be the projection. Giving a descent datum on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ along the morphism $\mathbb{S} \rightarrow \mathbb{S}/G_0$ amounts to giving an autoequivalence

$$\theta: \pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod} \xrightarrow{\sim} \pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod}$$

together with the cocycle isomorphisms. In turn, θ is determined by a bimodule $M_\theta \in (\mathcal{A}_{\mathbb{S}} \otimes \mathcal{A}_{\mathbb{S}}^{op})\text{-Mod}(G_0 \times \mathbb{S})$. We construct M_θ using the Beauville-Laszlo theorem.

The action ψ of G_0 on $\mathcal{A}_{\mathbb{S}}^b$ determines an autoequivalence

$$\theta^b: \pi^* \mathcal{A}_{\mathbb{S}}^b\text{-Mod} \xrightarrow{\sim} \pi^* \mathcal{A}_{\mathbb{S}}^b\text{-Mod}.$$

Consider the composition

$$(5.9) \quad \pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod} \xrightarrow{\pi^*(\Psi_*)} \pi^* \mathcal{A}_{\mathbb{S}}^b\text{-Mod} \xrightarrow{\theta^b} \pi^* \mathcal{A}_{\mathbb{S}}^b\text{-Mod} \xrightarrow{\pi^*(\Psi^*)} \pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod},$$

where Ψ_* , Ψ^* are the functors constructed in Lemma 5.4. Let

$$\hat{\theta}: \pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod} \xrightarrow{\sim} \pi^* \mathcal{A}_{\mathbb{S}}\text{-Mod}$$

be the functor determined by the action G_0 on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$. Since Ψ^* restricted to $\hat{\mathbb{S}}$ is G_0 -equivariant (by Lemma 5.4 (ii)), we have that

$$\pi^*(\hat{\Psi}^*) \circ \hat{\theta}^b \circ \pi^*(\hat{\Psi}_*) \xrightarrow{\sim} \pi^*(\hat{\Psi}^*) \circ \pi^*(\hat{\Psi}_*) \circ \hat{\theta} \longrightarrow \hat{\theta},$$

where the right arrow comes by adjunction. By Lemma 5.4 the cone of the right morphism is killed by a power h . Reformulating in terms of bimodules we have a $\mathcal{O}_{G_0 \times \mathbb{S}}$ -coherent bimodule $M' \in \pi^*(\mathcal{A}_{\mathbb{S}} \otimes \mathcal{A}_{\mathbb{S}}^{op})\text{-Mod}(G_0 \times \mathbb{S})$ determined by (5.9), a bimodule $M_{\hat{\theta}} \in \pi^*(\mathcal{A}_{\hat{\mathbb{S}}} \otimes \mathcal{A}_{\hat{\mathbb{S}}}^{op})\text{-Mod}(G_0 \hat{\times} \hat{\mathbb{S}})$ coherent as a $\mathcal{O}_{G_0 \hat{\times} \hat{\mathbb{S}}}$ -module over the formal scheme and an isogeny between $M_{\hat{\theta}}$ and M' restricted to $G_0 \hat{\times} \hat{\mathbb{S}}$. This determines the glueing datum over $\mathcal{O}(G_0)((h))$: an isomorphism between $M_{\hat{\theta}} \otimes_{\mathcal{O}(G_0)[[h]]} \mathcal{O}(G_0)((h))$ and the restriction of M' . By the Beauville-Laszlo theorem the latter defines $M_{\hat{\theta}}$. We leave it to the reader to glue the cocycle isomorphisms. This completes the proof of Theorem 1. \square

We shall denote by $\mathcal{A}_{\mathbb{S}}\text{-Mod}^{G_0}$ the quasi-coherent sheaves of categories over $\mathbb{S}/G_0 \simeq \mathbb{S} \times BG_0$ obtained by applying Theorem 1 to $H = G_0$ and $v = \text{Id}$.

We end the section with a proof of the result that we used in Remark (b) (during the proof of Theorem 1).

Lemma 5.5. *Let \mathcal{A} be a sheaf of \mathcal{O}_X -modules on a scheme X which is finite locally free as an \mathcal{O}_X -module. Then*

$$H_{\text{fpqc}}^1(X, \mathcal{A}^\times) \simeq H_{\text{Zar}}^1(X, \mathcal{A}^\times).$$

Proof. It suffices to show that any \mathcal{A} -module \mathcal{L} that is fpqc locally isomorphic to \mathcal{A} is also Zariski locally isomorphic to \mathcal{A} . Note that the \mathcal{O}_X -module underlying \mathcal{L} is Zariski locally free by flat descent. Let $\pi: \text{Tot}_X(\mathcal{L}) \rightarrow X$ denote the projection. Consider the open subscheme $U \subset \text{Tot}_X(\mathcal{L})$ classifying points v of $\text{Tot}_X(\mathcal{L})$ such that the map $\mathcal{A}|_{\pi(v)} \rightarrow \mathcal{L}|_{\pi(v)}$ defined by $a \mapsto a \cdot v$ is an isomorphism. Since \mathcal{L} is fpqc locally trivial the projection $U \rightarrow X$ is surjective. Now, pick a point $x \in X$; we claim that there exists a $k(x)$ -valued point of U over x . If the residue field $k(x)$ is infinite this is obvious. If the residue field of x is finite, note that it is true by the Lang theorem: $H_{\text{fpqc}}^1(x, \mathcal{A}^\times) \simeq H_{\text{et}}^1(x, \mathcal{A}^\times) = 0$. In both cases this point extends to a section of $U \rightarrow X$ over a Zariski neighbourhood of x which then defines a trivialization of \mathcal{L} over this neighbourhood. \square

5.2. The restriction of $\mathcal{A}_{\mathbb{S}}\text{-Mod}^{G_0}$ to $\{0\}/G_0$. In this subsection we shall identify the action of G_0 on $(\mathcal{A}_{\mathbb{S}}\text{-Mod})|_{\{0\}} \xrightarrow{\sim} \mathcal{A}_0\text{-Mod}$ from Theorem 1 with the one induced by the natural action of G_0 on \mathcal{A}_0 .

For applications in §6.5 we shall work in a more general setting of Theorem 1. Let H be an affine group scheme over \mathbb{F}_p equipped with a homomorphism $v: H \rightarrow G_0$.

Let $\iota: \mathbb{S} \rightarrow \mathbb{S}$ be the involution sending h to $-h$. There exists a unique isomorphism of algebras over \mathbb{S}

$$(5.10) \quad \alpha: \iota^* \mathcal{A}_{\mathbb{S}}^{op} \xrightarrow{\sim} \mathcal{A}_{\mathbb{S}}$$

such that, for every i , $\alpha(x_i) = x_i$, $\alpha(y_i) = y_i$. Observe that, for any algebra $\mathbb{C}_{\mathbb{S}}$, the group stack of autoequivalences of $\iota^* \mathbb{C}_{\mathbb{S}}^{op}\text{-Mod}$ is that of $\mathbb{C}_{\mathbb{S}}\text{-Mod}$ pulled back by ι . It follows that there is an equivalence of categories between the groupoid of actions of H on $\mathbb{C}_{\mathbb{S}}\text{-Mod}$ and on $\iota^* \mathbb{C}_{\mathbb{S}}^{op}\text{-Mod}$. Consequently, α defines an autoequivalence of the groupoid of actions of H on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$. We shall refer to the latter as twisting by α . Applying this autoequivalence to an action from Theorem 1 (equipped with a trivialization at ∞) we derive that its twist by α is uniquely isomorphic to the former one. We rephrase this using the language of group extensions: if the action is given by a diagram as in (5.1), then there exists a unique isomorphism $\hat{\tau}: \iota^* \hat{H}_{\mathbb{S}} \rightarrow \hat{H}_{\mathbb{S}}$ fitting in the diagram

$$(5.11) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \iota^* \underline{\mathcal{A}}_{\mathbb{S}}^* & \longrightarrow & \iota^* \hat{H}_{\mathbb{S}} & \longrightarrow & H_{\mathbb{S}} \longrightarrow 1 \\ & & \downarrow \tau & & \downarrow \hat{\tau} & & \downarrow Id \\ 1 & \longrightarrow & \underline{\mathcal{A}}_{\mathbb{S}}^* & \longrightarrow & \hat{H}_{\mathbb{S}} & \longrightarrow & H_{\mathbb{S}} \longrightarrow 1 \end{array}$$

in which the first row is the pullback of the second one via ι , $\tau(a) = \alpha(a)^{-1}$, that commutes with the homomorphisms to $\underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}) = \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}^{op}) \xrightarrow{\sim} \iota^* \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})$ and respects the sections at ∞ . In particular, restricting $\hat{\tau}$ to $h = 0$, we find an involution $\hat{\tau}_0$ of $\hat{H}_{\mathbb{S}} \times_{\mathbb{S}} \{0\}$ that carries $a \in \underline{A}_0^*$ to a^{-1} and equals the identity on the quotient H .

Proposition 5.6. *Assume that H is connected. There exists a unique $\hat{\tau}_0$ -invariant group scheme theoretic section*

$$(5.12) \quad H \rightarrow \hat{H}_{\mathbb{S}} \times_{\mathbb{S}} \{0\}.$$

of the projection $\hat{H}_{\mathbb{S}} \times_{\mathbb{S}} \{0\} \rightarrow H_0$. In particular, this applies to $H = G_0$.

Proof. Let $(\hat{H}_{\mathbb{S}} \times_{\mathbb{S}} \{0\})^{\hat{\tau}_0} \subset \hat{H}_{\mathbb{S}} \times_{\mathbb{S}} \{0\}$ be the subgroup of $\hat{\tau}_0$ -invariants. The homomorphism $\underline{A}_0^* \xrightarrow{a \rightarrow a^2} \underline{A}_0^*$ is surjective with kernel isomorphic to μ_2 . This gives an extension

$$1 \rightarrow \mu_2 \rightarrow (\hat{H}_{\mathbb{S}} \times_{\mathbb{S}} \{0\})^{\hat{\tau}_0} \rightarrow H \rightarrow 1.$$

We wish to show that the above has a unique splitting. The uniqueness follows from connectedness of H . For the existence, it suffices to prove the claim for $H = G_0$, $v = \text{Id}$. In this case the result follows from

Lemma 5.7. *We have $H_{et}^1(G_0 \times \text{Spec } \overline{\mathbb{F}}_p, \mu_2) = 0$.*

Proof. According to [BV, Lemma 4.1] the reduced subgroup of G_0 is an extension of $\text{Sp}(2n)$ by a (connected) unipotent group. By [FP] $H_{et}^1(\text{Sp}(2n) \times \text{Spec } \overline{\mathbb{F}}_p, \mu_2) \simeq H_{\text{sing}}^1(\text{Sp}(2n)(\mathbb{C}), \mathbb{F}_2)$ (recall that $p \neq 2$); the latter group is 0 since the topological group $\text{Sp}(2n)(\mathbb{C})$ is simply connected. \square

Indeed, this gives a section $s: G_0 \dashrightarrow \hat{G}_{\mathbb{S}} \times_{\mathbb{S}} \{0\}$ of the corresponding μ_2 -torsor over $\overline{\mathbb{F}}_p$. Changing the section by an element of $\mu_2(G_0) \simeq \mathbb{Z}/2$ so that $s(1) = 1$, one sees from connectedness of $(G_0)_{\overline{\mathbb{F}}_p}$ that s is in fact a group splitting. The uniqueness of the group splitting then guarantees that it descends to \mathbb{F}_p . \square

Section (5.12) identifies the G_0 -action on $A_0\text{-Mod}$ with one induced by the tautological action of G_0 on A_0 .

5.3. A $\mathbb{G}_m \times G_0$ -equivariant structure on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$. The next proposition shows that the action of G_0 on the quasi-coherent sheaf of categories $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ constructed in Theorem 1 extends to an equivariant structure with respect to a larger group acting on \mathbb{S} . Define an action $\lambda: A_0 \rightarrow A_0 \otimes \mathcal{O}(\mathbb{G}_m)$ of \mathbb{G}_m on A_0 by the formulae

$$(5.13) \quad \lambda(x_i) = x_i, \lambda(y_i) = zy_i, 1 \leq i \leq n.$$

Here z denotes the coordinate on \mathbb{G}_m . We have that $\lambda([\eta]) = z[\eta]$. Hence, the subgroup $\mathbb{G}_m \subset \text{Aut}(A_0)$ normalizes G_0 . Denote by

$$(5.14) \quad \mathbb{G}_m \times G_0 := \mathbb{G}_m \times_{\text{Ad}_\lambda} G_0 \subset \text{Aut}(A_0)$$

the subgroup generated by G_0 and \mathbb{G}_m . Endow \mathbb{S} with an action χ of \mathbb{G}_m by homotheties: $\chi(h) = zh$. The projection to the first factor defines an action of $\mathbb{G}_m \times G_0$ on \mathbb{S} . Formulae (5.13) yield a \mathbb{G}_m -equivariant structure on sheaf of algebras $\mathcal{A}_{\mathbb{S}}$. We denote by $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/\mathbb{G}_m}$ the corresponding quasi-coherent sheaf of categories on the stack \mathbb{S}/\mathbb{G}_m . Since the \mathbb{G}_m -action on the fiber of $\mathcal{A}_{\mathbb{S}}$ over the fixed point $\{\infty\} \in \mathbb{S}$ is trivial, we have that the restriction of $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/\mathbb{G}_m}$ to $\{\infty\}/\mathbb{G}_m$ is equivalent to $\mathcal{O}_{\{\infty\}/\mathbb{G}_m}\text{-Mod}$. Finally, let $q: \mathbb{S}/\mathbb{G}_m \rightarrow \mathbb{S}/(\mathbb{G}_m \times G_0)$ by the morphism of stacks induced by the inclusion $\mathbb{G}_m \hookrightarrow \mathbb{G}_m \times G_0$.

