Mod 2 power operations revisited

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Abstract

In this mostly expository note we take advantage of homotopical and algebraic advances to give a modern account of power operations on the mod 2 homology of $\mathbb{E}_\infty$-ring spectra. The main advance is a quick proof of the Adem relations utilizing the Tate-valued Frobenius as a homotopical incarnation of the total power operation. We also give a streamlined derivation of the action of power operations on the dual Steenrod algebra.

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Introduction

As someone who entered college at about the time that Netflix started automatically playing the next episode of a series, I cannot imagine discovering or verifying the Adem relations using the tools available to Adem [Ade52].[1] I even find it hard to *remember* the Adem and Nishida relations.

Luckily, there is a useful mnemonic device which utilizes the **total power operation**:

$$Q(t) := \sum_{i \in \mathbb{Z}} Q^i t^i.$$

Here t is an indeterminate, and the operation $Q^i : A_* \to A_{*+i}$ acts on the homotopy of any $\mathbb{E}_\infty$-$\mathbb{F}_2$-algebra A. The total power operation then produces a map:

$$Q(t) : A_* \to A_*(\!(t)\!).$$

We extend $Q(t)$ to a ring map

$$Q(t) : A_*(\!(s)\!) \to A_*[\![s,t]\!][s^{-1}, t^{-1}]$$

by requiring that

$$Q(t)(s) = s + s^2 t^{-1}.$$

With this convention, it is possible to restate the Adem relations, following Bullett-Macdonald [BM82], Steiner [Ste83], and Bisson-Joyal [BJ97] as:

- (Adem relations) For any $x \in A_*$, $Q(t)Q(s)x$ is symmetric in s and t.

The usual Adem relations are recovered using a trick with residues which we will review in §4.3. Steiner's proof that the above identity holds is to reduce it to one of the expressions met in the proof of the Adem relations as in [Ste62, p.119] and [May70, 4.7(e,g,i)].

In the case of Steenrod operations acting on the cohomology of a space X, there is a more conceptual argument due to Segal [BM82, §4]. One can use the diagonal map to produce an version of the total power operation taking values in $H^*(X \times \mathrm{B}\Sigma_2)$. Indeed, this is one of the earlier constructions of Steenrod operations [Ste62, Ch. VII]. The iterated total square then takes values in $H^*(X \times \mathrm{B}\Sigma_2 \times \mathrm{B}\Sigma_2) = H^*(X)[s,t]$ but factors through the total fourth power which takes values in $H^*(X \times \mathrm{B}\Sigma_4)$. The automorphism swapping s and t arises as an inner auromorphism of Σ_4 so the formula for the iterated square must be symmetric in s and t.

Our primary goal is to explain how the Tate diagonal (§2.3) on spectra allows for a similar argument for general power operations. The reader could probably reconstruct the argument themselves just from the observation that the total power operation is the effect on homotopy of the (non-$\mathbb{F}_2$-linear) map of spectra:

$$A \xrightarrow{\Delta} (A \otimes_{\mathbb{F}_2} A)^{t\Sigma_2} \to A^{t\Sigma_2}.$$

In fact, we take this as a definition, and develop all the basic properties of power operations efficiently from there. We hope that this note will give a mnemonic for the *proofs* of the standard identities for power operations in much the same way that the work of Steiner [Ste83], Bisson-Joyal [BJ97], and Baker [Bak15] has provided mnemonics for their *statements*.

[1] It was precisely while trying and failing multiple times to prove the Adem relations in equivariant homotopy theory that, in act of true laziness, I stumbled upon the technique explained in this note.

Outline

In §1 and §2 we review the facts we need about the Tate construction and the Tate diagonal, following Nikolaus-Scholze [NS18]. In §3 we give three definitions of the operations Q^i: the classical one, one due to Lurie [Lur11, §2.2], and one in terms of the Tate valued Frobenius. We then explain how to recover the first properties of power operations.

In §4 we turn to the Adem relations. The key thing to prove is that having a Σ_4-equivariant map $A^{\otimes 4} \to A$ produces a lift of the iterated total power operation through the Frobenius $A \to A^{t\Sigma_4}$. This takes a little bit of work but the reader could come up with the argument themselves if they remember to use the universal property of the Tate diagonal amongst natural transformations of exact, lax symmetric monoidal functors over and over again. Indeed, this proof is an excellent illustration of the computational utility of establishing such universal properties in the first place.

Finally, in §5, we show how the Bisson-Joyal and Baker formulations of the Nishida relations arise naturally from the perspective of the Tate-valued Frobenius. We end by explaining how to recover Steinberger's formulas [BMMS86, §III.2] for the action of power operations on the dual Steenrod algebra. This last step is mostly algebraic, and essentially due to Bisson-Joyal, but we have included it for completeness.

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1 The Tate construction

We review the Tate construction (§1.1) and its universal property (§1.3) as well as the important Warwick duality (§1.2) of Greenlees [Gre94] which allows an alternative computation of the Tate construction. We end (§1.4) by spelling out what happens in the case $G = \Sigma_2$.

1.1 Definitions

Let G be a finite group and k an $\mathbb{E}_\infty$-ring, and denote by

$$\mathsf{Mod}_k^{hG} := \mathsf{Psh}(\mathrm{B}G; \mathsf{Mod}_k)$$

the ∞-category of **Borel G-modules**. There is a fully faithful embedding $\mathsf{Mod}_k^{hG} \to \mathsf{Mod}_k^G$ from Borel G-modules to modules over k in *genuine* G-spectra whose essential image consists of the **Borel complete G-modules**, i.e. those X such that $X \to F(\mathrm{E}G_+, X)$ is an equivalence. Let $\mathcal{F}$ be a collection of subgroups closed under sub-conjugacy, and $\mathrm{E}\mathcal{F}$ the G-space characterized up to homotopy by the requirement

$$\mathrm{E}\mathcal{F}^H = \begin{cases} * & H \in \mathcal{F} \\ \varnothing & H \notin \mathcal{F} \end{cases}$$

and define $\widetilde{\mathrm{E}\mathcal{F}}$ as the cofiber of $\mathrm{E}\mathcal{F}_+ \to S^0$. Then the **$\mathcal{F}$-Tate spectrum** of a Borel G-spectrum can be computed as [Gre87, p.443]:

$$X^{t\mathcal{F}} = (\widetilde{\mathrm{E}\mathcal{F}} \wedge F(\mathrm{E}G_+, X))^G,$$

where the right hand side is computed in genuine G-spectra.

