

C_2 -equivariant Homology Operations: Results and Formulas

Dylan Wilson

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Abstract

In this note we state corrected and expanded versions of our previous results on power operations for C_2 -equivariant Bredon homology with coefficients in the constant Mackey functor on \mathbb{F}_2 . In particular, we give a version of the Adem relations. The proofs rely on certain results in equivariant higher algebra which we will supply in a longer version of this paper.

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Introduction

If X is a spectrum then the smash power $X^{\wedge 2}$ carries a Σ_2 -action. Equipping X with a map

$$X_{h\Sigma_2}^{\wedge 2} \rightarrow X$$

is a way of encoding a coherently symmetric multiplication. In the presence of this structure, there are natural operations

$$Q^i : H_*(X) \rightarrow H_{*+i}X$$

on the mod 2 homology of X . These operations play a crucial role in computations. For example, in the dual Steenrod algebra $\mathcal{A}_* = \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$, Steinberger [BMMS86, §III.2] proved that, for $i \geq 1$:

$$Q^{2^i} \bar{\zeta}_i = \bar{\zeta}_{i+1}.$$

This computation provides key non-formal input in many results, for example:

- Bökstedt's [Boe87] computation of the topological Hochschild homology of \mathbb{F}_2 and \mathbb{Z} .
- The Hopkins-Mahowald theorem identifying HF_p as a Thom spectrum.
- Lawson's [Law18] proof that the Brown-Peterson spectrum cannot be made \mathbb{E}_∞ .

Since the success of Hill-Hopkins-Ravenel [HHR16], there has been much recent activity in calculations in C_2 -equivariant homotopy theory. To aid in this endeavor, the author set out in [Wil16] to study C_2 -equivariant power operations on ordinary equivariant homology with mod 2 coefficients. As first applications, the author gave an equivariant, cellular construction of BP with its action by complex conjugation, and proved an analogue of the Hopkins-Mahowald theorem jointly with Mark Behrens [BW18]. Unfortunately, the treatment in [Wil16] is a bit clunky, contains some errors in various formulas, and did not include the Adem relations.

The purpose of this note is to give a corrected and expanded account of these power operations. In order to more quickly provide a reference for those interested in calculating, we have decided to state the main formulas governing the operations and sketch the proofs. We defer the verification of certain categorical statements to a more detailed, forthcoming paper. We believe our approach is interesting even nonequivariantly and have given a separate and complete treatment of that story in [Wil19].

Outline

In §1 we briefly recall the set-up for equivariant homotopy theory and then go on to state the properties of the equivariant power operations. This includes a Cartan formula (Proposition 1.3.1), Adem relations (Theorem 1.4.3), and Nishida relations (Proposition 1.5.1). We end by computing the action of power operations on the equivariant dual Steenrod algebra (Theorem 1.6.1).

In §2 we sketch the categorical and computational input necessary to prove the theorems from §1. The main idea is to adapt the method in [Wil19] to the equivariant setting using several basic calculations combined with tools from the theory of C_2 - ∞ -categories and higher equivariant algebra.

Much of this paper is devoted to explaining how the theory of power operations in C_2 -equivariant homotopy theory behaves like its nonequivariant counterpart (or, more accurately, like the *odd* primary nonequivariant counterpart). However, there are several important instances where the theory is different:

- The equivariant Cartan and Adem formulas contain terms which vanish after restriction to nonequivariant homotopy.
- The homology of equivariant extended powers of spheres is not so simple. For example, there is not, in general, a Thom isomorphism which reduces their computation to the homology of a classifying space.
- The power operations we build are necessarily stable. Nonequivariantly, all mod 2 power operations are stable. The analogous statement is false equivariantly, see §2.4.

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A historical remark

Our approach to power operations relies heavily on their relationship to Tate constructions. This has a long history. On the algebraic side, the relationship between Tate-like constructions and operations goes back to Singer [Sin80]. Jones and Wegmann realized this algebraic construction topologically in [JW83] (see especially their results 5.2(a) and 5.2(b)). An ad-hoc version of the Tate-valued Frobenius on HF_p makes an appearance in disguise during McClure’s proof of the Nishida relations in [BMMS86, §VIII.3], and there are hints of the same in Steinberger’s calculation of the action of power operations on the dual Steenrod algebra [BMMS86, §III.2]. More recent references, where the relationship is more explicit, include course notes of Lurie [Lur07], a paper of Kuhn and McCarty [KM13, Cor. 2.13], and especially a paper of Glasman-Lawson [LG19]. See also Theorem IV.1.15 in [NS18] (attributed to Lurie).

1 Results

In §1.1 we review enough equivariant homotopy theory to be able to state our results. In §1.2 we assert the existence of power operations satisfying certain basic properties. In the remaining sections we state the Cartan formulas for products and norms (Proposition 1.3.1), the Adem relations (Theorem 1.4.3), the Nishida relations (Proposition 1.5.1), and the action on the equivariant dual Steenrod algebra (Theorem 1.6.1). In each case we give a clean statement in terms of the total power operation and then extract more explicit formulas from there. The formulas in terms of power series identities are more than just a mnemonic; in §2 we explain how the set-up we have chosen produces these identities directly.

Remark 1.0.1. The answers we get below closely align with the properties of the motivic Steenrod operations [Voe03] upon making the substitutions:

$$Q^{i\rho} \leftrightarrow \mathrm{Sq}^{-2i}, \quad Q^{i\rho-1} \leftrightarrow \mathrm{Sq}^{-2i+1}, \quad a \leftrightarrow \rho, \quad u \leftrightarrow \tau$$

Of course, we allow i to be positive in our case, and the proofs are necessarily different. It would be interesting to know of a more precise relationship between the two stories.