Proposition 5.8. *There exists a unique (up to a unique equivalence) quasi-coherent sheaf of categories $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \times G_0)}$ on the stack $\mathbb{S}/(\mathbb{G}_m \times G_0)$ equipped with equivalences*

$$\Theta: q^*((\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \times G_0)}) \xrightarrow{\sim} (\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/\mathbb{G}_m},$$

$$\Xi_\infty: ((\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \times G_0)})|_{\infty/(\mathbb{G}_m \times G_0)} \xrightarrow{\sim} \mathcal{O}_{\{\infty\}/(\mathbb{G}_m \times G_0)}\text{-Mod},$$

and an isomorphism ϖ between the pullbacks of Ξ_∞ and Θ to ∞/\mathbb{G}_m , such that the action of $\mathbb{G}_m \times G_0$ on the center of $A_0\text{-Mod}$ induced by Θ comes from the embedding (5.14). Moreover, the $\mathbb{G}_m \times G_0$ -action on $A_0\text{-Mod}$ is isomorphic to the one given by (5.14).

Remark 5.9. Consider the diagram corresponding to the G_0 -action on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ (see Remark (a) in the proof of Theorem 1).

$$(5.15) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \underline{\mathcal{A}_{\mathbb{S}}}^* & \longrightarrow & \widehat{G}_{\mathbb{S}} & \longrightarrow & G_0 \times \mathbb{S} \longrightarrow 1 \\ & & \searrow \text{Ad} & & \downarrow \alpha & & \\ & & & & \underline{\text{Aut}(\mathcal{A}_{\mathbb{S}})} & & \end{array}$$

Morphisms λ , χ and Ad_λ determine a \mathbb{G}_m -equivariant structure on all the group schemes displayed above except for $\widehat{G}_{\mathbb{S}}$. The proposition asserts that there exists a unique \mathbb{G}_m -equivariant structure on $\widehat{G}_{\mathbb{S}}$ that makes the whole diagram \mathbb{G}_m -equivariant and such that the section $G_0 \times \infty \rightarrow \widehat{G}_{\mathbb{S}} \times_S \infty$ commutes with the \mathbb{G}_m -action.

Proof. We start with the existence assertion. Let $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0}$ be the quasi-coherent sheaf of categories on \mathbb{S}/G_0 constructed in Theorem 1. We construct the quadruple $((\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \times G_0)}, \Theta, \Xi_\infty, \varpi)$ by specifying the descent data for $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0}$ along the morphism

$$\text{pr}: \mathbb{S}/G_0 \rightarrow \mathbb{S}/(\mathbb{G}_m \times G_0).$$

Let $\text{pr}_{\mathbb{S}/\mathbb{G}_m}, \chi: (\mathbb{S}/G_0) \times \mathbb{G}_m \rightarrow \mathbb{S}/G_0$ be the projection and the action morphisms respectively. We have to construct an equivalence

$$(5.16) \quad \text{pr}_{\mathbb{S}/\mathbb{G}_m}^*((\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0}) \xrightarrow{\sim} \chi^*((\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0})$$

together with the cocycle datum. Note the \mathbb{G}_m -equivariant structure on $\mathcal{A}_{\mathbb{S}}$ identifies the pullback of the quasi-coherent sheaves in question to $\mathbb{S} \times \mathbb{G}_m$ with $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ lifted along $\text{pr}_{\mathbb{S}}: \mathbb{S} \times \mathbb{G}_m \rightarrow \mathbb{S}$. Thus, each of the corresponding quasi-coherent sheaves of categories determines an action of G_0 on $\text{pr}_{\mathbb{S}}^* \mathcal{A}_{\mathbb{S}}\text{-Mod}$, each of which comes with a trivialization over $\infty \times \mathbb{G}_m$. Using Remark 5.2 there exists a unique isomorphism between the two actions compatible with trivializations. This gives equivalence (5.16). The cocycle datum for this equivalence is defined using the “unique up to a unique isomorphism” part of Remark 5.2.

For uniqueness assertion, observe that if $((\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \times G_0)}, \Theta, \Xi_{\infty}, \varpi)$ is a quadruple from the Proposition, then, by Theorem 1, $\text{pr}^*((\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \times G_0)})$ is equivalent to $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0}$. Hence $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \times G_0)}$ arises from a descent data along $\text{pr}_{\mathbb{S}}$ as above and our assertion also follows from the “unique up to a unique isomorphism” part of Remark 5.2.

As in §5.2, to prove the last assertion of the Proposition it suffices to construct a \mathbb{G}_m -invariant section $G_0 \times 0 \rightarrow \widehat{G}_{\mathbb{S}} \times_{\mathbb{S}} 0$. In fact, the uniqueness of $\widehat{\tau}$ in (5.11) implies that the latter is \mathbb{G}_m -equivariant. It follows that the same is true for section (5.12). \square

5.4. Further remarks. The following corollary of Theorem 1 will be used in §6.5.

Proposition 5.10. *Let $f: T \rightarrow G_0 \times \mathbb{S}$ be a morphism of schemes, $T_0 \hookrightarrow T$ the scheme theoretic fiber of f over $G_0 \times 0$. Then Zariski locally on T there exists a morphism of \mathbb{S} -schemes $g: T \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})$ such that $f|_{T_0} = g|_{T_0}$.*

Proof. Consider the action of G_0 on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ from Theorem 1 and let

$$1 \rightarrow \underline{\mathcal{A}}_{\mathbb{S}}^* \rightarrow \widehat{G}_{\mathbb{S}} \rightarrow G_0 \times \mathbb{S} \rightarrow 1$$

be the corresponding extension (5.1). As explained in Remark (b) in the proof of Theorem 1, Zariski locally on T the map f admits a lifting $\tilde{f}: T \rightarrow \widehat{G}_{\mathbb{S}}$. The composition of \tilde{f} with $\alpha: \widehat{G}_{\mathbb{S}} \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})$ does the job. \square

References in the following have to be fixed. The following description of the extension (5.5) will be used in §6.5. For a group scheme H over \mathbb{F}_p , let LH be the sheaf of groups defined by $LH(R) = H(R((h)))$. Recall from Remark 3.18 that there is an isomorphism $A_h(h^{-1}) \xrightarrow{\sim} \text{Mat}_{p^n}(k((h)))$. This defines an embedding $G = \underline{\text{Aut}}(A_h) \hookrightarrow \underline{\text{Aut}}(A_h[h^{-1}]) \simeq \text{LPGL}(p^n)$, and a central extension

$$(5.17) \quad 1 \rightarrow L\mathbb{G}_m \rightarrow \widetilde{G} \rightarrow G \rightarrow 1, \quad \widetilde{G} = \text{LGL}(p^n) \times_{\text{LPGL}(p^n)} G$$

together with an embedding $\underline{\mathcal{A}}_h^* \hookrightarrow \widetilde{G}$ (see [BV, §3.2]) induced by the maps $\underline{\mathcal{A}}_h^* \rightarrow A_h[h^{-1}]^* \simeq \text{LGL}(p^n)$ and $\underline{\mathcal{A}}_h^* \rightarrow G$. By a result of Contou-Carrere (see for instance [BV, Lemma A.1]), the extension $L^+\mathbb{G}_m \rightarrow L\mathbb{G}_m \rightarrow L\mathbb{G}_m/L^+\mathbb{G}_m =: \text{Gr}_{\mathbb{G}_m}$ splits. Consider the commutative diagram

$$(5.18) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \underline{\mathcal{A}}_h^* & \longrightarrow & \widetilde{G}/\text{Gr}_{\mathbb{G}_m} & \longrightarrow & G_0 \longrightarrow 1 \\ & & & \searrow \text{Ad} & \downarrow & & \\ & & & & \underline{\text{Aut}}(A_h) & & \end{array}$$

Proposition 5.11. *There is an isomorphism $\widehat{G} \cong \widetilde{G}/\mathrm{Gr}_{\mathbb{G}_m}$ that induces an isomorphism between diagrams (5.5) and (5.18).*

Proof. We freely use notations from [BV, §3.2]. Consider the extension

$$(5.19) \quad 1 \rightarrow L\mathbb{G}_m \rightarrow \widetilde{G}^b \rightarrow G^b \rightarrow 1.$$

It suffices to show that its pullback via the homomorphism $\psi : G_0 \rightarrow G^b$ reduces to the subgroup $\mathrm{Gr}_{\mathbb{G}_m} \subset L\mathbb{G}_m$. In fact, we shall prove that the pullback extension reduces to the neutral connected component \widehat{W} of $\mathrm{Gr}_{\mathbb{G}_m}$. Using Lemma ?? we identify \widehat{W} with the kernel of the evaluation homomorphism $\mathrm{ev}_{h=\infty} : \underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathbb{G}_m) \rightarrow \mathbb{G}_m$. Let $A^b \subset A_h^b(h^{-1})$ be a k -subalgebra generated by $h^{-1}, x_i, y_i, h^{-1}v_i, h^{-1}u_i, 1 \leq i \leq n$. One readily sees that the action ψ on $A_h^b(h^{-1})$ preserves A^b and the left ideal $J \subset A^b$ generated by $h^{-1}, h^{-1}v_i, h^{-1}u_i$. As an algebra over the polynomial ring A^b is isomorphic to the matrix algebra $\mathrm{Mat}_{p^{2n}}(k[h^{-1}])$. In particular, its automorphism group, $\underline{\mathrm{Aut}}(A^b)$, is identified with $\underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathrm{PGL}(V))$, where V is a vector space of dimension of dimension p^{2n} . The subgroup of the former that consists of automorphisms preserving the ideal J is identified with the subgroup $\underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathrm{PGL}(V))'$ of the latter formed by elements f such that $\mathrm{ev}_{h=\infty}(f) \in \mathrm{PGL}(V)$ belongs to the subgroup $P \subset \mathrm{PGL}(V)$ of automorphisms that fix a given line span $e \subset V$. Summarizing, we see that $\psi : G_0 \rightarrow \underline{\mathrm{Aut}}(A_h^b(h^{-1})) \xrightarrow{\sim} L\mathrm{PGL}(V)$ factors through

$$(5.20) \quad G_0 \rightarrow \underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathrm{PGL}(V))'.$$

Let $\underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathrm{GL}(V))' \subset \underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathrm{GL}(V))$ be the subgroup formed by elements f such that $\mathrm{ev}_{h=\infty}(f) \in \mathrm{GL}(V)$ fixes the vector $e \in V$. The projection

$$(5.21) \quad \mathrm{GL}(V) \rightarrow \mathrm{PGL}(V)$$

defines a left exact sequence

$$1 \rightarrow \widehat{W} \rightarrow \underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathrm{GL}(V))' \rightarrow \underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathrm{PGL}(V))'.$$

To complete the proof of the Proposition it suffices to show that morphism (5.20) lifts locally, for the Zariski topology on G_0 (and consequently for the *fpc* topology), to $\underline{\mathrm{Mor}}(\mathrm{Spec} k[h^{-1}], \mathrm{GL}(V))'$. To this end consider the map

$$(5.22) \quad \Psi : G_0 \times \mathrm{Spec} k[h^{-1}] \rightarrow \mathrm{PGL}(V)$$

given by (5.20), and let L be the pullback of the \mathbb{G}_m -torsor (5.21) to $G_0 \times \mathrm{Spec} k[h^{-1}]$.

The morphism Ψ carries the closed subscheme $G_0 \xrightarrow{h=\infty} G_0 \times \mathrm{Spec} k[h^{-1}]$ to the subgroup P . Observe that the restriction of the \mathbb{G}_m -torsor (5.21) has a section. Hence L comes together with a trivialization $L|_{G_0} \xrightarrow{\sim} G_0 \times \mathbb{G}_m$. We have to check that there exists an open covering $G_0 = \cup U_i$ such that the restriction of (L, σ) to each $U_i \times \mathrm{Spec} k[h^{-1}]$ is trivial. Recall that G_0 is a group scheme of finite type over a perfect field. In particular, the reduced subscheme $G_{0,red} \subset G_0$ is smooth. It follows that, for every affine open subset $U \subset G_0$, we have that

$$\mathrm{Pic}(U \times \mathrm{Spec} k[h^{-1}]) \xrightarrow{\sim} \mathrm{Pic}(U_{red} \times \mathrm{Spec} k[h^{-1}]) \xrightarrow{\sim} \mathrm{Pic}(U_{red}).$$

Thus, there exists a covering $G_0 = \cup U_i$ such that the restriction of L to each $U_i \times \mathrm{Spec} k[h^{-1}]$ is trivial. Since the inclusion $U_i \hookrightarrow U_i \times \mathrm{Spec} k[h^{-1}]$ admits a retraction, one can choose the trivialization of L compatible with σ .