It will be more convenient for us to think of the above as a computation and not a definition. Instead, we opt to define the Tate construction by a universal property, following [NS18].

To that end, let

$$\left(\mathsf{Mod}_k^{hG}\right)_{\mathcal{F}-\mathrm{ind}} \subseteq \mathsf{Mod}_k^{hG}$$

be the smallest full, stable subcategory containing all objects which are left Kan extended from diagrams $\mathrm{B}H \to \mathsf{Sp}$ for some $H \in \mathcal{F}$.

Recall [NS18, §I.3] that, associated to any exact functor $F : \mathsf{Mod}_k^{hG} \to \mathcal{E}$ to a presentable stable ∞-category $\mathcal{E}$, there is a natural transformation

$$F \to L_{\mathcal{F}} F$$

which is initial amongst natural transformations to exact functors which annihilate the subcategory $\left(\mathsf{Mod}_k^{hG}\right)_{\mathcal{F}-\mathrm{ind}}$. Concretely, $L_{\mathcal{F}}F$ is specified by the formula [NS18, I.3.3]:

$$L_{\mathcal{F}}F(X) = \operatorname*{colim}_{\left(\mathsf{Mod}_k^{hG}\right)_{\mathcal{F}-\mathrm{ind}/X} \ni Y} F(\mathrm{cofib}(Y \to X)).$$

Definition 1.1.1. With notation as above, we define

$$(-)^{t\mathcal{F}} = L_{\mathcal{F}}((-)^{hG}) : \mathsf{Mod}_k^{hG} \to \mathsf{Mod}_k.$$

More generally, if $G \subseteq G'$ we define

$$(-)^{t\mathcal{F}} = L_{\mathcal{F}}((-)^{hG}) : \mathsf{Mod}_k^{hG'} \to \mathsf{Mod}_k^{hW_{G'}G}$$

where $W_{G'}G = N_{G'}G/G$ is the Weyl group of G in G'.

Example 1.1.2. When $\mathcal{F}$ consists only of the trivial subgroup, we denote $X^{t\mathcal{F}} = X^{tG}$. This can be computed as the cofiber of the trace map $X_{hG} \to X^{hG}$.

Example 1.1.3. Suppose $G \subseteq \Sigma_n$ is a subgroup and let $\mathcal{F} = \mathcal{T}$, be the family of subgroups of G which do *not* act transitively on $\{1, ..., n\}$. When $G = C_n$ this coincides with the more commonly seen family of proper subgroups; and when $G = C_p$ this coincides with the family consisting of only the trivial subgroup.

1.2 Warwick duality

We can dualize the construction in the previous section and define the **opposite $\mathcal{F}$-Tate spectrum**[2] as

$$X^{t^{\mathrm{op}}\mathcal{F}} := \operatorname*{holim}_{\left(\left(\mathsf{Mod}_k^{hG}\right)_{\mathcal{F}-\mathrm{ind}}\right)_{X/} \ni Y} \mathrm{fib}(X \to Y)_{hG}$$

Greenlees proved [Gre94, §B] that this construction is not really new:

Theorem 1.2.1 (Warwick duality)**.** *There is a canonical equivalence*

$$X^{t^{op}\mathcal{F}} \simeq \Sigma^{-1} X^{t\mathcal{F}}.$$

In particular, we obtain extra functoriality: if $\mathcal{F} \subseteq \mathcal{F}'$, then the original construction produces a canonical map $(-)^{t\mathcal{F}'} \to (-)^{t\mathcal{F}}$ while the opposite construction, composed with suspension, produces a map $(-)^{t\mathcal{F}} \to (-)^{t\mathcal{F}'}$.

[2]We stole this name from [LG19].

1.3 Monoidal structure

We will make much use of the following excellent description of the lax symmetric monoidal structure on the Tate construction.

Proposition 1.3.1. *There is a natural transformation of lax symmetric monoidal functors*

$$(-)^{hG} \to (-)^{t\mathcal{F}}$$

which is initial amongst natural transformations of lax symmetric monoidal functors with target an exact functor that annihilates $\left(\mathsf{Mod}_k^{hG}\right)_{\mathcal{F}-\mathrm{ind}}$.

This follows from the more general result [NS18, I.3.6] about the relationship between Verdier quotients and lax symmetric monoidal structures.

1.4 An example

Let k be a field of characteristic 2. Then we have

$$\pi_* k^{h\Sigma_2} \simeq \mathrm{H}^{-*}(\mathrm{B}\Sigma_2, k) = k[\![t]\!]$$

where $t \in \pi_{-1} k^{h\Sigma_2}$ is the Stiefel-Whitney class of the canonical line bundle. The Tate construction has the effect of inverting t and we can compute

$$\pi_* k^{t\Sigma_2} = k(\!(t)\!),$$

the algebra of Laurent series over k.

On the other side, the homotopy orbits $k_{h\Sigma_2}$ have a dual basis on homotopy

$$\pi_* k_{h\Sigma_2} = k\{e_0, e_1, ...\}$$

where e_i is the linear dual of t^i. The trace map

$$k_{h\Sigma_2} \to k^{h\Sigma_2}$$

is zero on homotopy groups and so we have a short exact sequence

$$0 \to k[\![t]\!] \to k(\!(t)\!) \to \pi_* \Sigma k_{h\Sigma_2} \to 0$$

which identifies the last term as the quotient $k(\!(t)\!)/k[\![t]\!]$. This provides another basis for the homotopy of $k_{h\Sigma_2}$, and the two are related by the correspondence

$$e_i \leftrightarrow t^{-i-1}.$$

Under this interpretation, the composite map

$$k^{t\Sigma_2} \to \Sigma k_{h\Sigma_2} \to \Sigma k$$

is given by sending a Laurent series $g(t) = \sum a_i t^i$ to the **residue** a_{-1}.