1.1 Preliminaries

We establish just as many notions and notations as we need to state our results. A nice treatment of most of the material we need is in [HHR16, §1-3, §A, §B].

1.1.1 Spaces and spectra

The homotopy theory of C_2 -spaces arises from the category \mathbf{Top}^{C_2} of topological spaces equipped with a C_2 -action by inverting (in the ∞ -categorical sense) the collection of maps W which become weak equivalences on underlying and fixed point spaces. We write:

$$\mathbf{Top}^{C_2}[W^{-1}] =: \mathbf{Spaces}^{C_2}.$$

Given a representation V we may form the one-point compactification S^V and thus define functors Σ^V and Ω^V on $\mathbf{Spaces}_*^{C_2}$. If we let σ denote the sign representation and $\rho = 1 + \sigma$ the regular representation, then the ∞ -category of (**genuine**) C_2 -**spectra** is the homotopy limit:

$$\mathbf{Sp}^{C_2} = \mathrm{holim} \left(\cdots \rightarrow \mathbf{Spaces}_*^{C_2} \xrightarrow{\Omega^\rho} \mathbf{Spaces}_*^{C_2} \right).$$

This limit comes with a canonical infinite loop functor $\Omega^\infty : \mathbf{Sp}^{C_2} \rightarrow \mathbf{Spaces}_*^{C_2}$ which admits a left adjoint, the suspension spectrum functor Σ^∞ . We abuse notation and write S^V for either the space or its suspension spectrum, and similarly if X is a C_2 -space then we often write X_+ for its suspension spectrum.

The ∞ -category \mathbf{Sp}^{C_2} is stable and hence canonically enriched in the ∞ -category of spectra [Lur17, 4.2.1.33, 4.8.2.18]. In particular, the objects S^0 and C_{2+} corepresent functors to spectra, denoted $(-)^{C_2}$ and $(-)^u$, respectively. We refer to X^{C_2} as the **genuine fixed points** and X^u as the **underlying spectrum**.

The ∞ -category \mathbf{Sp}^{C_2} is closed symmetric monoidal under the smash product. We denote the internal mapping object by $F(-, -)$.

1.1.2 Homotopy groups

If V is a representation, then S^V is an invertible object in \mathbf{Sp}^{C_2} , so we can make sense of $S^{a+b\sigma}$ for any $a, b \in \mathbb{Z}$. We define groups indexed over the representation ring $RO(C_2)$ by:

$$\pi_{a+b\sigma} X := [S^{a+b\sigma}, X],$$

and we denote the direct sum of these groups by $\pi_\star X$.

By dualizing the cofiber sequence $C_{2+} \rightarrow S^0 \rightarrow S^\sigma$ and using the shearing equivalence $C_{2+} \wedge S^{1-\sigma} \simeq C_{2+}$ we learn that C_{2+} is self-dual. In particular, we have a map $S^0 \rightarrow C_{2+}$ dual to the collapse $C_{2+} \rightarrow S^0$. These give rise to the transfer and restriction, respectively, after smashing with $S^{a+b\sigma}$:

$$\begin{aligned} \mathrm{tr}_{a+b\sigma} &: \pi_{a+b} X^u \rightarrow \pi_{a+b\sigma} X \\ \mathrm{res} &: \pi_{a+b\sigma} X \rightarrow \pi_{a+b} X \end{aligned}$$

If $b = 0$ then we omit the subscript on the transfer.

If X is equipped with a homotopy commutative product $X \wedge X \rightarrow X$ then the homotopy groups carry the structure of a graded ring with commutation rule given by [HK01, Lemma 2.12]:

$$xy = (-1)^{km} (1 - \mathrm{tr}(1))^{\ell n} yx, \quad x \in \pi_{k+\ell\sigma} X, y \in \pi_{m+n\sigma} X.$$

In particular, if $\mathrm{tr}(1) = 2$, this is the usual rule for a bigraded commutative ring.

1.1.3 Eilenberg-MacLane spectra

Associated to any abelian group M there is a C_2 -spectrum which we will denote by the same symbol and which is uniquely characterized by the requirements:

$$\begin{aligned} \text{res} : \pi_0 M &\xrightarrow{\cong} \pi_0 M^u = M \\ \pi_i M &= \pi_i M^u = 0, \quad i \neq 0. \end{aligned}$$

If the abelian group M is equipped with the structure of a commutative ring then the corresponding C_2 -spectrum is an \mathbb{E}_∞ -ring in \mathbf{Sp}^{C_2} . We also have that $\text{tr}(1) = 2$ so that any M -module equipped with a homotopy commutative multiplication satisfies the simpler bigraded commutativity rule on its homotopy groups.

Convention. We will denote by k both the field \mathbb{F}_2 and its corresponding C_2 -spectrum.

We will need two elements in the homotopy of k . The first comes from the homotopy of the sphere

$$a : S^{-\sigma} \rightarrow S^0$$

is adjoint to the inclusion of fixed points. The second arises from the identity $\text{tr}(1) = 0 \in \pi_0 k$, which produces a map

$$u : S^{1-\sigma} = \text{cofib}(S^0 \rightarrow C_{2+}) \rightarrow k$$

that is well-defined since $\pi_1 k = 0$.

1.1.4 Norms

The symmetric monoidal structure on \mathbf{Sp}^{C_2} can be enhanced to allow for tensor products indexed over finite C_2 -sets. This is one of the key innovations introduced in [HHR16, §B].