□

References in the above have to be fixed

Remark 5.12. We claim that the action of G_0 on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ constructed above does not lift to an action on the algebra $\mathcal{A}_{\mathbb{S}}$ even if \mathbb{S} is replaced by the formal completion $\hat{\mathbb{S}}$. To see this we shall prove that the surjection $r: G = \underline{\text{Aut}}(A_h) \rightarrow G_0$ has no sections *i.e.*, group homomorphisms $s: G_0 \rightarrow G$ with $r \circ s = \text{Id}$. Assuming the contrary consider the induced morphisms of the restricted Lie algebras. Write $k = \mathbb{F}_p$. The restricted Lie algebra $\text{Lie } G_0$ is the algebra of Hamiltonian vector fields on $\text{Spec } A_0$ ([BK, Lemma 3.3.]). The restricted Lie algebra $\text{Lie } G$ is isomorphic to the restricted Lie algebra of $k[[h]]$ -linear derivations of the associative algebra A_h . It follows from Proposition 3.45 that all derivations of A_h that are trivial modulo h are inner. This way we can identify $\text{Lie } G$ with the Lie subalgebra $\frac{1}{h}(A_h/k[[h]]) \subset A_h(h^{-1})/k((h))$:

$$0 \rightarrow \frac{1}{h}k[[h]] \rightarrow \frac{1}{h}A_h \xrightarrow{\text{ad}} \text{Der } A_h \rightarrow 0.$$

In particular, $\text{Lie } G$ has the structure of restricted Lie algebra over $k[[h]]$ and the differential of r is identified with projection

$$dr: \text{Lie } G \rightarrow \text{Lie } G/h \text{Lie } G = \text{Lie } G_0.$$

Thus, ds defines an isomorphism of restricted Lie algebras over $k[[h]]$

$$\text{Lie } G_0 \otimes_k k[[h]] \xrightarrow{\sim} \text{Lie } G.$$

Consequently, we have $\text{Lie } G_0 \otimes_k k((h)) \xrightarrow{\sim} \text{Lie } G \otimes_{k[[h]]} k((h))$. The isomorphism $A_h(h^{-1}) = \text{Mat}_{p^n}(k((h)))$ identifies $\text{Lie } G \otimes_{k[[h]]} k((h))$ with $\mathfrak{pgl}_{p^n}(k((h)))$, the quotient of $\mathfrak{gl}_{p^n}(k((h)))$ by its center. It is shown in [BV, Lemma 4.4] that there exists a split surjection $G_0 \rightarrow \mathbb{G}_a$. This yields a nonzero homomorphism of restricted Lie algebras $\text{Lie } G_0 \otimes_k k((h)) \rightarrow k((h))$, where $k((h))$ is equipped with the zero p -th power operation. We claim that $\mathfrak{pgl}_{p^n}(k((h)))$ does not admit such homomorphisms. Indeed, the quotient $\mathfrak{gl}_{p^n}/[\mathfrak{gl}_{p^n}, \mathfrak{gl}_{p^n}]$ is the restricted Lie algebra of the multiplicative group. In particular, p -th power operation on this quotient is non-trivial. This contradiction completes the proof.

6. CANONICAL QUANTIZATION OF $\text{QCoh}(X)$.

6.1. Construction of the canonical quantization and its uniqueness. For the duration of this subsection let S be a scheme³⁵ over \mathbb{F}_p with $p > 2$, and let $(X, [\eta])$ be a quasi-smooth scheme over S equipped a restricted symplectic structure

$$[\eta] \in H^0(X, \text{coker}(\mathcal{O}_X \xrightarrow{d} \Omega_{X/S}^1)).$$

We shall also assume that the rank of the vector bundle $\Omega_{X/S}^1$ is constant, say, $2n$, for some $n \in \mathbb{Z}$. Denote by $\omega := d[\eta]$ the corresponding symplectic form on X/S . Let $\pi: \mathcal{M}_{X, [\eta]} \rightarrow X^{\{p\}}$ be the G_0 -torsor of Darboux frames (see Construction 2.72). Denote by $\xi: X^{\{p\}} \rightarrow BG_0$ the morphism corresponding to the latter. In Theorem 1 we constructed an action of G_0 on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ and the corresponding quasi-coherent sheaf of categories $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0}$ on \mathbb{S}/G_0 . Let QCoh_h be its pullback via the morphism $\xi \times \text{Id}: X^{\{p\}} \times \mathbb{S} \rightarrow BG_0 \times \mathbb{S} = \mathbb{S}/G_0$:

$$\text{QCoh}_h := (\xi \times \text{Id})^*(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0}.$$

³⁵Do we need any finiteness assumptions on S ? — Doesn't seem so: all the object on X we are interested in are classified by maps from $X^{\{p\}}$ to finite-type stack over \mathbb{F}_p , so they'll (locally) be base-changed from $X'^{\{p\}} \rightarrow S' = \text{Spec } R'$ where R' is a finite-type \mathbb{F}_p -algebra.

In §5.2 we identified the action of G_0 restricted to $A_0\text{-Mod}$ with the one induced by the tautological G_0 -action on A_0 . In particular, section (5.12) defines a lift of the G_0 -torsor $\mathcal{M}_{X, [\eta]} \rightarrow X^{\{p\}}$ to a $\hat{G}_{\mathbb{S}} \times_{\mathbb{S}} \{0\}$ -torsor. Using Proposition 4.9 and the isomorphism $F_*^{\{p\}} \mathcal{O}_X \simeq \mathcal{M}_{X, [\eta]} \times^{G_0} A_0$ we derive an equivalence

$$(6.1) \quad \Xi_0 : (\text{QCoh}_h)_{|X^{\{p\}} \times \{0\}} \xrightarrow{\sim} F_*^{\{p\}} \mathcal{O}_X\text{-Mod}.$$

Observe that (6.1) yields, in particular, an equivalence between the category $\text{QCoh}_h(X^{\{p\}} \times \{0\})$ and the category of quasi-coherent sheaves on X . We shall refer to QCoh_h as *the canonical quantization of $\text{QCoh}(X)$* .

The trivialization of the G_0 -action on the fiber of $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ at $h = \infty$ yields an equivalence

$$\Xi_{\infty} : (\text{QCoh}_h)_{|X^{\{p\}} \times \{\infty\}} \xrightarrow{\sim} \mathcal{O}_{X^{\{p\}}}\text{-Mod}.$$

Moreover, the restriction of QCoh_h to $\mathbb{S} \setminus \{0\}$ can be described as modules over a central reduction of the algebra of twistor differential operators (see Construction 3.13):

$$(6.2) \quad (\text{QCoh}_h)_{|X^{\{p\}} \times \mathbb{S} \setminus \{0\}} \xrightarrow{\sim} \mathcal{D}_{X, h^{-1}[\eta], \mathbb{S} \setminus \{0\}}\text{-Mod},$$

where $\mathcal{D}_{X, h^{-1}[\eta], \mathbb{S} \setminus \{0\}}$ is the restriction of $\mathcal{D}_{X, h^{-1}[\eta], \mathbb{S}}$. To see this recall from the proof of Theorem 1 that the restriction of $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0}$ to $\mathbb{S} \setminus \{0\}/G_0$ is equivalent to the restriction of $(\mathcal{A}_{\mathbb{S}}^b\text{-Mod})_{\mathbb{S}/G_0}$, where the action of G_0 on $\mathcal{A}_{\mathbb{S}}^b\text{-Mod}$ arises from a G_0 -action on $\mathcal{A}_{\mathbb{S}}^b$. Proposition 4.9 and the isomorphism of algebras $\mathcal{M}_{X, [\eta]} \times^{G_0} \mathcal{A}_{\mathbb{S}}^b \simeq \mathcal{D}_{X, h^{-1}[\eta], \mathbb{S}}$ from Proposition 3.51 yields (6.2).

By construction, QCoh_h is functorial in $(X/S, [\eta])$. In particular, QCoh_h is an equivariant quasi-coherent sheaf of categories with respect to the group of automorphisms of $(X/S, [\eta])$ that acts naturally on $X^{\{p\}} \times \mathbb{S}$.

In the remaining part of this subsection we explain in what sense the canonical quantization QCoh_h is unique. Let H be a connected affine group scheme over \mathbb{F}_p equipped with a homomorphism $\nu: H \rightarrow G_0$ and let $\pi_H: \mathcal{M}_{X, [\eta], H} \rightarrow X^{\{p\}}$ be an H -torsor equipped an H -equivariant morphism $\mathcal{M}_{X, [\eta], H} \rightarrow \mathcal{M}_{X, [\eta]}$, where the action of H on $\mathcal{M}_{X, [\eta]}$ is given by ν .

Proposition 6.1. *There exists a unique (up to an equivalence unique up to a unique isomorphism) quadruple $(\text{QCoh}_h, \Theta_H, \Xi_{\infty}, \varpi_H)$, consisting of quasi-coherent sheaf of categories QCoh_h on $X^{\{p\}} \times \mathbb{S}$ equipped with the following pieces of structure:*

- (i) *An equivalence $\Theta_H: (\pi_H \times \text{Id})^* \text{QCoh}_h \xrightarrow{\sim} \text{pr}_{\mathcal{M}_{X, [\eta], H}}^* \mathcal{A}_{\mathbb{S}}\text{-Mod}$ of quasi-coherent sheaves of categories over $\mathcal{M}_{X, [\eta], H} \times \mathbb{S}$. Here $\text{pr}_{\mathcal{M}_{X, [\eta], H}}: \mathcal{M}_{X, [\eta], H} \times \mathbb{S} \rightarrow \mathbb{S}$ is the projection.*
- (ii) *An equivalence $\Xi_{\infty}: (\text{QCoh}_h)_{|X^{\{p\}} \times \{\infty\}} \xrightarrow{\sim} \mathcal{O}_{X^{\{p\}}}\text{-Mod}$ of quasi-coherent sheaves of categories over $X^{\{p\}}$ together with an isomorphism ϖ_H between the pullback of Ξ_{∞} to $\mathcal{M}_{X, [\eta], H}$ and the restriction of Θ_H to $\mathcal{M}_{X, [\eta], H} \times \{\infty\}$ ³⁶.*

that satisfies the following property

³⁶The restriction of Θ_H yields $\pi_H^*((\text{QCoh}_h)_{|X^{\{p\}} \times \{\infty\}}) \xrightarrow{\sim} (\text{pr}_{\mathcal{M}_{X, [\eta], H}}^* \mathcal{A}_{\mathbb{S}}\text{-Mod})_{|\mathcal{M}_{X, [\eta], H} \times \{\infty\}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}_{X, [\eta], H}}\text{-Mod}$, where the second equivalence comes from (3.18).

(P) *The isomorphism $\text{Center}(\pi_H^*((\text{QCoh}_h)|_{X^{(p)} \times \{0\}})) \simeq A_0 \otimes_{\mathbb{F}_p} \mathcal{O}_{\mathcal{M}_{X, [\eta], H}}$ determined by Θ_H descends³⁷ to $\text{Center}((\text{QCoh}_h)|_{X^{(p)} \times \{0\}}) \simeq F_*^{(p)} \mathcal{O}_X$.*

Proof. The existence assertion follows from the construction of QCoh_h explained above: Θ_H is the pullback of the equivalence $(\pi \times \text{Id})^* \text{QCoh}_h \xrightarrow{\sim} \text{pr}_{\mathcal{M}_{X, [\eta]}}^* \mathcal{A}_{\mathbb{S}\text{-Mod}}$ along the morphism $\mathcal{M}_{X, [\eta], H} \rightarrow \mathcal{M}_{X, [\eta]}$ and ϖ_H comes from the construction of Ξ_∞ as a decent of the equivalence on $\mathcal{M}_{X, [\eta]} \times \{\infty\}$. Property (P) follows from (6.1).