Finally, Warwick duality in this context translates to the computation [GM95, 16.1]

$$\Sigma^{-1} k^{t\Sigma_2} = \operatorname*{holim}_n (\Sigma^{-n\tau} k)_{h\Sigma_2} = \operatorname*{holim}_n k \wedge (\mathbb{R}P^\infty)^{-n\tau} = \operatorname*{holim}_n k \wedge \mathbb{R}P^\infty_{-n},$$

where τ is the sign representation.

2 Tate powers

The source of power operations is the symmetry present on $X^{\otimes n}$. In §2.1 we review several constructions based on this symmetry. In §2.2 we explain how the construction $X \mapsto (X^{\otimes n})^{t\mathcal{T}}$ arises as a Goodwillie derivative; in particular this construction is exact. In §2.3, following [NS18], we describe the spectral analog of the diagonal map we will use when defining power operations.

2.1 Variants of extended powers

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then there is a natural functor

$$\mathcal{C} \to \mathcal{C}^{h\Sigma_n} = \mathsf{Fun}(\mathrm{B}\Sigma_n, \mathcal{C})$$

given as the composite

$$\mathcal{C} \xrightarrow{\delta} \left(\mathcal{C}^{\times n}\right)^{h\Sigma_n} \to \mathcal{C}^{h\Sigma_n}$$

where the latter map is a choice of tensor product. In other words, for every $X \in \mathcal{C}$, the object $X^{\otimes n}$ has a Σ_n-action.

If $\mathcal{C}$ admits homotopy limits and colimits, we can form both a 'symmetric' power of an object and a 'divided' power of an object. We do this more generally for a fixed subgroup $G \subseteq \Sigma_n$.

Definition 2.1.1. We define symmetric and divided power functors as:

$$\mathrm{Sym}^G(X) := \left(X^{\otimes n}\right)_{hG}, \quad \Gamma^G(X) := \left(X^{\otimes n}\right)^{hG}.$$

Finally, if $\mathcal{C} = \mathsf{Mod}_k$ is the ∞-category of k-modules over an $\mathbb{E}_\infty$-ring k, then:

Definition 2.1.2. Let $G \subseteq \Sigma_n$ be a subgroup. We define the **Tate power** of X as

$$\mathrm{T}_G(X) := (X^{\otimes n})^{t\mathcal{T}}$$

where $\mathcal{T}$ is the family of non-transitive subgroups of G.

In each case we abbreviate G as n if $G = \Sigma_n$.

2.2 Tate powers as a Goodwillie derivative

Let $\mathcal{C}$ and $\mathcal{D}$ be stable, presentable ∞-categories. Then the full subcategory

$$\mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \subseteq \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

admits a left adjoint [Lur17, 6.1.1.10], the 1-excisive approximation:

$$P_1 : \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathsf{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}).$$

In the case where $F(0) = 0$, we may compute $P_1 F$ as [Lur17, 6.1.1.23,6.1.1.27]:

$$P_1 F(X) = \operatorname*{hocolim}_n \Omega^n_{\mathcal{D}} F(\Sigma^n_{\mathcal{C}} X).$$

I believe the following is well-known but do not know a reference.

Proposition 2.2.1. *With notation as in §2.1, There is an equivalence:*

$$P_1\Gamma^G \simeq T_G.$$

Proof. Let V denote the standard representation of Σ_n on $\mathbb{R}^n$ and $\overline{V}$ the reduced standard representation. By the formula above we have

$$\begin{aligned} P_1\Gamma^G(X) &= \operatorname*{hocolim}_j \Omega^j \Gamma^G(\Sigma^j X) \\ &\simeq \operatorname{hocolim} \Omega^j (\Sigma^{jV} X^{\otimes n})^{h\Sigma_n} \\ &\simeq \operatorname{hocolim} (\Sigma^{j\overline{V}} X^{\otimes n})^{h\Sigma_n} \\ &\simeq \operatorname{hocolim} (S^{j\overline{V}} \wedge F(\mathrm{E}G_+, X^{\otimes n}))^G \\ &\simeq (S^{\infty\overline{V}} \wedge F(\mathrm{E}G_+, X^{\otimes n}))^G. \end{aligned}$$

The last identification used that genuine fixed points commute with all homotopy limits and colimits. Finally, observe that $S^{\infty\overline{V}}$ is a model for $\widetilde{\mathrm{E}\mathfrak{T}}$. □

The same argument computes the Goodwillie *co*derivative of Sym^G:

Proposition 2.2.2. *The Goodwillie coderivative of* Sym^G *is* $((-)^{\otimes n})^{t^{op}\mathfrak{T}} = \Sigma^{-1}T_G$.

This last observation motivates the excellent account of stable power operations given by Glasman-Lawson [LG19].

2.3 The Tate diagonal

Recall the following result of Nikolaus [Nik16, Cor. 6.9]:

Proposition 2.3.1. *The forgetful functor $U : \mathsf{Mod}_k \to \mathsf{Sp}$ is initial amongst exact, lax symmetric monoidal functors to spectra.*

In the previous section we identified T_G as a Goodwillie derivative. In particular, T_G is exact. It also has a lax symmetric monoidal structure, being a composite of lax symmetric monoidal functors. So we get the following:

Corollary 2.3.2. *There is an essentially unique natural transformation of lax symmetric monoidal functors $U \to UT_G$.*

We refer to this map $\Delta_G : M \to T_G(M)$ as the **Tate diagonal**.

Remark 2.3.3. This is not the same as the Tate diagonal in [NS18] unless $k = S^0$, since we use the tensor product in Mod_k. Of course there is an evident relationship between the two: the Tate diagonal above is just the composite

$$M \to (M^{\wedge n})^{t\mathfrak{T}} \to (M^{\otimes n})^{t\mathfrak{T}}.$$

Warning 2.3.4. The Tate diagonal is *not* k-linear.

3 Power operations

We now fix a field k of characteristic 2 and let Mod_k be the ∞-category of k-module (spectra). In §3.1 we serve up power operations three ways, and then verify they agree in §3.5. In between we verify the first properties of power operations up to the Cartan formula. We emphasize that this section does not show off the utility of the approach via the Tate-valued Frobenius, but we have included the proofs since they are still pleasant.