To describe the homotopical features of this enhancement, we define, for each n , an ∞ -category $(\mathbf{Sp}^{C_2})^{h\Sigma[n]}$ as follows. Begin with $\mathbf{Top}^{C_2 \times \Sigma_n}$ and invert those maps which are C_2 -equivalences to form an ∞ -category $(\mathbf{Spaces}^{C_2})^{h\Sigma[n]}$. Now form the limit

$$(\mathbf{Sp}^{C_2})^{h\Sigma[n]} := \text{holim} \left(\cdots \xrightarrow{\Omega^\rho} (\mathbf{Spaces}^{C_2})_*^{h\Sigma[n]} \xrightarrow{\Omega^\rho} (\mathbf{Spaces}^{C_2})_*^{h\Sigma[n]} \right).$$

The inclusion $\mathbf{Top}^{C_2} \rightarrow \mathbf{Top}^{C_2 \times \Sigma_n}$ of C_2 -spaces with trivial Σ_n -action induces functors

$$\begin{aligned} \mathbf{Spaces}^{C_2} &\longrightarrow (\mathbf{Spaces}^{C_2})^{h\Sigma[n]} \\ \mathbf{Sp}^{C_2} &\longrightarrow (\mathbf{Sp}^{C_2})^{h\Sigma[n]} \end{aligned}$$

which admit both left and right adjoints, denoted $(-)_{{h\Sigma[n]}}$ and $(-)^{h\Sigma[n]}$, respectively.

We will need to know that the assignment $X \mapsto X^{\wedge n}$ can be refined to a functor

$$(-)^{\wedge n} : \mathbf{Sp}^{C_2} \longrightarrow (\mathbf{Sp}^{C_2})^{h\Sigma[n]},$$

which allows us to produce a homotopical symmetric power

$$\text{Sym}^{[n]}(X) := (X^{\wedge n})_{{h\Sigma[n]}}.$$

The latter functor, at least, can be found in [HHR16, §B.6.1]. In the case $n = 2$ we may form the **norm**:

$$N(X) := \left(X^{\wedge 2} \wedge \frac{C_2 \times \Sigma_2}{\Delta} \right)_{h\Sigma_{[2]}} \in \mathbf{Sp}^{C_2}.$$

Note that we have natural maps of C_2 -spectra:

$$X^{\wedge 2} \rightarrow \mathrm{Sym}^{[2]}(X), \quad N(X) \rightarrow \mathrm{Sym}^{[2]}(X).$$

We will also need the generalization of the above discussion to k -modules, where the only difference is that we use $\otimes := \otimes_k$ instead of the smash product.

1.2 The operations

The most general setting in which our operations exist is the following.

Definition 1.2.1. A k -module A is **equipped with an equivariant symmetric multiplication** if we provide a map $\mathrm{Sym}^{[2]}(A) \rightarrow A$, where we take the indexed symmetric square in Mod_k .

Note that we do not require this product to be unital or associative. An example of such a k -module is $F(X_+, k)$ where X is a C_2 -space and the product arises from the diagonal.

Definition 1.2.2. If A_\star is an $RO(C_2)$ -graded commutative k_\star -algebra, we denote by $A_\star((s, t))$ the algebra:

$$A_\star((s, t)) := (A_\star[s][[t]]/(s^2 = as + ut))[t^{-1}]$$

where $|s| = -\sigma$ and $|t| = \rho$. We observe that homogeneous elements in this algebra may be written uniquely in the form

$$\sum_{i \in \mathbb{Z}} x_i s t^{i-1} + \sum_{i \in \mathbb{Z}} y_i t^i$$

where $x_i, y_i \in A_\star$ vanish for i sufficiently negative.

In Construction 2.3.2 below, we will sketch the construction of operations $Q^{i\rho-\varepsilon} : A_\star \rightarrow A_{\star+i\rho-\varepsilon}$ for $\varepsilon = 0, 1$. These agree with the operations constructed in a different manner in our previous treatment [Wil16], and the following theorem summarizes their first properties. We draw attention to (iv)-(vi) which are corrected versions of the corresponding statements in [Wil16].

Theorem 1.2.3. *Let A be a k -module equipped with an equivariant symmetric multiplication. Then there is a natural operation:*

$$Q(s, t) : A_\star \rightarrow A_\star((s, t))$$

with coefficients

$$Q(s, t)x = \sum_i (Q^{i\rho-1}x) s t^{i-1} + \sum_i (Q^{i\rho}x) t^i$$

satisfying the following properties:

- (i) (Mackey) For $\varepsilon = 0, 1$, the operation $Q^{i\rho-\varepsilon}$ restricts to $Q^{2i-\varepsilon}$ on underlying homotopy groups, and commutes with addition and transfer.
- (ii) (Loops) For any representation V , we have $\Omega^V Q(s, t) = Q(s, t) \Omega^V$.