To prove the uniqueness, let $(\mathfrak{S}_i, \Theta_i, \Xi_{\infty, i}, \varpi_i)$, $i = 1, 2$, be quadruples with property (P). We shall prove that there exists a unique equivalence (up to a unique isomorphism) connecting the two. The argument is similar to the proof of the uniqueness part of Theorem 1. Giving a pair $(\mathfrak{S}_i, \Theta_i)$ is **equivalent** to specifying the descent data for $\text{pr}_{\mathcal{M}_{X, [\eta], H}}^* \mathcal{A}_{\mathbb{S}\text{-Mod}}$ along the morphism $\mathcal{M}_{X, [\eta], H} \times \mathbb{S} \rightarrow \mathcal{M}_{X, [\eta], H}/H \times \mathbb{S} = X^{(p)} \times \mathbb{S}$. In particular, $(\mathfrak{S}_i, \Theta_i)$ determines an autoequivalence ϕ_i of $\mathcal{A}_{\mathbb{S}\text{-Mod}}$ lifted to $G_0 \times \mathcal{M}_{X, [\eta], H} \times \mathbb{S}$. Property (P) implies that the composition $\phi_1 \otimes \phi_2^{-1}$ acts as Id on the center of $\mathcal{A}_{\mathbb{S}\text{-Mod}}$ pulled back to $H \times \mathcal{M}_{X, [\eta], H} \times \{0\}$. Applying Lemma 5.2 we conclude that $\phi_1 \otimes \phi_2^{-1}$ is the tensor product with a line bundle L over $H \times \mathcal{M}_{X, [\eta], H} \times \mathbb{S}$. The equivalence $\Xi_{\infty, i}$ yields an isomorphism between ϕ_i restricted to $H \times \mathcal{M}_{X, [\eta], H} \times \{\infty\}$ and the identity functor. In turn, the latter determines a trivialization of L restricted to $H \times \mathcal{M}_{X, [\eta], H} \times \infty$. The groupoid of line bundles over $H \times \mathcal{M}_{X, [\eta], H} \times \mathbb{S}$ equipped with a trivialization over $H \times \mathcal{M}_{X, [\eta], H} \times \{\infty\}$ is discrete and its π_0 is the group $H^0(H \times \mathcal{M}_{X, [\eta], H}, \mathbb{Z}) = \mathbb{Z}$ (since H is connected). Finally, the cocycle constraint implies that ϕ_i and, hence, $\phi_1 \otimes \phi_2^{-1}$ is isomorphic to Id when restricted to $1_H \times \mathcal{M}_{X, [\eta], H} \times \{\infty\}$. This defines a trivialization of L . \square

Let $(\text{QCoh}_h, \Theta_H, \Xi_\infty, \varpi_H)$ be a quadruple from Proposition 6.1. We shall explain how the uniqueness part of the Proposition can be used to construct equivalence (6.1). First, note that, by property (P), Θ_H determines a $F_* \mathcal{O}_X$ -linear structure on $(\text{QCoh}_h)|_{X^{(p)} \times \{0\}}$ (i.e., $(\text{QCoh}_h)|_{X^{(p)} \times \{0\}}$ is the direct image of a quasi-coherent sheaf of categories on X). Secondly, observe that, for any ring A , the groupoid of autoequivalences of $A\text{-Mod}$ is identified with the groupoid of autoequivalences of $A^{op}\text{-Mod}$. In particular, the descent data for $\text{pr}_{\mathcal{M}_{X, [\eta], H}}^* \mathcal{A}_{\mathbb{S}\text{-Mod}}$ along the morphism $\mathcal{M}_{X, [\eta], H} \times \mathbb{S} \rightarrow X^{(p)} \times \mathbb{S}$ specified by $(\text{QCoh}_h, \Theta_H)$ determines a descent data for $\text{pr}_{\mathcal{M}_{X, [\eta], H}}^* \mathcal{A}_{\mathbb{S}}^{op}\text{-Mod}$. We denote QCoh_h° the corresponding sheaf of categories over $X^{(p)} \times \mathbb{S}$ and by $\Theta_H^\circ: (\pi_H \times \text{Id})^* \text{QCoh}_h^\circ \xrightarrow{\sim} \text{pr}_{\mathcal{M}_{X, [\eta], H}}^* \mathcal{A}_{\mathbb{S}}^{op}\text{-Mod}$ the equivalence it comes with. Explicitly, QCoh_h° assigns to each affine $Z \xrightarrow{u} X^{(p)} \times \mathbb{S}$ the category of all right exact \mathcal{O} -linear functors $u^* \text{QCoh}_h \rightarrow \mathcal{O}_Z\text{-Mod}$ that commute with all direct sums³⁸. Equivalence Ξ_∞ and isomorphism ϖ_H determine $\Xi_\infty^\circ: (\text{QCoh}_h^\circ)|_{X^{(p)} \times \{\infty\}} \xrightarrow{\sim} \mathcal{O}_{X^{(p)}}\text{-Mod}$ and an isomorphism ϖ_H° between the pullback of Ξ_∞° to $\mathcal{M}_{X, [\eta], H}$ and the restriction of Θ_H° . Next, recall from (5.10) an isomorphism $\alpha: \iota^* \mathcal{A}_{\mathbb{S}}^{op} \xrightarrow{\sim} \mathcal{A}_{\mathbb{S}}$. By Proposition 6.1 there exists a unique equivalence of

³⁷For a quasi-coherent sheaf of categories \mathfrak{S} over a stack Y , we denote by $\text{Center}(\mathfrak{S})$ the presheaf of \mathcal{O}_Y -algebras on Y assigning to each affine scheme Z over Y the $\mathcal{O}(Z)$ -algebra $\text{Center}(\mathfrak{S}(Z))$. If $u: Z' \rightarrow Z$ is a flat morphism of affine schemes over Y then the canonical morphism $\text{Center}(\mathfrak{S}(Z)) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Z') \rightarrow \text{Center}(\mathfrak{S}(Z'))$ is an isomorphism. In particular, $\text{Center}(\mathfrak{S})$ is a sheaf for the *fpqc* topology on Y (but not necessarily a quasi-coherent sheaf on Y).

³⁸To see this observe that, for any ring A , the category of right A -modules is equivalent to the category of right exact functors from left A -modules to abelian groups that commute with all direct sums.

the triples

$$(6.3) \quad \Sigma: (\mathrm{QCoh}_h, \Theta_H, \Xi_\infty, \varpi_H) \xrightarrow{\sim} ((\mathrm{Id} \times \iota)^* \mathrm{QCoh}_h^\circ, \alpha \circ \Theta_H^\circ, \Xi_\infty^\circ, \varpi_H^\circ).$$

Denote by

$$\Sigma_0: (\mathrm{QCoh}_h)_{|X^{\{p\}} \times \{0\}} \xrightarrow{\sim} (\mathrm{QCoh}_h^\circ)_{|X^{\{p\}} \times \{0\}}$$

the restriction of the latter to $X^{\{p\}} \times \{0\}$. Observe that the pullback Σ_0 to $\mathcal{M}_{X, [\eta], H} \times \{0\}$ is naturally isomorphic to the identity functor $A_0 \otimes \mathcal{O}_{\mathcal{M}_{X, [\eta], H}}\text{-Mod} \simeq A_0^{op} \otimes \mathcal{O}_{\mathcal{M}_{X, [\eta], H}}\text{-Mod}$. Every $F_*\mathcal{O}_X$ -linear equivalence $\Xi_0: (\mathrm{QCoh}_h)_{|X^{\{p\}} \times \{0\}} \xrightarrow{\sim} F_*\mathcal{O}_X\text{-Mod}$ induces $(\mathrm{QCoh}_h^\circ)_{|X^{\{p\}} \times \{0\}} \xrightarrow{\sim} F_*\mathcal{O}_X\text{-Mod}$ denoted by Ξ_0° .

Proposition 6.2. *There exists a unique triple (Ξ_0, κ, v) , where*

$$(6.4) \quad \Xi_0: (\mathrm{QCoh}_h)_{|X^{\{p\}} \times \{0\}} \xrightarrow{\sim} F_*\mathcal{O}_X\text{-Mod}$$

is a $F_\mathcal{O}_X$ -linear equivalence of quasi-coherent sheaves of categories over $X^{\{p\}}$, κ is an isomorphism between $\pi_H^*(\Xi_0)$ and the composite*

$$\pi_H^*((\mathrm{QCoh}_h)_{|X^{\{p\}} \times \{0\}}) \xrightarrow{\Theta_H} A_0 \otimes \mathcal{O}_{\mathcal{M}_{X, [\eta], H}}\text{-Mod} \xrightarrow{\sim} \pi_H^*(F_*\mathcal{O}_X\text{-Mod}),$$

and v is an isomorphism $\Xi_0^\circ \circ \Sigma_0 \xrightarrow{\sim} \Xi_0$ compatible with κ .

Proof. Locally, for the *fpqc* topology, $(\mathrm{QCoh}_h)_{|X^{\{p\}} \times \{0\}}$, as a quasi-coherent sheaf of categories with a $F_*\mathcal{O}_X$ -linear structure, is equivalent, to $F_*\mathcal{O}_X\text{-Mod}$. Assigning to a scheme over $X^{\{p\}}$ the groupoid of such equivalences we define a gerbe on $X^{\{p\}}$ banded by the group scheme of invertible elements in $F_*\mathcal{O}_X$ or, equivalently, a gerbe \mathcal{E} on X banded by \mathbb{G}_m . The equivalence Σ_0 amounts to specifying an isomorphism between \mathcal{E} and the opposite gerbe *i.e.*, the reduction of \mathcal{E} to a gerbe \mathcal{E}' on X banded by $\mu_2 \subset \mathbb{G}_m$. By construction the pullback of \mathcal{E}' to $\mathcal{M}_{X, [\eta], H} \times_{X^{\{p\}}} X$ is trivialized. We wish to show that the latter descends uniquely to a trivialization of \mathcal{E}' .

For the uniqueness part it suffices to show that every μ_2 -torsor on X equipped with a trivialization over $\mathcal{M}_{X, [\eta], H} \times_{X^{\{p\}}} X$ admits a unique trivialization (compatible with the given trivialization over $\mathcal{M}_{X, [\eta], H} \times_{X^{\{p\}}} X$). The latter amounts to showing that the pullback map

$$(6.5) \quad H_{\text{ét}}^i(X, \mu_2) \rightarrow H_{\text{ét}}^i(\mathcal{M}_{X, [\eta]} \times_{X^{\{p\}}} X, \mu_2)$$

is an isomorphism $i = 0$ and injective for $i = 1$. The morphism $\pi \times \mathrm{Id}: \mathcal{M}_{X, [\eta]} \times_{X^{\{p\}}} X \rightarrow X$ is a H -torsor locally trivial for the étale topology. The claim follows from connectedness of H .

For the existence part we may assume that $H = G_0$. Using the vanishing of $H_{\text{ét}}^i(G_0 \times \mathrm{Spec} \overline{\mathbb{F}}_p, \mu_2)$, for $i = 0, 1$ (see Lemma 5.7), we conclude that $\mu_2 \xrightarrow{\sim} \tau_{<2} R(\pi \times \mathrm{Id})_* \mu_2$. This implies that, for $H = G_0$ the pullback map (6.5) is an isomorphism $i = 0, 1$ and injective for $i = 2$ as desired. \square

Remark 6.3. We leave it to the reader to verify that equivalence (6.3) comes as the pullback of $(\mathcal{A}_{\mathbb{S}}\text{-Mod})_{\mathbb{S}/G_0} \xrightarrow{\sim} (\iota^* \mathcal{A}_{\mathbb{S}}^{op}\text{-Mod})_{\mathbb{S}/G_0}$ given by diagram (5.11). Consequently, equivalence (6.1) defined using (5.11) comes equipped with κ and v and, thus, it is isomorphic to the one from Proposition 6.2.

6.2. \mathbb{G}_m -equivariant quantizations. Let S be a scheme over \mathbb{F}_p with $p > 2$, and let (X, ω) be a scheme quasi-smooth over S equipped with a symplectic 2-form $\omega \in \Omega_{X/S}^2(X)$ and a \mathbb{G}_m -action

$$(6.6) \quad \gamma: \mathbb{G}_m \times X \rightarrow X.$$

We assume that γ is a morphism of schemes over S and the following identity holds

$$(6.7) \quad \gamma^* \omega = z^m \operatorname{pr}_X^* \omega,$$

for some integer m invertible in \mathbb{F}_p . Here z denotes the coordinate on \mathbb{G}_m and $\operatorname{pr}_X: \mathbb{G}_m \times X \rightarrow X$ the projection. The \mathbb{G}_m -action on X defines a homomorphism from the Lie algebra of \mathbb{G}_m to the Lie algebra of vector fields on X . Denote by θ the image of the generator $z \frac{\partial}{\partial z}$ of $\operatorname{Lie} \mathbb{G}_m$. Formula (6.7) together with the identity $d\omega = 0$ imply that

$$d\iota_\theta \omega = m\omega.$$

Hence, 1-form $\eta = \frac{1}{m} \iota_\theta \omega$, defines a restricted Poisson structure on X . Observe that the composition

$$\mathbb{G}_m \times X^{(1)} \xrightarrow{F_{\mathbb{G}_m} \times \operatorname{Id}} \mathbb{G}_m \times X^{(1)} \xrightarrow{\gamma'} X^{(1)},$$

where $F_{\mathbb{G}_m}$ is the Frobenius morphism on \mathbb{G}_m , carries the closed subscheme $\mathbb{G}_m \times X^{\{p\}} \hookrightarrow \mathbb{G}_m \times X^{(1)}$ to $X^{\{p\}} \hookrightarrow X^{(1)}$ yielding a morphism

$$(6.8) \quad \gamma_p: \mathbb{G}_m \times X^{\{p\}} \rightarrow X^{\{p\}}.$$

Endow $X^{\{p\}}$ with a \mathbb{G}_m -action given by γ_p . Note that the Frobenius morphism $F: X \rightarrow X^{\{p\}}$ is \mathbb{G}_m -equivariant. Also, consider the action $\chi_m: \mathbb{G}_m \times \mathbb{S} \rightarrow \mathbb{S}$ given by the formula $\chi_m^*(h) = z^m h$. We shall see that the canonical quantization QCoh_h comes equipped with a \mathbb{G}_m -equivariant structure with respect to the diagonal action $(\gamma_p, \chi_m): \mathbb{G}_m \times (X^{\{p\}} \times \mathbb{S}) \rightarrow X^{\{p\}} \times \mathbb{S}$. Let us explain a construction of the corresponding quasi-coherent sheaf of categories $(\operatorname{QCoh}_h)_{(X^{\{p\}} \times \mathbb{S})/\gamma_p, \chi_m \mathbb{G}_m}$ on the quotient stack $(X^{\{p\}} \times \mathbb{S})/\gamma_p, \chi_m \mathbb{G}_m$.