3.1 Three definitions of operations

First we specify the objects on which power operations will act.

Definition 3.1.1. We say that $A \in \mathsf{Mod}_k$ is **equipped with a symmetric multiplication** if we have specified a map $\mathrm{Sym}^2(A) \to A$ of k-modules. Equivalently, if we have specified a map $A^{\otimes 2} \to A$ in $\mathsf{Mod}_k^{h\Sigma_2}$.

Remark 3.1.2. A k-module with a symmetric multiplication is the same as an object of $\mathscr{C}(2,\infty)$ in the notation of [May70].

To give the classical construction of power operations we'll need a computation.

Lemma 3.1.3. *For any integer n there is a canonical equivalence*

$$\mathrm{Sym}^2(\Sigma^n k) \simeq \Sigma^{2n} k_{h\Sigma_2}.$$

Proof. The object $(\Sigma^n k)^{\otimes 2} = \Sigma^{nV} k$ in $\mathsf{Mod}_k^{h\Sigma_2}$ corresponds to a map $\mathrm{B}\Sigma_2 \to \mathsf{Mod}_k^{h\Sigma_2}$ which is determined by a map

$$\Sigma_2 \to \mathsf{End}_k(\Sigma^{2n} k) \simeq \mathsf{End}_k(k,k) \simeq k.$$

of $\mathbb{E}_1$-monoids. The map factors through the units $k^\times$, but k has characteristic 2 and hence no nontrivial square roots of unity. So the action is trivial and the result follows. □

The following construction is the current standard definition of power operations.

Construction 3.1.4 (Hands-on power operations)**.** Let A be a k-module equipped with a symmetric multiplication. Given $x \in \pi_n A$ and $i \geqslant n$, define $Q^i(x) \in \pi_{n+i} A$ as the composite

$$S^{n+i} \xrightarrow{\Sigma^{2n} e_{i-n}} \Sigma^{2n} k_{h\Sigma_2} \simeq \mathrm{Sym}^2(\Sigma^n k) \xrightarrow{\mathrm{Sym}^2(x)} \mathrm{Sym}^2(A) \longrightarrow A\ .$$

This has the benefit of generalizing well to power operations for other cohomology theories, but in the case of mod 2 cohomology there is a more uniform option. The author learned this next approach from [Lur11, §2.2] and has not found an earlier reference, but a more recent and detailed account can be found in [LG19].

First we need a preliminary observation. Let $T_2' : \mathsf{Mod}_k \to \mathsf{Mod}_k$ denote the left Kan extension of the restriction of T_2 to the full subcategory of compact objects. This endomorphism commutes with all colimits and so ([Lur17, 7.1.2.4]) there is a bimodule B and an equivalence $T_2'(M) \simeq B \otimes M$. By evaluating on $M = k$ we deduce that $B = k^{t\Sigma_2}$ as a left k-module. Notice, by construction, we have a natural map $B \otimes M \to T_2(M)$.

Construction 3.1.5 (Stable power operations)**.** Let A be a k-module equipped with a symmetric multiplication. The element $t^{-i-i} \in \pi_{i+1}k^{t\Sigma_2}$ extends to a *right* module map $\Sigma^i k \to \Sigma^{-1}B$. We now define $Q^i : \Sigma^i A \to A$ as the (non k-linear!) composite:

$$\Sigma^i A = \Sigma^i k \otimes A \to \Sigma^{-1} B \otimes A \to \Sigma^{-1} T_2(A) \to \mathrm{Sym}^2(A) \to A.$$

This construction emphasizes the role of $\Sigma^{-1}k^{t\Sigma_2}$ as acting on A, but we can also record this information in a kind of co-action. For that we first need a computation.

Lemma 3.1.6. *For any k-module M equipped with the trivial Σ_2-action, there is a canonical equivalence of $\pi_* k^{t\Sigma_2}$-modules*

$$\pi_* M^{t\Sigma_2} \simeq M_*(\!(t)\!).$$

Proof. It suffices to prove $\pi_* M^{h\Sigma_2} \simeq M_*[\![t]\!]$. From the skeletal filtration on $\mathrm{B}\Sigma_2$ we have

$$M^{h\Sigma_2} \simeq \mathrm{holim}\, F(\mathrm{sk}_j \mathrm{B}\Sigma_2, k) \otimes M$$

and $\pi_* F(\mathrm{sk}_j \mathrm{B}\Sigma_2, k) \otimes M = M_*[t]/t^{j+1}$. The transition maps are surjective so there is no $\lim^1$ term in the Milnor exact sequence and the result follows. □

Construction 3.1.7 (Tate-valued Frobenius)**.** Let A be a k-module equipped with a symmetric multiplication. Define the **total power operation** as the composite:

$$Q(t) : A \xrightarrow{\Delta_2} T_2(A) = (A^{\otimes 2})^{t\Sigma_2} \to A^{t\Sigma_2}.$$

We then define $Q^i : A \to \Sigma^{-i} A$ as the composite

$$A \to A^{t\Sigma_2} \overset{t^{-i-1}}{\to} \Sigma^{-i-1} A^{t\Sigma_2} \to \Sigma^{-i} A_{h\Sigma_2} \to \Sigma^{-i} A.$$

In §3.5 we will verify that the two definitions of the endomorphism $Q^i : \Sigma^i A \to A$ coincide and that each induce the operation $Q^i : \pi_n A \to \pi_{n+i} A$ on homotopy. For now we will assume this compatibility.

Remark 3.1.8 (Naturality of Frobenius)**.** The Tate-valued Frobenius can be defined for any *spectrum* equipped with a symmetric multiplication, as the composite $A \to (A \wedge A)^{t\Sigma_2} \to A^{t\Sigma_2}$. Since the k-module Tate diagonal factors through the spectrum Tate diagonal, we learn that the Tate valued Frobenius only depends on the underlying $\mathbb{E}_\infty$-ring. In particular, the Tate-valued Frobenius is natural for maps $A \to B$ of $\mathbb{E}_\infty$-rings, independent of any compatibility with k-module structures.