- (iii) (Squaring) *If $|x| = n\rho - \varepsilon$ for $\varepsilon = 0, 1$ then $Q^{n\rho-\varepsilon}(x) = x^2$.*
- (iv) (Vanishing) *Suppose $|x| = a + b\rho$ and $\varepsilon = 0, 1$. Then $Q^{i\rho-\varepsilon}x = 0$ if $i < a + \varepsilon$ and $i \leq b$.*
- (v) (Cohomology) *If $A = F(X, k)$ for a pointed C_2 -space X , then $Q^{i\rho-\varepsilon}$ acts by zero if $2i - \varepsilon > 0$.*
- (vi) (Bockstein) *Suppose that A arises as the mod 2 reduction of a $\mathbb{Z}/4$ -module. Then $Q^{i\rho-1}$ acts by $\beta Q^{i\rho}$ where β is the Bockstein.*

1.3 Cartan formulas

In [Wil16] we stated an obviously incorrect version of the Cartan formula (it did not even restrict to the usual version on underlying homotopy). The correct version of the Cartan formula, as well as a new formula for the value of operations on a norm class, is as follows:

Proposition 1.3.1.

$$\begin{aligned} Q(s, t)(x \otimes y) &= Q(s, t)x \otimes Q(s, t)y \\ Q(s, t)(Nx) &= N(Q(w)x) \end{aligned}$$

Here we have used Nx to denote the class obtained from $x : \Sigma^i k^u \rightarrow A^u$ as $\Sigma^{i\rho} k = N(\Sigma^i k^u) \rightarrow N(A^u)$. We also use w to denote a power series generator for the underlying total power operation $Q(w) = \sum Q^i(-)w^i$ and define $N(w) = t$.

We may extract more explicit formulas by comparing coefficients of $s^\varepsilon t^{n-\varepsilon}$ and using the distributive law for the norm of a sum [HHR16, Prop. A.37]:

Corollary 1.3.2 (Explicit Cartan formulas).

$$\begin{aligned} Q^{n\rho}(x \otimes y) &= \sum_{i+j=n} Q^{i\rho}x \otimes Q^{j\rho}y + u \sum_{i+j=n+1} Q^{i\rho-1}x \otimes Q^{j\rho-1}y \\ Q^{n\rho-1}(x \otimes y) &= \sum_{i+j=n} (Q^{i\rho-1}x \otimes Q^{j\rho}y + Q^{i\rho}x \otimes Q^{j\rho-1}y) + a \sum_{i+j=s+1} Q^{i\rho-1}x \otimes Q^{j\rho-1}y \\ Q^{n\rho}(Nx) &= N(Q^n x) + \sum_{i < j, i+j=2n} \text{tr}_{n\rho}(Q^i x \otimes Q^j x) \\ Q^{n\rho-1}(Nx) &= \sum_{i < j, i+j=2n-1} \text{tr}_{n\rho-1}(Q^i x \otimes Q^j y) \end{aligned}$$

There are internal versions of these formulas if the maps $N(A^u) \rightarrow \text{Sym}^{[2]}(A) \rightarrow A$ and $A^{\otimes 2} \rightarrow \text{Sym}^{[2]}(A) \rightarrow A$ commute with the equivariant symmetric multiplication.

1.4 Adem relations

The author's motivation for the change in our treatment of power operations was to prove the Adem relations. We state them first in a form reminiscent of Bullett-Macdonald [BM82], Steiner [Ste83], and Bisson-Joyal [BJ97], but we need some preliminaries.

Definition 1.4.1. An **Adem object** is a k -module A equipped with an equivariant symmetric multiplication such that there exists a dotted arrow

$$\begin{array}{ccc} \mathrm{Sym}^{[2]}(\mathrm{Sym}^{[2]}(A)) & \longrightarrow & \mathrm{Sym}^{[2]}(A) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^{[4]}(A) & \dashrightarrow & A \end{array}$$

making the diagram commute up to homotopy.

Definition 1.4.2. Given $Q(s, t) : A_\star \rightarrow A_\star((s, t))$ we extend to a ring homomorphism

$$Q(s, t) : A_\star((c, d)) \rightarrow A_\star((c, s, d, t))$$

by the rule:

$$Q(s, t)c = c + dst^{-1}, \quad Q(s, t)d = d + d^2t^{-1}.$$

Here the target ring is defined as:

$$A_\star((c, s, d, t)) := (A_\star[c, d][[d, t]]/(c^2 = ac + ud, s^2 = as + ut)) [d^{-1}, t^{-1}]$$

and the operation lands in this ring because of the vanishing property in Theorem 1.2.3(iv).

Theorem 1.4.3 (Adem relations). *Suppose A is an Adem object. Then, for any $x \in A$, the series*

$$Q(s, t) \circ Q(c, d)x \in A_\star((c, s, d, t))$$

is invariant under the transformation $c \leftrightarrow s, d \leftrightarrow t$.

Comparing coefficients of $s^\varepsilon c^{\varepsilon'}$ we get four different identities between series in d and t . Using a residue argument analogous to that in [BM82] or [BJ97, §1], we can extract the following:

Corollary 1.4.4 (Explicit Adem relations). *With assumptions as above, we have (with x omitted):*

$$\begin{aligned} Q^{i\rho-1}Q^{j\rho-1} &= \sum \binom{\ell-j-1}{2\ell-i} Q^{(i+j-\ell)\rho-1} Q^{\ell\rho-1} \\ Q^{i\rho}Q^{j\rho} &= \sum \binom{\ell-j-1}{2\ell-i} Q^{(i+j-\ell)\rho} Q^{\ell\rho} + u \sum \binom{\ell-j-1}{2\ell-1-i} Q^{(i+j-\ell+1)\rho-1} Q^{\ell\rho-1} \\ Q^{i\rho-1}Q^{j\rho} &= \sum \binom{\ell-j-1}{2\ell-i} Q^{(i+j-\ell)\rho-1} Q^{\ell\rho} + a \sum \binom{\ell-j-1}{2\ell-i-1} Q^{(i+j-\ell+1)\rho-1} Q^{\ell\rho-1} \\ Q^{i\rho}Q^{j\rho-1} &= \sum_\ell \binom{\ell-j}{2\ell-i} Q^{(i+j-\ell)\rho-1} Q^{\ell\rho} + \sum_\ell \binom{\ell-j}{2\ell+1-i} Q^{(i+j-\ell-1)\rho} Q^{(\ell+1)\rho-1} \\ &\quad + a \sum \binom{\ell-j}{2\ell+1-i} Q^{(i+j-\ell)\rho-1} Q^{(\ell+1)\rho-1} \end{aligned}$$