Recall from §5.3 the homomorphism $\lambda: \mathbb{G}_m \rightarrow \operatorname{Aut}(A_0)$ and denote by λ_m its pre-composition with the isogeny $t_m: \mathbb{G}_m \rightarrow \mathbb{G}_m$, $t_m^*(z) = z^m$. The action of \mathbb{G}_m on A_0 normalizes $G_0 \subset \operatorname{Aut}(A_0)$. Denote by $\mathbb{G}_m \ltimes_{\operatorname{Ad}_{\lambda_m}} G_0$ the corresponding semidirect product. Let $\pi: \mathcal{M}_{X, [\eta]} \rightarrow X^{\{p\}}$ be the G_0 -torsor of Darboux frames. Then, using λ_m and γ , the action of G_0 on $\mathcal{M}_{X, [\eta]}$ extends to an action of $\mathbb{G}_m \ltimes_{\operatorname{Ad}_{\lambda_m}} G_0$ making π equivariant with respect to this larger group. The product $\mathcal{M}_{X, [\eta]} \times \mathbb{S}$ equipped with diagonal action of $\mathbb{G}_m \ltimes_{\operatorname{Ad}_{\lambda_m}} G_0$ yields a morphism of stacks

$$(6.9) \quad (X^{\{p\}} \times \mathbb{S})/\gamma_p, \chi_m \mathbb{G}_m \xrightarrow{\sim} (\mathcal{M}_{X, [\eta]} \times \mathbb{S})/(\mathbb{G}_m \ltimes_{\operatorname{Ad}_{\lambda_m}} G_0) \rightarrow \mathbb{S}/_{\chi_m} (\mathbb{G}_m \ltimes_{\operatorname{Ad}_{\lambda_m}} G_0)$$

The homomorphism

$$\mathbb{G}_m \ltimes_{\operatorname{Ad}_{\lambda_m}} G_0 \xrightarrow{t_m \times \operatorname{Id}} \mathbb{G}_m \ltimes_{\operatorname{Ad}_\lambda} G_0$$

and the identity map on \mathbb{S} induce a morphism

$$\mathbb{S}/_{\chi_m} (\mathbb{G}_m \ltimes_{\operatorname{Ad}_{\lambda_m}} G_0) \rightarrow \mathbb{S}/_\chi (\mathbb{G}_m \ltimes_{\operatorname{Ad}_\lambda} G_0),$$

where $\chi = \chi_0$. Let

$$\bar{\xi}_S: (X^{\{p\}} \times \mathbb{S})/\gamma_p, \chi_m \mathbb{G}_m \rightarrow \mathbb{S}/(\mathbb{G}_m \ltimes_{\operatorname{Ad}_\lambda} G_0)$$

be the pre-composition of the latter with (6.9). Set $(\operatorname{QCoh}_h)_{(X^{\{p\}} \times \mathbb{S})/\gamma_p, \chi_m \mathbb{G}_m} = \bar{\xi}_S^*((\mathcal{A}_S\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \ltimes G_0)})$, where $(\mathcal{A}_S\text{-Mod})_{\mathbb{S}/(\mathbb{G}_m \ltimes G_0)}$ is constructed in Proposition 5.8. By construction, the pullback of $(\operatorname{QCoh}_h)_{(X^{\{p\}} \times \mathbb{S})/\gamma_p, \chi_m \mathbb{G}_m}$ to $X^{\{p\}} \times \mathbb{S}$ is QCoh_h . Also,

the category of global sections of $(\mathrm{QCoh}_h)_{(X^{[p]} \times \mathbb{S})/\gamma_p, x_m \mathbb{G}_m}$ on $(X^{[p]} \times \{0\})/\gamma_p \mathbb{G}_m$ is the category of quasi-coherent sheaves on $X/\gamma \mathbb{G}_m$.

6.3. The canonical quantization of the cotangent bundle. Let Y be a quasi-smooth scheme over S . We shall describe the canonical quantization of $\mathrm{QCoh}(X)$, where $X = T^*Y$ is the cotangent bundle equipped with the restricted symplectic structure given by the Liouville form $\eta \in \Omega_{X/S}^1(X)$. Recall from Definition 3.10 the sheaf of twistor differential operators obtained by applying the Rees construction to the sheaf of differential operators on Y/S equipped with the Hodge and reindexed conjugate filtrations. Twistor differential operators $\mathcal{D}_{Y,\mathbb{S}}$ form a \mathbb{G}_m -equivariant coherent sheaf of algebras on $X^{[p]} \times \mathbb{S}$. We wish to show, using Proposition 6.1, that the canonical quantization of $\mathrm{QCoh}(X)$ is given by $\mathcal{D}_{Y,\mathbb{S}}\text{-Mod}$.

Let $H \subset G_0$ be the subgroup of automorphisms of A_0 that carry the subalgebra

$$C_0 = \mathbb{F}_p[x_1, \dots, x_n]/(x_i^p, 1 \leq i \leq n) \subset A_0$$

to itself and let $\mathcal{M}_{X, [\eta], H} \hookrightarrow \mathcal{M}_{X, [\eta]}$ be a closed subscheme of Darboux frames compatible with the projection $X \rightarrow Y$. That is, a T -point of $\mathcal{M}_{X, [\eta], H}$ is a pair (i_ν, α) , where $i_\nu : T \rightarrow X^{[p]}$ is a morphism (that itself can be thought as a pair consisting of a morphism $T \rightarrow Y^{[p]}$ and global section ν of the sheaf $\Omega_{(Y \times_{Y^{[p]}} T)/T}^{1, [p]}$) and α is an isomorphism $X \times_{X^{[p]}} T \xrightarrow{\sim} \mathrm{Spec} A_0 \times T$ of restricted symplectic schemes over T that fits into a commutative diagram

$$(6.10) \quad \begin{array}{ccc} X \times_{X^{[p]}} T & \xrightarrow{\alpha} & \mathrm{Spec} A_0 \times T \\ \downarrow & & \downarrow \\ Y \times_{Y^{[p]}} T & \xrightarrow{\beta} & \mathrm{Spec} C_0 \times T, \end{array}$$

for some β (uniquely determined by α). The group H acts on $\mathcal{M}_{X, [\eta], H}$ and $\pi_H : \mathcal{M}_{X, [\eta], H} \rightarrow X^{[p]}$ is an H -torsor. We wish to construct an isomorphism of algebras

$$(6.11) \quad (\pi_H \times \mathrm{Id})^* \mathcal{D}_{Y,\mathbb{S}} \xrightarrow{\sim} \mathrm{pr}_{\mathcal{M}_{X, [\eta], H}}^* \mathcal{A}_{\mathbb{S}}.$$

To do this observe that a pair (i_ν, α) as above determines a 1-form μ on $Y \times_{Y^{[p]}} T$ relative to T with $\mu^{[p]} = \nu$: the graph $i_\mu : Y \times_{Y^{[p]}} T \hookrightarrow X \times_{X^{[p]}} T$ of μ is the preimage of the closed subscheme of $\mathrm{Spec} A_0 \times T$ specified by equations $y_i = 0$, $1 \leq i \leq n$. The image $[\mu]$ in

$$H^0(Y \times_{Y^{[p]}} T, \mathrm{coker}(\mathcal{O}_{Y \times_{Y^{[p]}} T} \xrightarrow{d} \Omega_{Y \times_{Y^{[p]}} T}^1))$$

is 0. (Conversely, (ν, β, μ) as above determines α .) The pullback of $\mathcal{D}_{Y,\mathbb{S}}$ along the map $T \times \mathbb{S} \rightarrow X^{[p]} \times \mathbb{S}$ is the central reduction of twistor differential operators

$$\mathcal{D}_{(Y \times_{Y^{[p]}} T)/T, \frac{\nu}{h^p}, \mathbb{S}} \xrightarrow{\sim} \mathcal{D}_{(Y \times_{Y^{[p]}} T)/T, \frac{[\mu]}{h}, \mathbb{S}} \xrightarrow{\sim} \mathcal{D}_{(Y \times_{Y^{[p]}} T)/T, [0], \mathbb{S}}.$$

Using β we construct $\mathcal{D}_{(Y \times_{Y^{[p]}} T)/T, [0], \mathbb{S}} \xrightarrow{\sim} \mathcal{A}_{\mathbb{S}} \boxtimes \mathcal{O}_T$. This gives (6.11) and, thus, an equivalence

$$\Theta_H : (\pi_H \times \mathrm{Id})^* \mathcal{D}_{Y,\mathbb{S}}\text{-Mod} \xrightarrow{\sim} \mathrm{pr}_{\mathcal{M}_{X, [\eta], H}}^* \mathcal{A}_{\mathbb{S}}\text{-Mod}.$$

In addition, as explained in Remark 3.11 the restriction of $\mathcal{D}_{Y,\mathbb{S}}$ to $X^{[p]} \times \{\infty\}$, being isomorphic to the pullback of $\mathrm{End}_{\mathcal{O}_{Y^{[p]}}} (F_{Y^*}^{[p]} \mathcal{O}_Y)$ via the projection $\mathrm{pr}^{[p]} : X^{[p]} \rightarrow$

$Y^{\{p\}}$, is a canonically split Azumaya algebra. The splitting yields an equivalence

$$\Xi_\infty : (\mathcal{D}_{Y,\mathbb{S}}\text{-Mod})|_{X^{\{p\}} \times \{\infty\}} \xrightarrow{\sim} \mathcal{O}_{X^{\{p\}}}\text{-Mod}.$$

Furthermore, the trivialization of the vector bundle $F_*^{\{p\}}\mathcal{O}_Y$ pulled back to $\mathcal{M}_{X,[\eta],H}$ yields an isomorphism ϖ_H making $(\mathcal{D}_{Y,\mathbb{S}}\text{-Mod}, \Theta_H, \Xi_\infty, \varpi_H)$ the canonical quantization of $\text{QCoh}(X)$ ³⁹.

Next, we wish to compute Ξ_0 . The description of $\mathcal{D}_{X,\mathbb{S} \setminus \{\infty\}}$ as the Rees construction applied to \mathcal{D}_Y endowed with the order filtration (also called the Hodge filtration) yields an isomorphism (see Remark 3.11)

$$(6.12) \quad (\mathcal{D}_{Y,\mathbb{S}})|_{X^{\{p\}} \times \{0\}} \xrightarrow{\sim} F_{X^*}^{\{p\}}\mathcal{O}_X.$$

Proposition 6.4. *The canonical equivalence Ξ_0 from Proposition 6.2 is isomorphic to the composition*

$$(6.13) \quad \Xi_0 : (\mathcal{D}_{Y,\mathbb{S}}\text{-Mod})|_{X^{\{p\}} \times 0} \xrightarrow{\gamma} F_{X^*}^{\{p\}}\mathcal{O}_X\text{-Mod} \xrightarrow{\lambda} F_{X^*}^{\{p\}}\mathcal{O}_X\text{-Mod}.$$

Here γ is induced by isomorphism (6.12) and λ is given by the formula

$$\lambda(M) = M \otimes_{F_{X^*}^{\{p\}}\mathcal{O}_X} F_{X^*}^{\{p\}} \text{pr}^* K_Y^{\frac{1-p}{2}},$$

where $K_Y := \Omega_Y^n$ and $\text{pr}: X \rightarrow Y$ is the projection.

Proof. Recall that if M is a left \mathcal{D}_Y -module then $M \otimes_{\mathcal{O}_Y} K_Y$ has the structure of a right \mathcal{D}_Y -module. This extends to an equivalence $\mathcal{D}_{Y,\mathbb{S}}\text{-Mod} \xrightarrow{\sim} (\text{Id} \times \iota)^* \mathcal{D}_{Y,\mathbb{S}}^{op}\text{-Mod} \cong (\text{Id} \times \iota)^*(\mathcal{D}_{Y,\mathbb{S}}\text{-Mod})^\circ$. The same is true for the functor that carries a left \mathcal{D}_Y -module M to the right \mathcal{D}_Y -module $M \otimes_{\mathcal{O}_Y} K_Y^{1-p} \xrightarrow{\sim} (M \otimes_{\mathcal{O}_Y} K_Y^{-p}) \otimes_{\mathcal{O}_Y} K$, where K_Y^{-p} is equipped with the Frobenius descent connection. We wish to promote the corresponding equivalence $\Sigma: \mathcal{D}_{X,\mathbb{S}}\text{-Mod} \xrightarrow{\sim} (\text{Id} \times \iota)^*(\mathcal{D}_{X,\mathbb{S}}\text{-Mod})^\circ$ to an equivalence of quadruples

$$(\mathcal{D}_{Y,\mathbb{S}}\text{-Mod}, \Theta_H, \Xi_\infty, \varpi_H) \xrightarrow{\sim} ((\text{Id} \times \iota)^* \mathcal{D}_{Y,\mathbb{S}}\text{-Mod}^\circ, \alpha \circ \Theta_H^\circ, \Xi_\infty^\circ, \varpi_H^\circ).$$

Let us just explain how to make the following diagram of functors 2-commutative.

$$(6.14) \quad \begin{array}{ccc} (\mathcal{D}_{Y,\mathbb{S}})|_{X^{\{p\}} \times \infty}\text{-Mod} & \xrightarrow{\Sigma} & (\mathcal{D}_{Y,\mathbb{S}}^{op})|_{X^{\{p\}} \times \infty}\text{-Mod} \\ \downarrow \Xi_\infty & & \downarrow \Xi_\infty^{op} \\ \text{QCoh}_{X^{\{p\}}} & \xrightarrow{\text{Id}} & \text{QCoh}_{X^{\{p\}}} \end{array}.$$

Equivalence Ξ_∞ is determined by the $(\mathcal{D}_{Y,\mathbb{S}})|_{X^{\{p\}} \times \infty}$ -module $\text{pr}^{\{p\}*} F_{Y^*}^{\{p\}}\mathcal{O}_Y$, equivalence Ξ_∞^{op} is given by $\mathcal{H}om_{\mathcal{O}_{X^{\{p\}}}}(\text{pr}^{\{p\}*} F_{Y^*}^{\{p\}}\mathcal{O}_Y, \mathcal{O}_{X^{\{p\}}})$. Thus, the commutativity datum in (6.14) amounts to an isomorphism of right $F_{Y^*}\mathcal{D}_Y$ -modules

$$(6.15) \quad F_{Y^*}^{\{p\}} K_Y^{1-p} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{Y^{\{p\}}}}(F_{Y^*}^{\{p\}}\mathcal{O}_Y, \mathcal{O}_{Y^{\{p\}}}).$$

The latter is constructed in Remark 2.43.