3.2 First properties

The first properties follow easily from the Tate-valued Frobenius description, with the exception of the squaring property, which is most readily seen through the classical definition.

Proposition 3.2.1. *The operations Q^i satisfy the following properties.*

(i) *(Additivity)* $Q^i(x + y) = Q^i(x) + Q^i(y)$*.*

(ii) *(Suspension)* $\Omega Q^i(x) = Q^i(\Omega x)$*.*

(iii) *(Squaring)* $Q^{|x|}(x) = x^2$.

(iv) *(Instability)* $Q^i(x) = 0$ *if* $i < |x|$.

(v) *(Action on cohomology) If* $A = F(X, k)$ *where* X *is a pointed space, then* $Q^i(x) = 0$ *for* $i > 0$ *and* $Q^0(x) = x$.

Proof. (i) (Additivity) Since Q^i is induced by a map of spectra, it is automatically additive.

(ii) (Suspension) The Tate diagonal is a natural transformation of exact functors, so $\Delta_{2,\Omega A} \simeq \Omega\Delta_2$. Exactness of T_2 then ensures that $\Omega T_2(A) \to T_2(\Omega A)$ is an equivalence, and composing with the multiplication on ΩA identifies $\Omega Q(t)$ with the total power operation for ΩA, which was to be shown.

(iii) (Squaring) Using Construction 3.1.4, observe that $Q^{|x|}(x)$ is the image of the bottom class in $\mathrm{Sym}^2(\Sigma^n k)$, which is the left vertical arrow in the diagram:

$$\begin{tikzcd}
\Sigma^n k \otimes \Sigma^n k \arrow[r, "x\otimes x"] \arrow[d] & M \otimes M \arrow[d] \\
\mathrm{Sym}^2(\Sigma^n k) \arrow[r] & \mathrm{Sym}^2(M)
\end{tikzcd}$$

The result follows by chasing the diagram clockwise.

(iv) (Instability) By (ii) we may replace A by $\Omega^{|x|-i}A$ and thereby reduce to the case that $A = \Omega B$ and $i = |x|$. By (iii), $Q^i x = x^2$, but the multiplication on ΩB is always trivial, since $S^1 \to S^1 \wedge S^1$ is null.

(v) (Action on cohomology) By naturality we may replace X with $K(k, n)$ and x with the fundamental class. Now the result follows for degree reasons.

□

3.3 Cartan formula

If A and A' are equipped with symmetric multiplications then $A\otimes A'$ inherits a canonical symmetric multiplication as well. In this case we have an external Cartan formula:

Proposition 3.3.1 (Cartan formula)**.**

$$Q(t)(x \otimes y) = Q(t)(x) \otimes Q(t)(y) \in (A \otimes A')(\!(t)\!).$$

Proof. The formula is equivalent to commutativity of the square:

$$\begin{tikzcd}
A \otimes A' \arrow[r] \arrow[d, equal] & T_2(A) \otimes T_2(A') \arrow[r] \arrow[d] & A^{t\Sigma_2} \otimes A^{t\Sigma_2} \arrow[d] \\
A \otimes A' \arrow[r] & T_2(A \otimes A') \arrow[r] & (A \otimes A')^{t\Sigma_2}
\end{tikzcd}$$

The left square commutes because the Tate diagonal is a transformation of lax symmetric monoidal functors. The right hand square commutes by naturality of the lax structure map

$$(-)^{t\Sigma_2} \otimes (-)^{t\Sigma_2} \to (- \otimes -)^{t\Sigma_2}$$

applied to $(A \otimes A')^{\otimes 2} \simeq A^{\otimes 2} \otimes A'^{\otimes 2} \to A \otimes A'$. □

Corollary 3.3.2. $Q^n(x \otimes y) = \sum_{i+j=n} Q^i(x) \otimes Q^j(y)$.

As a corollary of the proof, we see:

Corollary 3.3.3. *If $A \otimes A \to A$ is a map of objects equipped with symmetric multiplications, then $Q(t) : A \to A^{t\Sigma_2}$ is also a map of objects equipped with symmetric multiplications.*

3.4 An example

We revisit our example $k^{t\Sigma_2}$, but to avoid confusion we change the name of the generator: $k_*^{t\Sigma_2} = k(\!(s)\!)$. From the equivalence $k^{h\Sigma_2} = F(\mathrm{B}\Sigma_{2+}, k)$ together with properties (iii), (iv), and (v), we see that

$$Q(t)(s) = s + s^2 t^{-1}.$$

The Cartan formula now determines the behavior of $Q(t)$ in general:

$$Q(t) \sum_i a_i s^i = \sum_i a_i (s + s^2 t^{-1})^i.$$

3.5 Comparing the definitions

Let B denote the bimodule from Construction 3.1.5, which is equivalent to $k^{t\Sigma_2}$ as a left k-module. Let $k \to B$ extend $1 \in \pi_0 k^{t\Sigma_2}$ as a right module map.

Lemma 3.5.1. *The composite*

$$A \to B \otimes A \to T_2(A)$$

above is equivalent to the Tate diagonal Δ_2.

Proof. Indeed, first observe that by the universal property of spectra [Lur17, 1.4.2.23], we have

$$\Omega^\infty : \mathsf{Fun}^{\mathrm{ex}}(\mathsf{Mod}_k, \mathsf{Sp}) \xrightarrow{\simeq} \mathsf{Fun}^{\mathrm{lex}}(\mathsf{Mod}_k, \mathsf{Spaces}).$$

Now let $U : \mathsf{Mod}_k \to \mathsf{Sp}$ be the forgetful functor. Then $\Omega^\infty U$ is corepresented by k, so the Yoneda lemma applied to the previous observation implies that

$$\mathrm{Map}_{\mathsf{Fun}^{\mathrm{ex}}(\mathsf{Mod}_k, \mathsf{Sp})}(U, UT_2) \simeq \Omega^\infty k^{t\Sigma_2}.$$