1.5 Nishida relations

Recall that Hu and Kriz [HK01] computed the equivariant dual Steenrod algebra to be

$$\mathcal{A}_\star^{C_2} := \pi_\star(k \wedge k) = k_\star[\tau_i, \xi_{i+1} | i \geq 0] / (\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1})$$

for certain generators with

$$|\tau_i| = 2^i \rho - \sigma, \quad |\xi_i| = (2^i - 1)\rho.$$

If X is a C_2 -spectrum then $k \wedge X$ is equipped with a right coaction

$$k_\star X \rightarrow (k_\star X) \otimes_{k_\star} \mathcal{A}_\star^{C_2}$$

arising from the unit $k \wedge X \rightarrow k \wedge X \wedge k$. The Nishida relations describe the relationship between power operations on $k_\star X$, if they exist, and this coaction.

The Nishida relations from [Wil16] are correct as stated, but we restate them here in a slightly different form. We will use formal series:

$$\tau(s, t) = s + \sum_{i \geq 0} \tau_i t^{2^i}, \quad \xi(t) = t + \sum_{i \geq 1} t^{2^i}$$

as well as similar series for the conjugates $\bar{\tau}(s, t)$ and $\bar{\xi}(t)$. Note $\tau(s, t) + s$ is a power series in t .

Proposition 1.5.1. *Suppose X is a C_2 -spectrum equipped with an equivariant symmetric multiplication and $x \in \pi_\star(k \wedge X)$ is arbitrary. Then*

$$\sum \psi_R(Q^{i\rho-1}x)st^{i-1} + \sum \psi_R(Q^{i\rho}x)t^i = Q(\bar{\tau}(s, t), \bar{\xi}(t))\psi_R(x).$$

Remark 1.5.2. The theorem applies more generally to limits of spectra of the form $k \wedge X$ equipped with the corresponding *completed* right coaction, e.g. to $k^{h\Sigma[2]}$.

Comparing coefficients yields the formulas:

Corollary 1.5.3. *With assumptions as above,*

$$\begin{aligned} \sum_i \psi_R(Q^{i\rho-1}x)t^{i-1} &= \sum_r Q^{r\rho-1}(\psi_R(x))\bar{\xi}(t)^{r-1} \\ \sum_i \psi_R(Q^{i\rho}x)t^i &= \sum_r Q^{r\rho}(\psi_r(x))\bar{\xi}(t)^r + \sum_r Q^{r\rho-1}(\psi_R(x))(\bar{\tau}(s, t) + s)\bar{\xi}(t)^{r-1} \end{aligned}$$

1.6 Action on dual Steenrod algebra

In [Wil16] we gave a partial computation of the action of the power operations on the dual Steenrod algebra. The proof given was incomplete, but the results are correct. Now we are able to state a *complete* calculation of the action.

Theorem 1.6.1. *The total power operation on $\mathcal{A}_\star^{C_2}$ is uniquely determined by the identities:*

$$\tau(c, d) + \xi(d)\tau(s, t)\xi(t)^{-1} = (c + dst^{-1}) + \sum_{i \geq 0} (Q(s, t)\tau_i)(d^{2^i} + d^{2^{i+1}}t^{-2^i})$$

$$\xi(d) + \xi(d)^2\xi(t)^{-1} = \sum_{i \geq 0} (Q(s, t)\xi_i)(d^{2^i} + d^{2^{i+1}}t^{-2^i})$$

Comparing coefficients leads to a recursive description of $Q(s, t)\tau_i$ and $Q(s, t)\xi_i$ which can be solved explicitly. We record this solution and a few other more explicit corollaries of the above formula, including the main result from [Wil16].

Corollary 1.6.2. *The action of power operations on τ_n and ξ_n is given by the identities:*

$$\begin{aligned} t^{2^n} \sum_r Q^{r\rho}(\tau_n)t^r &= \left(\sum_{i \geq n+1} \tau_i t^{2^i} \right) + (\tau(s, t) + s)\xi(t)^{-1} \left(\sum_{i \geq n+1} \xi_i t^{2^i} \right) \\ t^{2^n} \sum_r Q^{r\rho-1}(\tau_n)t^{r-1} &= \xi(t)^{-1} \left(\sum_{i \geq n+1} \xi_i t^{2^i} \right) \\ t^{2^n} \sum_r Q^{r\rho}(\xi_n)t^r &= \sum_{i \geq n+1} \xi_i t^{2^i} + \xi(t)^{-1} \sum_{i \geq n} \xi_i^2 t^{2^{i+1}} \\ \sum_r Q^{r\rho-1}(\xi_n)t^{r-1} &= 0 \end{aligned}$$

Corollary 1.6.3. *The following formulas, for $k \geq 1$, hold for the action on τ_0 :*

$$\begin{aligned} Q^{(2^k-1)\rho}\tau_0 &= \bar{\tau}_k, \\ Q^{(2^k-1)\rho-1}\tau_0 &= \bar{\xi}_k. \end{aligned}$$