Let us return to the proof of Proposition. Since Ξ_0 is $F_{X^*}^{\{p\}}\mathcal{O}_X$ -linear it suffices to show that Ξ_0 carries the free module $(\underline{\mathcal{D}}_{Y,S})|_{X^{\{p\}} \times 0}$ to $F_{X^*}^{\{p\}} \text{pr}^* K^{\frac{1-p}{2}}$. Set $\Xi_0(\underline{\mathcal{D}}_{Y,S})|_{X^{\{p\}} \times 0} = L$. Then, by Proposition 6.2, L is locally free $F_{X^*}^{\{p\}}\mathcal{O}_X$ -module of rank 1 equipped with a trivialization of its pullback to $\mathcal{M}_{X,[\eta],H}$ and an isomorphism

³⁹To apply Proposition 6.1 we need to verify that H is connected. In fact, the maximal reduced subgroup $H_{red} \subset H$ is $\underline{\text{Aut}}(C_0)_{red}$.

$L \otimes_{F_{X^*}^{[p]} \mathcal{O}_X} L \xrightarrow{\sim} F_{X^*}^{[p]} \text{pr}^* K^{1-p}$ of $F_{X^*}^{[p]} \mathcal{O}_X$ -modules compatible with above trivialization. The module $F_{X^*}^{[p]} \text{pr}^* K^{\frac{1-p}{2}}$ comes equipped with the above structure, thus the uniqueness part of Proposition 6.2 completes the proof. \square

6.4. Quantizations of $\text{QCoh}(X)$ vs quantizations of \mathcal{O}_X . In this subsection we shall explain how, under a certain assumption, (6.18) below, $(\text{QCoh}_h)_{|X^{[p]} \times \hat{\mathbb{S}}}$ can be described as modules over a Frobenius-constant quantization of the algebra \mathcal{O}_X . For the sake of simplicity we shall assume that S is the spectrum of a field k of characteristic $p > 2$ and X is a smooth S -scheme.

For an integer $l \geq 0$, set $G_l = \text{coker}((\underline{A}_h^*)^{\geq l-1} \xrightarrow{\text{Ad}} G)$, where $(\underline{A}_h^*)^{\geq l-1} \subset \underline{A}_h^*$ is the subgroup of elements equal to 1 modulo h^l . Recall from [BK] that a Frobenius-constant quantization of \mathcal{O}_X of level l is a G_l -torsor $\mathcal{M}_{X, [\eta], l}$ over $X^{[p]}$ together with a G_l -equivariant morphism $\mathcal{M}_{X, [\eta], l} \rightarrow \mathcal{M}_{X, [\eta]}$ of schemes over $X^{[p]}$. The latter determines a coherent sheaf of algebras $\mathcal{O}_l = \mathcal{M}_{X, [\eta], l} \times^{G_l} \underline{A}_h / (h^{l+1})$ flat over $\mathcal{O}_{X^{[p]}}[h]/(h^{l+1})$. (Note that the canonical map $G_l \rightarrow \underline{\text{Aut}}(\underline{A}_h / (h^{l+1}))$ is not surjective. Consequently, the torsor $\mathcal{M}_{X, [\eta], l}$ carries more information than sheaf of algebras \mathcal{O}_l .) A section

$$(6.16) \quad G_0 \rightarrow G_1$$

of the projection $G_1 \twoheadrightarrow G_0$ constructed in [BK] identifies the groupoid of level 1 Frobenius-constant quantization with the groupoid of $\mathcal{O}_X^* / \mathcal{O}_X^{*p}$ -torsors on the étale site of X . Thus, every Frobenius-constant quantization \mathcal{O}_l , ($l > 0$), determines a class $\rho([\mathcal{O}_l]) \in H_{\text{ét}}^1(X^{[p]}, \mathcal{O}_X^* / \mathcal{O}_X^{*p})$.

Proposition 6.5. *For every integer $l \geq 0$, the following data are equivalent.*

- (i) A $\widehat{G}_{\mathbb{S}_l}$ -torsor $\widehat{\mathcal{M}}_{X, [\eta], l} \rightarrow X^{[p]} \times \mathbb{S}_l$ together with a $\widehat{G}_{\mathbb{S}_l}$ -equivariant morphism $\widehat{\mathcal{M}}_{X, [\eta], l} \rightarrow \mathcal{M}_{X, [\eta]} \times \mathbb{S}_l$ of schemes over $X^{[p]} \times \mathbb{S}_l$.
- (ii) A Frobenius-constant quantization $\mathcal{M}_{X, [\eta], l}$ of \mathcal{O}_X of level l together with an equivalence

$$(6.17) \quad \mathcal{O}_l\text{-Mod} \xrightarrow{\sim} (\text{QCoh}_h)_{|X^{[p]} \times \mathbb{S}_l}.$$

- (iii) An object $\mathcal{O}_l \in \text{QCoh}_h(X^{[p]} \times \mathbb{S}_l)$ such that $\Theta(\pi \times \text{Id})^*(\mathcal{O}_l) \in \mathcal{A}_{\mathbb{S}}\text{-Mod}(\mathcal{M}_{X, [\eta]} \times \mathbb{S}_l)$ is locally isomorphic to the free $\mathcal{A}_{\mathbb{S}}$ -module $\underline{\mathcal{A}}_{\mathbb{S}}$ pulled back to $\mathcal{M}_{X, [\eta]} \times \mathbb{S}_l$.

Moreover, in the setting of (ii) and (iii), $\rho([\mathcal{O}_l])$ is equal to the class of the line bundle $[\Xi_0((\mathcal{O}_l)_{|X^{[p]} \times \{0\}})] \in \text{Pic}(X)$ mapped to $H_{\text{ét}}^1(X^{[p]}, \mathcal{O}_X^* / \mathcal{O}_X^{*p})$.

Proof. (i) \Rightarrow (ii). The morphism $\widehat{G}_{\mathbb{S}_l} \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}_l})$ displayed in diagram (5.7) yields a coherent sheaf of algebras $\mathcal{O}_l = \widehat{\mathcal{M}}_{X, [\eta], l} \times^{\widehat{G}_{\mathbb{S}_l}} \mathcal{A}_{\mathbb{S}_l}$ and Proposition 4.9 yields equivalence (6.17). It remains to construct $\mathcal{M}_{X, [\eta], l}$ the algebra \mathcal{O}_l is associated with. Let \widehat{G} be a group scheme over k such that, for any scheme T over k , $\widehat{G}(T)$ is the subgroup of $\text{Mor}_{\mathbb{S}_l}(T \times \mathbb{S}_l, \widehat{G}_{\mathbb{S}_l})$ of morphisms whose composition with $\widehat{G}_{\mathbb{S}_l} \rightarrow G_0 \times \mathbb{S}_l$ is constant along \mathbb{S}_l . Smoothness of $\underline{\mathcal{A}}_{\mathbb{S}_l}^*$ implies that, for any $\widehat{G}_{\mathbb{S}_l}$ -torsor $\widehat{\mathcal{M}}_{X, [\eta], l} \rightarrow X^{[p]} \times \mathbb{S}_l$ that lifts $\mathcal{M}_{X, [\eta]} \times \mathbb{S}_l$, there exists a faithfully flat morphism $T \rightarrow X^{[p]}$ such that the pullback of $\widehat{\mathcal{M}}_{X, [\eta], l}$ to $T \times \mathbb{S}_l$ is trivial. It follows that the groupoid of such torsors is equivalent to the groupoid of torsors under $\widehat{G} / (\underline{A}_h^*)^{\geq l}$ that lift $\mathcal{M}_{X, [\eta]}$. The

torsor $\mathcal{M}_{X, [\eta], l}$ is constructed from $\widehat{G}/(\underline{A}_h^*)^{\geq l}$ -torsor corresponding to $\widehat{\mathcal{M}}_{X, [\eta], l}$ using the homomorphism $\widehat{G}/(\underline{A}_h^*)^{\geq l} \rightarrow G_{l+1} \rightarrow G_l$.

(ii) \Rightarrow (iii). The image of the free module $\underline{Q}_m \in O_m\text{-Mod}(X^{\{p\}} \times \mathbb{S}_m)$ under equivalence (6.17) does the job.

(iii) \Rightarrow (i). For a scheme over $X^{\{p\}} \times \mathbb{S}_m$, $f: T \rightarrow X^{\{p\}} \times \mathbb{S}_m$, let $\widehat{G}_{\mathbb{S}_m}(T)$ be the $\widehat{G}_{\mathbb{S}_m}(T)$ -set of pairs (\tilde{f}, α) , where $\tilde{f}: T \rightarrow \mathcal{M}_{X, [\eta]} \times \mathbb{S}_m$ is a morphism lifting f , and α is an isomorphism between $\tilde{f}^* \Theta(\pi \times \text{Id})^*(\underline{Q}_m) \in \mathcal{A}_S\text{-Mod}(T)$ and the free module \mathcal{A}_S pulled back to T . This is enough.

For the last statement of the Proposition, consider the extension

$$1 \rightarrow \mathbb{G}_m \rightarrow \widehat{G}_{\mathbb{S}_0} \rightarrow G_1 \rightarrow 1.$$

The equivalence Ξ_0 is constructed using the section $G_0 \rightarrow \widehat{G}_{\mathbb{S}_0}$ of the projection $\widehat{G}_{\mathbb{S}_0} \rightarrow G_0$ (see Remark 6.3). The assertion follows from the fact that the composition of the section $G_0 \rightarrow \widehat{G}_{\mathbb{S}_0}$ with the map $G_{\mathbb{S}_0} \rightarrow G_1$ is equal to (6.16). We leave the verification of this fact to the reader. \square

Remark 6.6. Assume that

$$(6.18) \quad H^2(X, \mathcal{O}_X) = 0.$$

Then every $\widehat{G}_{\mathbb{S}_{l+1}}$ -torsor. Indeed, as explained along the proof of Proposition 6.5 it suffices to check that every $\widehat{G}/(\underline{A}_h^*)^{\geq l}$ -torsor lifts to a $\widehat{G}/(\underline{A}_h^*)^{\geq l+1}$ -torsor. The obstruction to the lifting lives in $H^2(X^{\{p\}}, \mathcal{M}_{X, [\eta]} \times^{G_0} A_0) = H^2(X, \mathcal{O}_X) = 0$. Also under the same assumption on X and a Frobenius-constant quantization O_l with $\rho(\mathcal{O}_l) = 0$, ($l \geq 1$), one has an equivalence $O_{l-1}\text{-Mod} \xrightarrow{\sim} (\text{QCoh}_h)_{|X^{\{p\}} \times \mathbb{S}_{l-1}}$. To prove this consider the surjective homomorphism

$$\widehat{G}/(\underline{A}_h^*)^{\geq l-1} \rightarrow G_l \times_{G_1} \widehat{G}_{\mathbb{S}_0} \rightarrow 1$$

A Frobenius-constant quantization with $\rho(\mathcal{O}_l) = 0$ defines a torsor under the fiber product displayed above. The kernel of the above homomorphism is an abelian unipotent group. Thus, using the vanishing of $H^2(X, \mathcal{O}_X)$ our torsor lifts to a torsor under $\widehat{G}/(\underline{A}_h^*)^{\geq l-1}$ and the claim follows.

6.5. Canonical quantization of Lagrangian subschemes. For the duration of this subsection let $(X, [\eta])$ be a smooth symplectic variety of dimension $2n$ over a field k of characteristic $p > 2$ endowed with a restricted structure. Recall from [Mu] that a smooth Lagrangian subvariety Y of X is called restricted if $[\eta]|_Y = 0$ in $H_{Zar}^0\left(Y, \text{coker}\left(\mathcal{O}_Y \xrightarrow{d} \Omega_Y^1\right)\right)$. Let $J \subset A_0$ be the ideal generated by y_i , $1 \leq i \leq n$, and let $G_{0, J} \subset G_0$ be the subgroup of automorphisms preserving J . In ([Mu][Theorem 2.10]), Mundinger proves that there exists a $fpqc$ cover $Z \rightarrow Y^{\{p\}}$ and an isomorphism

$$Z \times_{X^{\{p\}}} X \xrightarrow{\sim} Z \times \text{Spec } A_0$$

of restricted Poisson schemes over Z that induces an isomorphism between the closed subscheme $Z \times_{Y^{\{p\}}} Y \hookrightarrow Z \times_{X^{\{p\}}} X$ and $Z \times \text{Spec } A_0/J \hookrightarrow Z \times \text{Spec } A_0$. This defines a $G_{0, J}$ -torsor $\pi_Y: \mathcal{M}_{X, Y, [\eta]} \rightarrow Y^{\{p\}}$ fitting into the commutative diagram

$$(6.19) \quad \begin{array}{ccc} \mathcal{M}_{X, Y, [\eta]} & \hookrightarrow & \mathcal{M}_{X, [\eta]} \\ \downarrow \pi_Y & & \downarrow \pi \\ Y^{\{p\}} & \hookrightarrow & X^{\{p\}}, \end{array}$$

where $\pi: \mathcal{M}_{X, [\eta]} \rightarrow X^{\{p\}}$ be the G_0 -torsor of Darboux frames. Let $(\mathrm{QCoh}_h, \Theta, \Xi_\infty, \varpi)$ be the canonical quantization of $\mathrm{QCoh}(X)$. The equivalence Θ induces $\mathrm{QCoh}_h(\mathcal{M}_{X, [\eta]} \times \mathbb{S}) \xrightarrow{\sim} \mathcal{A}_\mathbb{S}\text{-Mod}(\mathcal{M}_{X, [\eta]} \times \mathbb{S})$ also denoted by Θ . Let $\mathrm{pr}_{\mathbb{S}, Y}: \mathcal{M}_{X, Y, [\eta]} \times \mathbb{S} \rightarrow \mathbb{S}$ be the projection. Recall from Construction 3.19 a $\mathcal{A}_\mathbb{S}$ -module $\mathcal{V}_\mathbb{S}$.