Since the Tate diagonal is a transformation of lax symmetric monoidal functors, the transformation $U \to UT_2$ evaluates on k to the unit $k \to k^{t\Sigma_2}$. Combining this with the previous observation we learn that the Tate diagonal is the unique transformation $U \to UT$ which corresponds to the element $1 \in \pi_0 k^{t\Sigma_2}$. This completes the proof. $\square$

Thus the map

$$A \to B \otimes A \to T_2(A) \to A^{t\Sigma_2}$$

coincides with the Tate valued Frobenius. Now observe that the last three terms are left modules over $k^{t\Sigma_2}$, so multiplication by t^{-i-1} and naturality of $(-)^{t\Sigma_2} \to \Sigma(-)_{h\Sigma_2}$ gives a commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & B \otimes A \arrow[r] \arrow[d] & T_2(A) \arrow[r] \arrow[d] & A^{t\Sigma_2} \arrow[d] \\
& \Sigma^{-i-1} B \otimes A \arrow[r] & \Sigma^{-i-1} T_2(A) \arrow[r] \arrow[d] & \Sigma^{-i-1} A^{t\Sigma_2} \arrow[d] \\
& & \Sigma^{-i}\mathrm{Sym}^2(A) \arrow[r] & \Sigma^{-i} A
\end{tikzcd}$$

Chasing the diagram around clockwise gives the definition of Q^i in terms of the total power operation. Chasing the diagram around clockwise gives the definition of Q^i in terms of Construction 3.1.5. So these two constructions agree.

Now we compare with the classical construction. The equivalence $(\Sigma^n k)^{\otimes 2} \simeq \Sigma^{2n} k$ in $\mathsf{Mod}_k^{h\Sigma_2}$ gives a commutative diagram

$$\begin{tikzcd}
\Sigma^{-1} T_2(\Sigma^n k) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Sym}^2(\Sigma^n k) \arrow[d, "\simeq"] \\
\Sigma^{2n-1} k^{t\Sigma_2} \arrow[r] & \Sigma^{2n} k_{h\Sigma_2}
\end{tikzcd}$$

Since the bottom horizontal map is surjective on homotopy, so is the top, and we see that $\Sigma^{2n} e_{i-n}$ on the lower right corresponds to $t^{-i-1} y$ on the top left, where $y \in \pi_n \Sigma^n k$ is the generator. Now let $x : S^n \to A$ be a class and form the diagram:

$$\begin{tikzcd}
S^{i+n} \arrow[r, "t^{-i-1}y"] \arrow[dr, "t^{-i-1}x"'] & \Sigma^{-1} T_2(\Sigma^n k) \arrow[r] & \mathrm{Sym}^2(\Sigma^n k) \arrow[d] \\
& \Sigma^{-1} T_2(A) \arrow[r] & \mathrm{Sym}^2(A)
\end{tikzcd}$$

Traversing clockwise gives $Q^i(x)$ as in Construction 3.1.4 and traversing counterclockwise gives the image of x under Q^i as in Construction 3.1.5, and this completes the argument.

4 Adem relations

The Adem relations arise from relating the iterated total power operation to a total fourth power operation. In §4.1 we first explain how to lift the iterated total power operation to an intermediate Tate spectrum. In §4.2 we show that the existence of extra symmetry on iterated multiplication allows us to factor further through a total fourth power operation. This implies a version of the Adem relations as an identity between formal Laurent series in two variables, and in §4.3 we essentially perform the maneuver from [BM82] to recover the usual Adem relations.

For notational ease we adopt the following convention in this section:

Convention 4.0.1. If $G \subseteq \Sigma_n$ is a subgroup, and $\mathcal{T}$ denotes the family of non-transitive subgroups of G, then we denote $(-)^{t\mathcal{T}}$ by $(-)^{\tau G}$.

4.1 Iterated power operations

Suppose A is a k-module equipped with a symmetric multiplication. Iterating the multiplication gives a map

$$A^{\otimes 4} \to A$$

which need not admit an Σ_4-equivariant structure. However, it can be made $\Sigma_2 \wr \Sigma_2$-equivariant, so we may define a map:

$$A \to T_{\Sigma_2 \wr \Sigma_2}(A) \to A^{\tau \Sigma_2 \wr \Sigma_2}.$$

Our first goal is to show that this lifts the iterated total power operation.

Proposition 4.1.1. *Let A be a k-module equipped with a symmetric multiplication. Then there is a canonical commutative diagram:*

$$\begin{tikzcd} & A^{\tau \Sigma_2 \wr \Sigma_2} \arrow[d] \\ A \arrow[ur] \arrow[r, "Q(t) \circ Q(s)"'] & (A^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

Proof. First consider the following diagram:

$$\begin{tikzcd} T_2(A) \arrow[r] \arrow[d] & T_2(T_2(A)) \arrow[r] \arrow[d] & ((A^{\otimes 4})^{t\Sigma_2})^{t\Sigma_2} \arrow[d] \\ A^{t\Sigma_2} \arrow[r] & T_2(A^{t\Sigma_2}) \arrow[r] & (A^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

The first square commutes by naturality of the Tate diagonal applied to the map $T_2(A) \to A$. The second square commutes by naturality of the lax structure map for $(-)^{t\Sigma_2}$.

It follows that $Q(t) \circ Q(s)$ can be written as the composite:

$$A \to T_2(T_2(A)) \to ((A^{\otimes 4})^{t\Sigma_2})^{t\Sigma_2} \to (A^{t\Sigma_2})^{t\Sigma_2}.$$

Now consider both $(-)^{\tau \Sigma_2 \wr \Sigma_2}$ and $((-)^{t\Sigma_2})^{t\Sigma_2}$ as exact functors $\mathsf{Mod}_k^{h\Sigma_4} \to \mathsf{Mod}_k$. We have a natural transformation

$$(-)^{h\Sigma_2 \wr \Sigma_2} \to (-)^{h\Sigma_2 \times \Sigma_2} = ((-)^{h\Sigma_2})^{h\Sigma_2} \to ((-)^{t\Sigma_2})^{t\Sigma_2}$$

where the first map is induced by the inclusion $\Sigma_2 \times \Sigma_2 \to (\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2 = \Sigma_2 \wr \Sigma_2$ given by the diagonal on the first factor. By the universal property of the Tate construction (§1.1), we get a natural transformation $(-)^{\tau \Sigma_2 \wr \Sigma_2} \to ((-)^{t\Sigma_2})^{t\Sigma_2}$. In particular, applied to the multiplication map $A^{\otimes 4} \to A$, we get a commutative diagram:

$$\begin{tikzcd} T_{\Sigma_2 \wr \Sigma_2}(A) \arrow[r] \arrow[d] & A^{t\Sigma_2 \wr \Sigma_2} \arrow[d] \\ ((A^{\otimes 4})^{t\Sigma_2})^{t\Sigma_2} \arrow[r] & (A^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