Corollary 1.6.4. *The following formulas hold for the action on τ_i, ξ_i , and their conjugates:*

$$\begin{aligned} Q^{2^k\rho}\tau_k &= \tau_{k+1} + \tau_0\xi_{k+1} & Q^{2^k\rho}\bar{\tau}_k &= \bar{\tau}_{k+1} \\ Q^{2^k\rho-1}\tau_k &= \xi_{k+1} & Q^{2^k\rho-1}\bar{\tau}_k &= \bar{\xi}_{k+1} \\ Q^{2^k\rho}\xi_k &= \xi_{k+1} + \xi_1\xi_k^2 & Q^{2^k\rho}\bar{\xi}_k &= \bar{\xi}_{k+1} \end{aligned}$$

As an algebraic corollary, used in [BW18], we have:

Corollary 1.6.5. *The algebra $\mathcal{A}_\star^{C_2}$ is generated by τ_0 as a ring equipped with the operations $Q^{i\rho-\varepsilon}$. More specifically, the elements $Q^{2^k\rho}Q^{2^{k-1}\rho} \dots Q^\rho\tau_0$ and their Bocksteins generate $\mathcal{A}_\star^{C_2}$.*

2 Sketch of the proof

2.1 Review of the classical case

We begin by reviewing the classical setting, with some modifications. It is possible [Wil19] to develop the entire theory of mod 2 power operations based on the following categorical inputs:

- (A) (Universal property of the Tate construction) [NS18, §I.3] Let \mathcal{F} be a collection of subgroups of a group G , closed under sub-conjugacy. Then the generalized Tate construction [Gre87, p.443]

$$(-)^{t\mathcal{F}} : \mathbf{Sp}^{hG} \rightarrow \mathbf{Sp}$$

has the following universal property: the map $(-)^{hG} \rightarrow (-)^{t\mathcal{F}}$ is initial amongst all natural transformations of exact, lax symmetric monoidal functors whose target annihilates the stable subcategory of objects induced from subgroups $H \in \mathcal{F}$.

- (B) (Universal property of the forgetful functor) [Nik16] The forgetful functor $U : \mathbf{Mod}_{\mathbb{F}_2} \rightarrow \mathbf{Sp}$ is initial amongst all exact, lax symmetric monoidal functors from $\mathbf{Mod}_{\mathbb{F}_2}$ to \mathbf{Sp} .
- (C) (Universal property of Tate powers) Let $G \subseteq \Sigma_n$ be a subgroup and \mathcal{J} the family of non-transitive subgroups of G . Define the Tate powers by $T_G(X) := (X^{\otimes n})^{t\mathcal{J}}$ and the divided power by $\Gamma^G(X) := (X^{\otimes n})^{hG}$. Then the map $\Gamma^G \rightarrow T_G$ is initial amongst natural transformations from Γ^G to an exact functor, i.e. T_G is the Goodwillie derivative of Γ^G and is, in particular, exact. We do not know a reference that states the result in these words, but the actual calculation is well-known; we spell this out in [Wil19, 2.2.1].

We also need the following computational inputs:

- (I) As a ring,

$$\pi_* \mathbb{F}_2^{t\Sigma_2} = \mathbb{F}_2((w))$$

where w is the first Stiefel-Whitney class.

- (II) There is a canonical equivalence $(\Sigma^k \mathbb{F}_2)^{\otimes 2} \simeq \Sigma^{2k} \mathbb{F}_2$ in $\mathbf{Mod}_{\mathbb{F}_2}^{h\Sigma_2}$.

- (III) The map $\mathbb{F}_2 = \mathbb{F}_2 \wedge S^0 \rightarrow \mathbb{F}_2 \wedge \mathbb{F}_2$ induces, upon taking Tate fixed points for the trivial action of \mathbb{F}_2 , the completed coaction:

$$\psi_R : \mathbb{F}_2((w)) \rightarrow \mathcal{A}_*((w))$$

determined by

$$\psi_R(w) = \sum \zeta_i w^{2^i} =: \zeta(w).$$

We now review how these facts imply the desired properties of power operations. A more detailed treatment is given in [Wil19].

Suppose we are given an \mathbb{F}_2 -module A equipped with a symmetric multiplication $\mathrm{Sym}^2(A) \rightarrow A$. By (A), (B), and (C) there is an essentially unique transformation of lax symmetric monoidal functors $U \rightarrow UT_2$, and so we may form the **Tate-valued Frobenius**:

$$A \rightarrow T_2(A) = (A^{\otimes 2})^{t\Sigma_2} \rightarrow A^{t\Sigma_2}.$$

We can then define operations $Q^i : A \rightarrow \Sigma^{-i} A$, using (I), as the composite

$$A \rightarrow A^{t\Sigma_2} \xrightarrow{w^{-i-1}} \Sigma^{-i-1} A^{t\Sigma_2} \rightarrow \Sigma^{-i} A_{h\Sigma_2} \rightarrow \Sigma^{-i} A$$

The Frobenius then induces a total power operation

$$Q(w) : A_* \rightarrow A_*^{t\Sigma_2} \simeq A_*((w))$$

given as $Q(w)x = \sum_i (Q^i x) w^i$. The first properties follow immediately from this definition, with the exception of the squaring property, which eventually requires the computation (II) above. The Cartan formula follows from the lax symmetric monoidal structures on every functor in sight. The Adem relations follow from an important lemma: every \mathbb{F}_2 -module equipped with a symmetric multiplication admits a canonical lift of the iterated Frobenius

$$\begin{array}{ccc} & A^{\tau\Sigma_2 t\Sigma_2} & \\ & \nearrow & \downarrow \\ A & \longrightarrow & (A^{t\Sigma_2})^{t\Sigma_2} \end{array}$$

where $(-)^{\tau G}$ denotes $(-)^{t^J}$ when $G \subseteq \Sigma_n$ is a subgroup. The proof of this uses the universality assertions, naturality, and lax symmetric monoidal properties in each of (A), (B), and (C) to verify that various diagrams commute.