Theorem 2. *There exists a unique (up to a unique isomorphism) pair $(\mathcal{V}_{Y^{\{p\}} \times \mathbb{S}}, \vartheta)$, where $\mathcal{V}_{Y^{\{p\}} \times \mathbb{S}} \in \mathrm{QCoh}_h(Y^{\{p\}} \times \mathbb{S}) \subset \mathrm{QCoh}_h(X^{\{p\}} \times \mathbb{S})$ and*

$$\vartheta: \Xi_\infty(\mathcal{V}_{Y^{\{p\}} \times \mathbb{S}})|_{Y^{\{p\}} \times \{\infty\}} \xrightarrow{\sim} \mathcal{O}_{Y^{\{p\}}}$$

such that

$$(6.20) \quad \zeta: \Theta((\pi_Y \times \mathrm{Id})^* \mathcal{V}_{Y^{\{p\}} \times \mathbb{S}} \xrightarrow{\sim} \mathrm{pr}_{\mathbb{S}, Y}^* \mathcal{V}_\mathbb{S}.$$

Moreover, one has an isomorphism

$$(6.21) \quad \Xi_0(\mathcal{V}_{Y^{\{p\}} \times \mathbb{S}})|_{Y^{\{p\}} \times \{0\}} \xrightarrow{\sim} F_*^{[p]} K_Y^{\frac{1-p}{2}},$$

where $K_Y = \Omega_Y^n$ is the canonical sheaf.

Remark 6.7. Note that K_Y^{1-p} is the relative dualizing sheaf $F^! \mathcal{O}_{Y^{\{p\}}}$ of the Frobenius morphism $F: Y \rightarrow Y^{\{p\}}$. Indeed, we have that

$$F^! \mathcal{O}_{Y^{\{p\}}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(F^* K_{Y^{\{p\}}}, K_Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(K_Y^p, K_Y) \xrightarrow{\sim} K_Y^{1-p}.$$

In particular, there is a non-degenerate pairing

$$F_*^{[p]} K_Y^{\frac{1-p}{2}} \otimes_{F_*^{[p]} \mathcal{O}_Y} F_*^{[p]} K_Y^{\frac{1-p}{2}} \rightarrow F_*^{[p]} K_Y^{1-p} \xrightarrow{\mathrm{tr}} \mathcal{O}_{Y^{\{p\}}}.$$

Proof. We start with the existence part. In §5 we defined a quasi-coherent sheaf of categories $(\mathcal{A}_\mathbb{S}\text{-Mod})_{\mathbb{S}/G_0}$ on $BG_0 \times \mathbb{S}$ whose pullback to \mathbb{S} is $\mathcal{A}_\mathbb{S}\text{-Mod}$. We shall construct an object $\mathcal{V}_{\mathbb{S}/G_0, J} \in (\mathcal{A}_\mathbb{S}\text{-Mod})_{\mathbb{S}/G_0}(BG_{0, J} \times \mathbb{S})$ whose pullback to \mathbb{S} is $\mathcal{V}_\mathbb{S}$ and then define $\mathcal{V}_{Y^{\{p\}} \times \mathbb{S}}$ to be the pullback of $\mathcal{V}_{\mathbb{S}/G_0, J}$ along the map $Y^{\{p\}} \times \mathbb{S} \rightarrow BG_{0, J} \times \mathbb{S}$ determined by $\mathcal{M}_{X, Y, [\eta]}$. Explicitly, using Remark (a) in the beginning of the proof of Theorem 1, let

$$(6.22) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \underline{\mathcal{A}}_\mathbb{S}^* & \longrightarrow & \widehat{G}_{\mathbb{S}, J} & \longrightarrow & G_{0, J} \times \mathbb{S} \longrightarrow 1 \\ & & \searrow \mathrm{Ad} & & \downarrow \alpha & & \\ & & & & \underline{\mathrm{Aut}}(\underline{\mathcal{A}}_\mathbb{S}) & & \end{array}$$

be the diagram corresponding to the $G_{0, J} \times \mathbb{S}$ -action on $\mathcal{A}_\mathbb{S}\text{-Mod}$. Also, let $\underline{\mathrm{Aut}}(\underline{\mathcal{A}}_\mathbb{S}, \underline{\mathcal{V}}_\mathbb{S})$ be the group scheme parametrizing pairs $(\phi_{\underline{\mathcal{A}}_\mathbb{S}}, \phi_{\underline{\mathcal{V}}_\mathbb{S}})$, where $\phi_{\underline{\mathcal{A}}_\mathbb{S}} \in \underline{\mathrm{Aut}}(\underline{\mathcal{A}}_\mathbb{S})$, $\phi_{\underline{\mathcal{V}}_\mathbb{S}} \in \underline{\mathrm{Aut}}_{\mathcal{O}_\mathbb{S}}(\underline{\mathcal{V}}_\mathbb{S})$ with $\phi_{\underline{\mathcal{V}}_\mathbb{S}}(av) = \phi_{\underline{\mathcal{A}}_\mathbb{S}}(a)\phi_{\underline{\mathcal{V}}_\mathbb{S}}(v)$, for all $a \in \underline{\mathcal{A}}_\mathbb{S}$, $v \in \underline{\mathcal{V}}_\mathbb{S}$. Giving an object $\mathcal{V}_{\mathbb{S}/G_0, J}$ as above is equivalent to giving a homomorphism

$$(6.23) \quad \widehat{G}_{\mathbb{S}, J} \rightarrow \underline{\mathrm{Aut}}(\underline{\mathcal{A}}_\mathbb{S}, \underline{\mathcal{V}}_\mathbb{S}),$$

whose composition with the projection $\underline{\mathrm{Aut}}(\underline{\mathcal{A}}_\mathbb{S}, \underline{\mathcal{V}}_\mathbb{S}) \rightarrow \underline{\mathrm{Aut}}(\underline{\mathcal{A}}_\mathbb{S})$ is α and whose restriction to $\underline{\mathcal{A}}_\mathbb{S}^*$ carries $a \in \underline{\mathcal{A}}_\mathbb{S}^*$ to (Ad_a, a) . The uniqueness part of Theorem 1 asserts that (6.22) together with the splitting at $h = \infty$ is uniquely characterized by the requirement that the restriction of α to $\widehat{G}_{\mathbb{S}, J} \times_{\mathbb{S}} 0$ factors through the identity morphism $G_{0, J} \rightarrow \underline{\mathrm{Aut}}(\underline{\mathcal{A}}_0)$. To produce (6.23) we shall explain a convenient construction of (6.22).

Call a scheme T over \mathbb{S} *locally constant* if, for every point $x \in T$, there exists an open neighborhood $x \in U \subset T$ and a k -scheme P such that the map $U \rightarrow \mathbb{S}$ factors

as an étale map $U \rightarrow \mathbb{S} \times P$ followed by the projection $\mathbb{S} \times P \rightarrow \mathbb{S}$. Observe that a scheme smooth over a locally constant scheme is also locally constant and that the fiber product $T_1 \times_{\mathbb{S}} T_2$ of locally constant schemes is again locally constant. Denote by \mathcal{C} the site whose underlying category is formed by locally constant schemes over \mathbb{S} equipped with the Zariski topology. Define a sheaf $\widehat{G}_{\mathbb{S},J}$ of groups on \mathcal{C} sending $T \in \mathcal{C}$ to

$$\widehat{G}_{\mathbb{S},J}(T) = \{(\phi_{\mathcal{A}_{\mathbb{S}}}, \phi_{\mathcal{V}_{\mathbb{S}}}) \in \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}, \mathcal{V}_{\mathbb{S}})(T), \psi \in G_{0,J}(T) | (\phi_{\mathcal{A}_{\mathbb{S}}})|_{T_0} = \psi|_{T_0}\},$$

where T_0 is the scheme theoretic fiber of the structure morphism $T \rightarrow \mathbb{S}$ over $0 \in \mathbb{S}$. Consider the sequence of sheaves on \mathcal{C} .

$$(6.24) \quad 1 \rightarrow \underline{\mathcal{A}}_{\mathbb{S}}^* \rightarrow \widehat{G}_{\mathbb{S},J} \rightarrow G_{0,J} \times \mathbb{S} \rightarrow 1,$$

where the second map sends $a \in \underline{\mathcal{A}}_{\mathbb{S}}^*$ to $\{(\text{Ad}_a, a), \text{Id}\}$ and the third one carries $(\phi_{\mathcal{A}_{\mathbb{S}}}, \phi_{\mathcal{V}_{\mathbb{S}}}, \psi) \in H_{\mathbb{S}}$ to ψ .

Lemma 6.8. *The sequence (6.24) is exact for the Zariski topology on \mathcal{C} .*

Proof. Every scheme $T \in \mathcal{C}$ is flat over \mathbb{S} . In particular, the restriction map $\underline{\mathcal{A}}_{\mathbb{S}}^*(T) \rightarrow \underline{\mathcal{A}}_{\mathbb{S}}^*(T \setminus T_0)$ is injective. Hence, the injectivity of the second morphism in (6.24) follows from the diagram

$$\begin{array}{ccc} \underline{\mathcal{A}}_{\mathbb{S}}^*(T) & \longrightarrow & \underline{\text{Aut}}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}})(T) \\ \downarrow & & \downarrow \\ \underline{\mathcal{A}}_{\mathbb{S}}^*(T \setminus T_0) & \xrightarrow{\cong} & \underline{\text{Aut}}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}})(T \setminus T_0). \end{array}$$

Let us show exactness at the middle term. We have to check that Zariski locally on T every $(\phi_{\mathcal{A}_{\mathbb{S}}}, \phi_{\mathcal{V}_{\mathbb{S}}}) \in \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}, \mathcal{V}_{\mathbb{S}})(T)$ with $(\phi_{\mathcal{A}_{\mathbb{S}}})|_{T_0} = \text{Id}$ comes from a section of $\underline{\mathcal{A}}_{\mathbb{S}}^*$. Using Lemma 3.47 we may assume that $\phi_{\mathcal{A}_{\mathbb{S}}} = \text{Ad}_{a'}$, for some $a' \in \underline{\mathcal{A}}_{\mathbb{S}}^*(T)$. Since $\underline{\text{Aut}}_{\mathcal{A}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}})(T) \cong \mathcal{O}^*(T)$ we conclude that $\phi_{\mathcal{V}_{\mathbb{S}}} = fa'$, for some $a' \in \mathcal{O}^*(T)$. Hence, the pair $(\phi_{\mathcal{A}_{\mathbb{S}}}, \phi_{\mathcal{V}_{\mathbb{S}}})$ is the image of $fa' \in \underline{\mathcal{A}}_{\mathbb{S}}^*(T)$.