Finally, the composite

$$\Gamma^{\Sigma_2 \wr \Sigma_2} \to \Gamma^{\Sigma_2 \times \Sigma_2} \simeq \Gamma^2 \circ \Gamma^2 \to T_2 \circ T_2$$

yields a natural transformation $T_{\Sigma_2 \wr \Sigma_2} \to T_2 \circ T_2$ from the universal property of $T_{\Sigma_2 \wr \Sigma_2}$ as the Goodwillie derivative of $\Gamma^{\Sigma_2 \wr \Sigma_2}$. The diagram

$$\begin{tikzcd} T_{\Sigma_2 \wr \Sigma_2} \arrow[d] \arrow[dr] & \\ T_2 \circ T_2 \arrow[r] & (((-)^{\otimes 4})^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

commutes by the same universal property, and the result follows. □

4.2 Adem objects

For the Adem relations to hold we need the symmetric multiplication to satisfy an extra condition.

Definition 4.2.1. We say that k-module A equipped with a symmetric multiplication is an **Adem object** if there exists a map $\mathrm{Sym}^4(A) \to A$ such that the diagram

$$\begin{tikzcd} \mathrm{Sym}^2(\mathrm{Sym}^2(A)) \arrow[r] \arrow[d] & \mathrm{Sym}^2(A) \arrow[d] \\ \mathrm{Sym}^4(A) \arrow[r] & A \end{tikzcd}$$

commutes up to homotopy.

Proposition 4.2.2. *If A is an Adem object, then we have a commutative diagram:*

$$\begin{tikzcd} & A^{\tau\Sigma_4} \arrow[d] \\ & A^{\tau\Sigma_2 \wr \Sigma_2} \arrow[d] \\ A \arrow[uur] \arrow[ur] \arrow[r, "Q(t)\circ Q(s)"'] & (A^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

Proof. By Proposition 4.1.1, the bottom triangle commutes. Factor the top triangle as:

$$\begin{tikzcd} & T_4(A) \arrow[r] \arrow[d] & A^{\tau\Sigma_4} \arrow[d] \\ A \arrow[ur] \arrow[r] & T_{\Sigma_2 \wr \Sigma_2}(A) \arrow[r] & A^{\tau\Sigma_2 \wr \Sigma_2} \end{tikzcd}$$

The triangle commutes because each arrow is a transformation of exact, lax symmetric monoidal functors, and $U : \mathsf{Mod}_k \to \mathsf{Sp}$ is initial amongst such functors (Proposition 2.3.1). The square commutes by the definition of an Adem object, i.e. the structure of a Σ_4-equivariant map $A^{\otimes 4} \to A$ refining the given $(\Sigma_2 \wr \Sigma_2)$-equivariant structure. □

Theorem 4.2.3 (Adem relations)**.** *If A is an Adem object and $x \in \pi_* A$ is an element, then $Q(t)(Q(s)x)$ is symmetric in the variables s and t. Explicitly:*

$$\sum_{i,j} (Q^i Q^j x)(s + s^2 t^{-1})^j t^i = \sum_{i,j} (Q^i Q^j x)(t + t^2 s^{-1})^j s^i.$$

Proof. By Proposition 4.2.2, the iterated total power operation factors through $A^{\tau \Sigma_4}$ and the operation which swaps s and t arises from an inner automorphism of Σ_4 which thus acts trivially on the Tate constructs, whence the claim. The explicit formula follows from the basic properties of power operations, the Cartan formula, and the computation in §3.4. $\square$

4.3 Residues and relations

Now we recall how to recover the individual Adem relations using the power series identity above.

Proposition 4.3.1. *Let A be an Adem object and $x \in A_*$ a homotopy class. Then:*

$$Q^i Q^j(x) = \sum_{\ell} \binom{\ell - j - 1}{2\ell - i} Q^{i+j-\ell} Q^{\ell}(x).$$

Proof. In the previous section we showed

$$\sum_{j} Q(t)(Q^j x)(s + s^2 t^{-1})^j = \sum_{i,j} (Q^k Q^j x)(t + t^2 s^{-1})^j s^k.$$

Let $u = s + s^2 t^{-1}$ and observe that this is composition invertible as a power series in s with coefficients in $k(\!(t)\!)$. Now,

$$Q(t)(Q^j x) = \sum_{i} (Q^i Q^j x) t^i$$

is the coefficient of u^j on the left hand side, so we would like to compute the coefficient of u^j on the right hand side. It will be convenient to reindex the right hand side, for fixed j, as:

$$\sum_{i,\ell} (Q^{i+j-\ell} Q^{\ell} x)(t + t^2 s^{-1})^{\ell} s^{i+j-1}.$$

Observe that $du = ds$ since $2 = 0$ in k, and hence

$$\operatorname{res}(u^{-j-1}(Q^{i+j-\ell} Q^{\ell} x)(t + t^2 s^{-1})^{\ell} s^{i+j-1} du) = \operatorname{res}(u^{-j-1}(Q^{i+j-\ell} Q^{\ell} x)(t + t^2 s^{-1})^{\ell} s^{i+j-1} ds).$$

Fixing i and ℓ and writing $u = st^{-1}(t + s)$ and $(t + t^2 s^{-1}) = s^{-1} t(t + s)$, we have

$$u^{-j-1}(t + t^2 s^{-1})^{\ell} s^{i+j-1} = t^{\ell+j+1} s^{i-2\ell-1}(t + s)^{\ell-j-1}.$$

The coefficient of s^{-1} in the previous expression is then

$$\binom{\ell - j - 1}{2\ell - i} t^i$$

and the result follows. $\square$

5 Relationship to the Steenrod algebra

In this section we restrict to the case $k = \mathbb{F}_2$ for ease of exposition. In §5.1 we recall the Steenrod coaction on the Tate spectrum, then in §5.2 we use this to give a succinct proof of the Nishida relations. Finally, in §5.3 we show how this determines the action of $Q(t)$ on the dual Steenrod algebra, following an idea of Bisson-Joyal.

