Symmetric monoidal structures and the Frobenius

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If R is a commutative ring then we can write down a sort of diagonal, which is multiplicative:

$$r \mapsto r^{\otimes p} \in R^{\otimes p}$$

This lands in the symmetric tensors, $(R^{\otimes p})^{\Sigma_p}$. This map is not usually additive. For example, we have

$$(x+y)^{\otimes 2} = x \otimes x + y \otimes y + (x \otimes y + y \otimes x)$$

Here, the error term $x \otimes y + y \otimes x$ is in the image of the transfer map $R \otimes R \to (R \otimes R)^{\Sigma_2}$. And in fact, in general, the composite

$$R \to (R^{\otimes p})^{\Sigma_p} \to (R^{\otimes p})^{\Sigma_p}/\mathrm{tr}$$

is additive.

The goal of this lecture and part of the next is to set up a derived or homotopy coherent analogue of this story, and then see how it plays out in the case $R = \mathbb{F}_2 \wedge \mathbb{F}_2$.

(1) First we need to make sense of commutative rings. It's easier to do commutative monoids (i.e. commutative algebras in a cartesian monoidal category) first, so let's do that.

Example 1.1. Let $\mathrm{Gr} = \coprod_{n \geqslant 0} \mathrm{Gr}_n(\mathbb{R}^\infty)$ be the space of finite dimensional subspaces of $\mathbb{R}^\infty$. This kind of has a binary operation. If I have $V, W \subseteq \mathbb{R}^\infty$ then if I *choose* an embedding $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \hookrightarrow \mathbb{R}^\infty$, we may view $V \oplus W$ as a new element of Gr. But this depends on a choice. More generally, the natural thing to write down for any finite set J is a zig-zag

$$\mathrm{Gr}^{\times J} \leftarrow \mathrm{Isom}(\mathbb{R}^{\infty J}, \mathbb{R}^\infty) \times \mathrm{Gr}^{\times J} \to \mathrm{Gr}$$

Here, Isom denotes the space of isometric embeddings, the left-hand arrow is projection, and the right hand arrow is the operation where we take direct sums and then use our embedding. Since $\mathrm{Isom}(\mathbb{R}^{\infty J}, \mathbb{R}^\infty)$ is contractible, we see that there is a contractible space of multiplications on Gr. In fact, if we let

$$\mathrm{Gr}(J) := \mathrm{Isom}(\mathbb{R}^{\infty J}, \mathbb{R}^\infty) \times \mathrm{Gr}^{\times J}$$

what we have really done is defined a functor on finite sets with two types of functoriality:

- For every map of finite sets $I \to J$, we have a map $\mathrm{Gr}(I) \to \mathrm{Gr}(J)$ which is an amalgam of the direct sum operations above for each $j \in J$.
- For every inclusion $I_0 \subseteq I$ we have $\mathrm{Gr}(I) \to \mathrm{Gr}(I_0)$ which projects onto $\mathrm{Gr}^{\times I_0}$ and restricts isometries along the inclusion $\mathbb{R}^{\infty I_0} \subseteq \mathbb{R}^{\infty I}$.

Definition 1.2. The category Fin_* of finite pointed sets can be thought of in either of the following equivalent ways.

- objects are finite sets, and maps are spans $I \supseteq I_0 \to J$; or
- objects are finite pointed sets I_+, and maps are maps $I_+ \to J_+$ of finite pointed sets.

The morphisms of the form $I \supseteq I \to J$ (i.e. those obtained from maps of finite sets by adjoining a disjoint basepoint) are called **active** and the morphisms of the form $I \supseteq I_0 = I_0$ (i.e. those obtained as the 'Pontryagin-Thom collapse map' of an inclusion) are called **inert**.