If A is, moreover, an Adem object, then there is a further lift to $A^{\tau \Sigma_4}$ and this immediately implies the Adem relations in the form stated by [Ste83] after Bullett-Macdonald [BM82].

The computation (III) immediately implies the Nishida relations by examining the diagram

$$\begin{array}{ccc} (\mathbb{F}_2 \wedge X) & \longrightarrow & (\mathbb{F}_2 \wedge X) \otimes (\mathbb{F}_2 \wedge \mathbb{F}_2) \\ \downarrow & & \downarrow \\ (\mathbb{F}_2 \wedge X)^{t\Sigma_2} & \longrightarrow & ((\mathbb{F}_2 \wedge X) \otimes (\mathbb{F}_2 \wedge \mathbb{F}_2))^{t\Sigma_2} \end{array}$$

Finally, the action on the dual Steenrod algebra is obtained by applying the Nishida relations, in the form above, to the element $w \in \pi_{-1} \mathbb{F}_2^{h\Sigma_2}$.

2.2 The equivariant setting: categorical input

The C_2 -equivariant setting proceeds in an identical fashion. The idea is to first make the following adjustments to the categorical input:

- Replace the use of ∞ -categories with C_2 - ∞ -categories, in the sense of [BDG⁺16].
- Replace the use of exact functors with C_2 -exact functors in the sense of [Nar16, 2.14].
- Replace the use of symmetric monoidal ∞ -categories and lax symmetric monoidal functors with their indexed analogues. One can define a C_2 -symmetric monoidal ∞ -category as a product preserving functor

$$\text{Span}(\text{Fin}^G) \rightarrow \text{Cat}_\infty$$

but the notion of lax C_2 -symmetric monoidal functors takes a little longer to define. A suitable definition can be found in [Wil17, 2.1.4.2].

- Replace the group Σ_n with the C_2 -groupoid $\Sigma_{[n]}$, i.e. the functor $\Sigma_{[n]} : \text{Orbit}(C_2)^{\text{op}} \rightarrow \text{Gpd}$ which assigns to an orbit T the groupoid of maps of $(C_2 \times \Sigma_n)$ -sets $U \rightarrow T$ which are Σ_n torsors. This records the natural symmetries on indexed tensor powers in a C_2 -symmetric monoidal ∞ -category.

With these changes, the claims (A), (B), and (C) remain true in the C_2 -equivariant setting, and the proofs are mostly the same as their nonequivariant analogues. We note that, without the symmetric monoidal structures, these statements are readily provable using tools from [Nar16] and [Sha18]. However, the additional multiplicative structure requires more set-up.

2.3 The equivariant setting: computational input

The computation (I) is replaced by

$$\pi_\star k^{t\Sigma[2]} = k_\star((s, t))$$

with notation as in §1.2. Here some care is required. First, recall that there is a C_2 -space $B_{C_2}\Sigma_2$ which classifies Σ_2 -torsors on C_2 -spaces, and a corresponding universal torsor $E_{C_2}\Sigma_2$. The map $E_{C_2}\Sigma_{2+} \rightarrow S^0$ is an equivalence in $(\mathbf{Sp}^{C_2})^{h\Sigma_{[2]}}$, so we get equivalences:

$$F(E_{C_2}\Sigma_{2+}, X)^{h\Sigma_{[2]}} \simeq X^{h\Sigma_{[2]}}$$

$$(X \wedge E_{C_2}\Sigma_{2+})_{h\Sigma_{[2]}} \simeq X_{h\Sigma_{[2]}}$$

for any $X \in (\mathbf{Sp}^{C_2})^{h\Sigma_{[2]}}$. We will use this to define a filtration on these functors.

If we let τ denote the sign representation of Σ_2 , then an explicit model for $E_{C_2}\Sigma_2$ is the unit sphere $S(\infty(\rho \otimes \tau))$ in the $C_2 \times \Sigma_2$ -representation $(\rho \otimes \tau)^{\oplus \infty}$. The filtration

$$\Sigma_{2+} = S(1 \otimes \tau) \subseteq S(\rho \otimes \tau) \subseteq S((\rho + 1) \otimes \tau) \subseteq \dots$$

then produces a filtration on the functors $(-)^{h\Sigma_{[2]}}$ and $(-)_{h\Sigma_{[2]}}$. Now, in general, if λ is a one-dimensional representation and W is any representation, then the map

$$S(\lambda) \times \text{Hom}(\lambda, W) \rightarrow S(\lambda \oplus W) - S(W), \quad (x, \phi) \mapsto \frac{(x, \phi(x))}{|(x, \phi(x))|}$$

is an equivariant homeomorphism and so induces an equivalence

$$S(\lambda)_+ \wedge S^{\lambda^\vee \otimes W} \simeq S(\lambda \oplus W)/S(W)$$

upon taking one-point compactifications. Applying this to $S(\tau) = \Sigma_2$ and $S(\sigma \otimes \tau) = \frac{C_2 \times \Sigma_2}{\Delta}$, where Δ is the diagonal subgroup, we learn that:

$$\text{gr}_{2j} \text{Sym}^{[2]}(M) \simeq \Sigma^{j\rho} M^{\otimes 2},$$

$$\text{gr}_{2j+1} \text{Sym}^{[2]}(M) \simeq \Sigma^{(j+1)\rho-1} N(M).$$

Unlike the classical setting, this filtration does not split in general. For example, it does not split when M is of the form $\Sigma^{j\rho+1}k$ or $\Sigma^{j\rho-2}k$ (see below §2.4).