5.1 Coaction on the Tate spectrum

The map $k = k \wedge S^0 \to k \wedge k$ gives rise to a map $k^{t\Sigma_2} \to (k \wedge k)^{t\Sigma_2}$ if we equip the source and target with trivial Σ_2 action.

This induces a completed coaction:

$$\psi_R : k(\!(t)\!) \to \mathcal{A}_*(\!(t)\!).$$

Now recall that Milnor defined generators[3] of the dual Steenrod algebra by the identity

$$\psi_R(t) = \sum \zeta_i t^{2^i}.$$

5.2 Nishida relations

The easier version of the Nishida relations in this context is in terms of the coaction.

Theorem 5.2.1 (Bisson-Joyal, Baker)**.** *Let X be a spectrum equipped with an equivariant symmetric multiplication $X^{\wedge 2}_{h\Sigma_2} \to X$. Then*

$$\sum_i \psi_R(Q^i x)t^i = Q(\overline{\zeta}(t))\psi_R(x) \in (\mathrm{H}_* X \otimes \mathcal{A}_*)(\!(t)\!).$$

Proof. Let $k \wedge X$. Then the right coaction $k \wedge X \to (k \wedge X) \otimes_k (k \wedge k)$ is a map of *spectra* equipped with symmetric multiplications (though it is not a map of k-modules equipped with symmetric multiplications). By Remark 3.1.8 this yields a commutative diagram:

$$\begin{tikzcd}
k \wedge X \arrow[r, "\psi_R"] \arrow[d] & (k \wedge X) \otimes_k (k \wedge k) \arrow[d] \\
(k \wedge X)^{t\Sigma_2} \arrow[r, "(\psi_R)^{t\Sigma_2}"'] & ((k \wedge X) \otimes_k (k \wedge k))^{t\Sigma_2}
\end{tikzcd}$$

The bottom map arises by applying $(-)^{t\Sigma_2}$ to $k \wedge X \to k \wedge X \wedge k$ and this precisely gives the completed coaction on $(k \wedge X)^{t\Sigma_2}$. In other words:

$$\psi_R(Q(t)x) = Q(t)(\psi_R(x)).$$

Since ψ_R is a ring map, and $\psi_R(t) = \zeta(t)$, this becomes:

$$\sum \psi_R(Q^i x)\zeta(t)^i = Q(t)(\psi_R(x)).$$

Now substitute the conjugate series $\overline{\zeta}(t)$ for t and use the defining relation $\zeta(\overline{\zeta}(t)) = t$. $\square$

[3] Note that we are following Milnor's convention and not the more recent trend of using ζ_i to denote the *conjugates* of Milnor's generators.

5.3 Action on the dual Steenrod algebra

The following description of the action of the Q^i on $\mathcal{A}_*$ is essentially that of Bisson-Joyal [BJ97, §1, Prop. 6].

Theorem 5.3.1 (Bisson-Joyal)**.** *The total power operation on the Milnor generators ζ_i is determined implicitly by the identity:*

$$\zeta(s) + \zeta(s)^2\zeta(t)^{-1} = \sum_i (Q(t)\zeta_i)(s^{2^i} + s^{2^{i+1}}t^{-2^i}) \tag{1}$$

$$t^{2^n} Q(t)\zeta_n = \left(\sum_{i \geqslant n+1} \zeta_i t^{2^i}\right) + \zeta(t)^{-1}\left(\sum_{i \geqslant n} \zeta_i^2 t^{2^{i+1}}\right). \tag{2}$$

Proof. Write $\pi_* k^{h\Sigma_2} = k[\![s]\!]$. Then:

$$\psi_R(Q(t)s) = Q(t)\psi_R(s).$$

Now use the identities $Q(t)s = s + s^2t^{-1}$ and $\psi_R(s) = \zeta(s)$. Comparing coefficients for s^{2^n} gives a recursion for $Q(t)\zeta_n$ starting with $Q(t)\zeta_0 = Q(t)1 = 1$ and (2) is solves the recursion. □

It is not difficult to extract the earlier results of Steinberger [BMMS86, §III.2].

Corollary 5.3.2 (Steinberger)**.** *For $i \geqslant 2$, $Q^{2^i-2}\zeta_1 = \overline{\zeta}_i$.*

Proof. From Theorem 5.3.1(2) above in the case $n = 1$ we get

$$Q(t)\zeta_1 = t^{-1} + \zeta_1 + \zeta(t)^{-1}.$$

So, for $i \geqslant 2$, change of variables and a quick computation gives:

$$\begin{aligned} Q^{2^i-2}\zeta_1 &= \mathrm{res}(t^{-2^i+1}\zeta(t)^{-1}dt) \\ &= \mathrm{res}(\overline{\zeta}(u)^{-2^i+1}u^{-1}du) = \overline{\zeta}_i. \end{aligned}$$

□

Corollary 5.3.3 (Steinberger)**.** *We have $Q^{2^i}\zeta_i = \zeta_{i+1} + \zeta_i^2\zeta_1$ and $Q^{2^i}\overline{\zeta}_i = \overline{\zeta}_{i+1}$.*

Proof. The case $i = 0$ is evident, so assume $i \geqslant 1$. The coefficient of $t^0 s^{2^{i+1}}$ on the right hand side of Theorem 5.3.1(1) is visibly $Q^{2^i}\zeta_i + Q^0(\zeta_i) = Q^{2^i}\zeta_i$. The constant term of $\zeta(t)^{-1}$ is ζ_1, so the coefficient of $t^0 s^{2^{i+1}}$ on the left hand side is $\zeta_{i+1} + \zeta_i^2\zeta_1$. The other identity follows from this one by induction and the defining relation for conjugation. □

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