Lastly, let us check the surjectivity of the map to $G_{0,J} \times \mathbb{S}$. Let $\psi \in G_{0,J}(T)$, for some $T \in \mathcal{C}$. Using Proposition 5.10, Zariski locally on T , there exists $\phi_{\mathcal{A}_{\mathbb{S}}} \in \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})(T)$ with $(\phi_{\mathcal{A}_{\mathbb{S}}})|_{T_0} = \psi|_{T_0}$. To construct $\phi_{\mathcal{V}_{\mathbb{S}}}$ consider the embedding $\mathcal{A}_{\mathbb{S}} \subset \text{End}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}})$ given by the action of $\mathcal{A}_{\mathbb{S}}$ on $\mathcal{V}_{\mathbb{S}}$. Note that as a $\mathcal{O}_{\mathbb{S}}$ -algebra $\text{End}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}})$ is generated by global sections $x_i, \frac{y_i}{h}, 1 \leq i \leq n$ that are rational sections of $\mathcal{A}_{\mathbb{S}}$. It follows that every automorphism $\phi_{\mathcal{A}_{\mathbb{S}}} \in \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})(T)$ whose restriction to T_0 preserves the ideal generated by y_i 's extends to an automorphism of the algebra $\text{End}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}})$ pulled back to T . Using that, Zariski locally, every automorphism of an Azumaya algebra is inner, we infer, locally on T , the existence of $\phi_{\mathcal{V}_{\mathbb{S}}} \in \underline{\text{Aut}}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}})(T)$ with $\text{Ad}_{\phi_{\mathcal{V}_{\mathbb{S}}}} = \phi_{\mathcal{A}_{\mathbb{S}}}$ in $\underline{\text{Aut}}(\text{End}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}}))(T)$ as desired. \square

Next, we claim that the sheaf $\widehat{G}_{\mathbb{S},J}$ is representable by a locally constant scheme over $G_{0,J} \times \mathbb{S}$. Indeed, using Lemma 6.8 we may view $\widehat{G}_{\mathbb{S},J}$ as a $\underline{\mathcal{A}}_{\mathbb{S}}^*$ -torsor over $G_{0,J} \times \mathbb{S}$ locally trivial for the Zariski topology. The total space of this torsor, denoted, abusing notation, also by $\widehat{G}_{\mathbb{S},J}$ is locally constant since $\mathcal{A}_{\mathbb{S}}$ is smooth over \mathbb{S} . By the Yoneda Lemma $\widehat{G}_{\mathbb{S},J}$ acquires a group scheme structure over \mathbb{S} making (6.24) a group scheme extension. Together with the natural homomorphism $\widehat{G}_{\mathbb{S},J} \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}, \mathcal{V}_{\mathbb{S}}) \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}})$ it defines an action of $G_{0,J} \times \mathbb{S}$ on $\mathcal{A}_{\mathbb{S}}\text{-Mod}$ and makes $\mathcal{V}_{\mathbb{S}}$ a $G_{0,J} \times \mathbb{S}$ -equivariant $\mathcal{A}_{\mathbb{S}}$ -module with respect to this action. To complete the proof, it remains to observe that (6.24) splits over $S \setminus \{0\}$ and, in particular, over $h = \infty$.

To construct isomorphism (6.21) we shall use the description of Ξ_0 from Section 5.2 and Remark 6.3. We need the following.

Lemma 6.9. *The group $G_{0,J}$ is connected.*

Proof. Let $G_{0,J}^0 \subset G_{0,J}$ be the stabilizer of $0 \in \text{Spec } A_0$. Since $(G_{0,J}^0)_{red} = (G_{0,J})_{red}$ it suffices to prove that $G_{0,J}^0$ is connected. The subgroup $\mathbb{G}_m \subset \underline{\text{Aut}}(A_0)$ of homotheties normalizes $G_{0,J}^0$. Moreover, the map $\mathbb{G}_m \times G_{0,J}^0 \rightarrow G_{0,J}^0$, $(\lambda, g) \mapsto \lambda^{-1} \circ g \circ \lambda$, extends to a morphism $\Phi: \mathbb{A}^1 \times G_{0,J}^0 \rightarrow G_{0,J}^0$. The restriction $\Phi|_{0 \times G_{0,J}^0}$ carries $G_{0,J}^0$ to the subgroup $P \subset \text{Sp}(2n) \subset G_{0,J}^0$ of linear symplectic transformations preserving J which is connected. The lemma follows. \square

The isomorphism $\alpha: \iota^* \mathcal{A}_{\mathbb{S}}^{op} \xrightarrow{\sim} \mathcal{A}_{\mathbb{S}}$ makes $\iota^* \mathcal{V}_{\mathbb{S}}$ into a right $\mathcal{A}_{\mathbb{S}}$ -module. Explicitly, the right action of $\mathcal{A}_{\mathbb{S}}$ on $\iota^* \mathcal{V}_{\mathbb{S}}$ is given by the formula $fx_i = x_i f$, $v(hy_i) = -\frac{\partial f}{\partial x_i}$, for all $f \in A_0/J = \Gamma(\mathbb{S}, \iota^* \mathcal{V}_{\mathbb{S}})$ and $1 \leq i \leq n$. Define a perfect pairing

$$(6.25) \quad B: \iota^* \mathcal{V}_{\mathbb{S}} \otimes_{\mathcal{O}_{\mathbb{S}}} \mathcal{V}_{\mathbb{S}} \rightarrow \iota^* \mathcal{V}_{\mathbb{S}} \otimes_{\mathcal{A}_{\mathbb{S}}} \mathcal{V}_{\mathbb{S}} \rightarrow \mathcal{O}_{\mathbb{S}}$$

as follows. The dualizing sheaf on $\text{Spec } A_0/J$ is $(\Omega_{\text{Spec } A_0/J}^n)^{1-p}$. Hence, the trace map defines a non-degenerate bilinear form:

$$(6.26) \quad (\Omega_{\text{Spec } A_0/J}^n)^{\frac{1-p}{2}} \otimes_k (\Omega_{\text{Spec } A_0/J}^n)^{\frac{1-p}{2}} \rightarrow (\Omega_{\text{Spec } A_0/J}^n)^{1-p} \xrightarrow{\text{tr}} k.$$

Using the identification

$$(6.27) \quad A_0/J \xrightarrow{\sim} (\Omega_{\text{Spec } A_0/J}^n)^{\frac{1-p}{2}}, \quad f \mapsto f(dx_1 \cdots dx_n)^{\frac{1-p}{2}}$$

the latter yields $A_0/J \otimes_k A_0/J \rightarrow k$ which by the $\mathcal{O}_{\mathbb{S}}$ -linearity extends to a bilinear form $B: \iota^* \mathcal{V}_{\mathbb{S}} \otimes_{\mathcal{O}_{\mathbb{S}}} \mathcal{V}_{\mathbb{S}} \rightarrow \mathcal{O}_{\mathbb{S}}$. We need to check that B factors through $\iota^* \mathcal{V}_{\mathbb{S}} \otimes_{\mathcal{A}_{\mathbb{S}}} \mathcal{V}_{\mathbb{S}}$. This amounts to the following identity:

$$B\left(\frac{\partial f_1}{\partial x_i}, f_2\right) + B\left(f_1, \frac{\partial f_2}{\partial x_i}\right) = 0,$$

for $f_j \in A_0/J$, $1 \leq i \leq n$, which in turn follows from the invariance property of the trace map

$$\text{tr} \circ L_{\frac{\partial}{\partial x_i}} = 0.$$

For $\phi_{\mathcal{V}_{\mathbb{S}}} \in \text{Aut}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}})$, let $\phi_{\mathcal{V}_{\mathbb{S}}}^t \in \text{Aut}_{\mathcal{O}_{\mathbb{S}}}(\iota^* \mathcal{V}_{\mathbb{S}})$ be the adjoint automorphism with respect to bilinear form (6.25) and $\nu(\phi_{\mathcal{V}_{\mathbb{S}}}) = (\phi_{\mathcal{V}_{\mathbb{S}}}^t)^{-1}$. Define an isomorphism $\hat{\tau}$ in the diagram below

$$(6.28) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \iota^* \mathcal{A}_{\mathbb{S}}^* & \longrightarrow & \iota^* \widehat{G}_{\mathbb{S},J} & \longrightarrow & G_{0,J} \times \mathbb{S} \longrightarrow 1 \\ & & \downarrow \tau & & \downarrow \hat{\tau} & & \downarrow Id \\ 1 & \longrightarrow & \mathcal{A}_{\mathbb{S}}^* & \longrightarrow & \widehat{G}_{\mathbb{S},J} & \longrightarrow & G_{0,J} \times \mathbb{S} \longrightarrow 1 \end{array}$$

sending $(\phi_{\mathcal{A}_{\mathbb{S}}}, \phi_{\mathcal{V}_{\mathbb{S}}}, \psi) \in \iota^* \widehat{G}_{\mathbb{S},J}(T)$, for $T \in \mathcal{C}$, to $(\alpha \phi_{\mathcal{A}_{\mathbb{S}}} \alpha^{-1}, \nu(\phi_{\mathcal{V}_{\mathbb{S}}}), \psi)$. Also, recall that $\tau(a) = \alpha(a)^{-1}$. We need to check $(\alpha \phi_{\mathcal{A}_{\mathbb{S}}} \alpha^{-1}, \nu(\phi_{\mathcal{V}_{\mathbb{S}}})) \in \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}, \mathcal{V}_{\mathbb{S}})(T)$. Since T is flat over \mathbb{S} it is enough to check this after replacing T by T/T_0 . Over $\mathbb{S} \setminus \{0\}$ the morphism $\mathcal{A}_{\mathbb{S}}^* \rightarrow \underline{\text{Aut}}(\mathcal{A}_{\mathbb{S}}, \mathcal{V}_{\mathbb{S}})$ is an isomorphism. Thus, we may assume that $(\phi_{\mathcal{A}_{\mathbb{S}}}, \phi_{\mathcal{V}_{\mathbb{S}}}) = (\text{Ad}_a, a)$, for some $a \in \iota^* \mathcal{A}_{\mathbb{S}}^*(T)$. Thus, using that $B(av, v') = B(v, \alpha(a)v')$, we have that $(\alpha \text{Ad}_a \alpha^{-1}, \nu(a)) = (\text{Ad}_{\tau(a)}, \tau(a))$, as desired.

By construction, the following diagram is commutative.

$$(6.29) \quad \begin{array}{ccc} \iota^* \widehat{G}_{\mathbb{S}, J} & \longrightarrow & \text{Aut}_{\mathcal{O}_{\mathbb{S}}}(\iota^* \mathcal{V}_{\mathbb{S}}) \\ \downarrow \widehat{\tau} & & \downarrow \nu \\ \widehat{G}_{\mathbb{S}, J} & \longrightarrow & \text{Aut}_{\mathcal{O}_{\mathbb{S}}}(\mathcal{V}_{\mathbb{S}}). \end{array}$$

Using Lemma 6.9 and discussion in 5.2 there exists a unique section $G_{0, J} \rightarrow \widehat{G}_{\mathbb{S}, J} \times_{\mathbb{S}} \{0\}$ invariant under $\widehat{\tau}$. This section defines an action ρ of $G_{0, J}$ on the fiber $V_0 = A_0/J$ of $\mathcal{V}_{\mathbb{S}}$ over 0. satisfying the following properties:

- (i) $\rho_g(av) = g(a)\rho_g(v)$,
- (ii) $B(\rho_g(v) \otimes \rho_g(v')) = B(v \otimes v')$.

Using Remark 6.3, we have that $\Xi_0(\mathcal{V}_{Y^{\{p\}} \times \mathbb{S}})_{|Y^{\{p\}} \times \{0\}} \xrightarrow{\sim} \mathcal{M}_{X, Y, [\eta]}^b \times^{G_{0, J}} V_0$. Now isomorphism (6.21) is a consequence of the following.

Lemma 6.10. *Isomorphism (6.27) carries $G_{0, J}$ -action ρ on V_0 to the natural action of $\underline{\text{Aut}}(A_0/J)$ on $(\Omega_{\text{Spec } A_0/J}^n)^{\frac{1-p}{2}}$ restricted to the subgroup $G_{0, J} \subset \underline{\text{Aut}}(A_0/J)$*

Proof. Since the trace map $\text{tr}: (\Omega_{\text{Spec } A_0/J}^n)^{1-p} \rightarrow k$ is $\underline{\text{Aut}}(A_0/J)$ -invariant the second $G_{0, J}$ -action on V_0 also satisfies properties (i) and (ii). We prove the Lemma by showing that there exists at most one $G_{0, J}$ -action on V_0 satisfying (i) and (ii). Indeed, using (i), any two such actions ρ, ρ' differ by 1-cocycle $c_g \frac{\rho_g(1)}{\rho'_g(1)}$ of $G_{0, J}$ with values in $(A_0/J)^*$. Using (ii) we see that $c_g^2 \equiv 1$, that is c_g is a homomorphism from $G_{0, J}$ to μ_2 . Since $G_{0, J}$ is connected it follows that $c_g \equiv 1$. \square

Let us prove the uniqueness assertion of Theorem (2). Observe that, for any flat morphism $f: W \rightarrow \mathbb{S}$ the group of automorphisms of $f^* \mathcal{V}_{\mathbb{S}} \in \mathcal{A}_{\mathbb{S}}\text{-Mod}(W)$ is $\mathcal{O}^*(W)$. Let $(\mathcal{V}_{Y^{\{p\}} \times \mathbb{S}}, \vartheta), (\mathcal{V}'_{Y^{\{p\}} \times \mathbb{S}}, \vartheta')$ be two pairs as in Theorem (2). By the observation above, assigning to a flat morphism $q: T \rightarrow Y^{\{p\}} \times \mathbb{S}$ the set $L(T)$ of isomorphisms $q^* \mathcal{V}_{Y^{\{p\}} \times \mathbb{S}} \xrightarrow{\sim} q^* \mathcal{V}'_{Y^{\{p\}} \times \mathbb{S}}$, we get a \mathbb{G}_m -torsor L over $Y^{\{p\}} \times \mathbb{S}$. Isomorphisms ϑ, ϑ' define a trivialization γ of L over $Y^{\{p\}} \times \{\infty\}$. The groupoid of \mathbb{G}_m -torsors over $Y^{\{p\}} \times \mathbb{S}$ with equipped with a trivialization over $Y^{\{p\}} \times \infty$ is discrete and its π_0 is the group $H^0(Y^{\{p\}} \times \mathbb{S}, \mathbb{Z})$ (cf. proof of the uniqueness part of Theorem 1). Using (6.20) and the inclusion

$$(\pi_Y \times \text{Id})^*: H^0(Y^{\{p\}} \times \mathbb{S}, \mathbb{Z}) \hookrightarrow H^0(\mathcal{M}_{X, Y, [\eta]} \times \mathbb{S}, \mathbb{Z})$$

we conclude that (L, γ) is trivial. \square

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