However, in the case $M = k$ or, more generally, $M = \Sigma^{j\rho-\varepsilon}k$ for $\varepsilon = 0, 1$, the filtration splits for degree reasons stemming from a computation of the homology of a point, k_\star . This gives an additive calculation

$$k_{h\Sigma_{[2]}} \simeq \bigoplus_{j \geq 0} \Sigma^{j\rho} k \oplus \bigoplus_{j \geq 1} \Sigma^{j\rho-1} k$$

Dualizing produces an additive calculation for $k^{h\Sigma_{[2]}}$ and it is not difficult to prove that the ring structure is given by

$$k_\star^{h\Sigma_2} = k_\star[s][[t]]/(s^2 = as + ut)$$

This calculation was obtained in a different way by Hu-Kriz [HK01, 6.27], but the filtration above is useful in other contexts.

Remark 2.3.1. Here there is a subtle point: there are two possible generators in degree $-\sigma$ which satisfy the above relation. The choice is determined by restriction to $S(\rho \otimes \tau)/\Sigma_2 = S^\sigma \subseteq B_{C_2}\Sigma_2$. It is standard to choose the generator which restricts to $1 \in \pi_0 k = \pi_{-\sigma} \Sigma^{-\sigma} k$. We will *not* make

that choice. Instead we choose s to be the generator which restricts to a . The other generator is given by

$$\bar{s} := s + a.$$

These two generators behave in an almost identical fashion, but with our choice of generator the map

$$k^{t\Sigma_{[2]}} \rightarrow \Sigma k_{h\Sigma_{[2]}} \rightarrow \Sigma k$$

will, on homotopy, record the coefficient of st^{-1} (as opposed to $(s+a)t^{-1}$), which is more convenient.

To finish the analogue of (I), one checks that the Tate construction inverts the class t . The filtration can also be used to show that, for M equipped with trivial Σ_2 -action,

$$M_\star^{t\Sigma_{[2]}} = M_\star((s, t))$$

even if M is not free as a k -module. An equivariant symmetric multiplication on a k -module A is now enough to form the **equivariant Tate-valued Frobenius**:

$$A \rightarrow T_{[2]}(A) := (A^{\otimes 2})^{t\Sigma_{[2]}} \rightarrow A^{t\Sigma_{[2]}}.$$

Construction 2.3.2. The individual power operations are defined as the composites:

$$Q^{i\rho-\varepsilon} : A \rightarrow A^{t\Sigma_{[2]}} \xrightarrow{\bar{s}^\varepsilon t^{-i-1+\varepsilon}} \Sigma^{-i\rho-1+\varepsilon} A^{t\Sigma_{[2]}} \rightarrow \Sigma^{-i\rho+\varepsilon} A_{h\Sigma_{[2]}} \rightarrow \Sigma^{-i\rho+\varepsilon} A.$$

The analogous computation to (II) is an equivalence $(\Sigma^{j\rho}k)^{\otimes 2} \simeq \Sigma^{2j\rho}k$ in $\text{Mod}_k^{h\Sigma_{[2]}}$. One can show this either by establishing the orientability of a bundle on $B_{C_2}\Sigma_2$ or using the computation of $\text{Sym}^2(\Sigma^{j\rho}k)$ referenced above. The analogous fact does not hold for other representations, but this turns out to suffice. Finally, the analogue of (III) is essentially the definition of the Milnor generators from Hu-Kriz [HK01, p.381]. The main theorems in §1 now follow as in §2.1.

2.4 A counterexample

In the previous section we constructed a filtration on $\text{Sym}^{[2]}(M)$ and remarked that it does not split in general. Let us explain what happens in two examples to see how things differ from the classical case.

First, in the case $M = \Sigma^{n\rho+1}k$ it turns out that, while the given filtration does not split, we can still describe the answer:

Proposition 2.4.1.

$$\text{Sym}^{[2]}(\Sigma^{n\rho+1}k) \simeq \bigoplus_{j \geq n+1} \Sigma^{(n+j)\rho+1}k \oplus \bigoplus_{j \geq n+1} \Sigma^{(n+j)\rho}k$$

This situation is not too bad because the generators can actually be described in terms of the power operations we have. That is, we can choose generators so that, if A is equipped with an equivariant symmetric multiplication, and $x : \Sigma^{n\rho+1}k \rightarrow A$ is a class, then the image of the generator in degree $(n\rho + 1) + (j\rho - \varepsilon)$ under the map

$$\text{Sym}^{[2]}(\Sigma^{n\rho+1}k) \rightarrow \text{Sym}^{[2]}(A) \rightarrow A$$

is equal to $Q^{j\rho-\varepsilon}x$.

We also have a computation of $\text{Sym}^{[2]}(\Sigma^{n\rho-2}k)$:

Proposition 2.4.2.

$$\mathrm{Sym}^{[2]}(\Sigma^{n\rho-2}k) \simeq \Sigma^{2n\rho-4}k \oplus \bigoplus_{j \geq n} \Sigma^{(n+j)\rho-2}k \oplus \bigoplus_{j \geq n} \Sigma^{(n+j)\rho-3}k$$

In this case, most classes will correspond to named power operations, but the bottom class cannot, for degree reasons. In other words: there is no stable operation which refines the square $x \mapsto x^2$ for classes in degree $n\rho - 2$. We imagine the situation becomes worse the further one strays from regular representations.

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