Either way, we have the following definition:

Definition 1.3 (Segal)**.** Let $\mathcal{C}$ be an ∞-category with finite products. We denote by $\mathsf{CMon}(\mathcal{C}) \subseteq \mathsf{Fun}(\mathrm{Fin}_*, \mathcal{C})$ the full subcategory of functors X such that

- $X(\varnothing_+)$ is final.
- If $J = J_0 \amalg J_1$, then the natural map arising from the collapse maps
$$X(J_+) \to X(J_{0+}) \times X(J_{1+})$$
is an equivalence.

Remark 1.4. We think $X(\{1\}_+)$ as the underlying object of X, and usually just also denote it by X. We think of $X(\{1, ..., n\}_+)$ as parameterizing n-tuples in X: we have supplied an equivalence to $X(\{1\}_+)^{\times n}$. That is, functoriality for inert morphisms provides us with a way to list the 'components' of an element in $X(\{1, ..., n\}_+)$. The functoriality for active maps like $\{1, 2\} \to \{1\}$ then encode the multiplication. The functoriality also encodes the coherent associativity and commutativity of these operations, via composition of morphisms and that various automorphism groups of objects in Fin_*.

We can now make the following definition:

Definition 1.5. A **symmetric monoidal ∞-category** is a commutative monoid object in Cat_∞, and a **symmetric monoidal functor** is then a morphism in $\mathsf{CMon}(\mathsf{Cat}_\infty)$.

This is mostly okay, except for two things:

- How do we cook up examples? It's pretty hard to write down a functor into Cat_∞ and, moreover, this is not usually how symmetric monoidal categories arise in nature. We often do not *choose* once and for all a notion of tensor product that is functorial, instead we characterize tensor products by a universal property. It would be nice if the formalism reflected the same agnosticism about choices that arises in nature.

- How do we define lax symmetric monoidal functors? That's possible but a little hard to do in this set-up. And then it becomes harder when we'd like to build a *category* of lax-symmetric monoidal functors (i.e. discuss non-invertible natural transformations between lax symmetric monoidal functors, compatible with the lax structure maps, etc.)

To address both of these problems requires an important digression.

(2) One of the more important ideas in higher category theory is the **Grothendieck construction**. Let's start with two examples of this phenomenon in nature.

Example 2.6. Let **Mod** be the category of pairs (R, M) where R is a ring and M is a left module over R. A morphism $(R, M) \to (R', M')$ is a map of rings $\theta : R \to R'$ and a map of abelian groups $f : M \to M'$ satisfying the formula $f(rm) = \theta(r)f(m)$. We call such a map f a θ-linear map There is a forgetful functor **Mod** → **Ring**. Suppose we're given a map $\theta : R \to R'$ and an R-module M. Then there is a certain best possible θ-linear map $M \to M'$. Specifically, I claim that there is a θ-linear map $f : M \to M'$ such that, given any $\phi : R \to R''$, factorization

$$\begin{tikzcd}
R \arrow[rr, "\phi"] \arrow[dr, "\theta"'] & & R'' \\
& R' \arrow[ur, "\theta'"'] &
\end{tikzcd}$$

and ϕ-linear map $h : M \to M''$, there is a *unique* factorization

$$\begin{tikzcd}
M \arrow[rr, "h"] \arrow[dr, "f"'] & & M'' \\
& M' \arrow[ur, "g"'] &
\end{tikzcd}$$

through a θ'-linear map g. This universal property is exactly the one satisfied by an extension of scalars $M' = R' \otimes_R M$. Another way of stating this universal property is that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Mod}}(R' \otimes_R M, M'') \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{Mod}}(M, M'') \arrow[d] \\
\mathrm{Hom}_{\mathbf{Ring}}(R', R'') \arrow[r] & \mathrm{Hom}_{\mathbf{Ring}}(R, R'')
\end{tikzcd}$$

is cartesian for every pair (R'', M'').

Example 2.7. Let $\mathcal{M} \to [1]$ be a functor and let $\mathcal{C} = \mathcal{M}_0$ be the (homotopy) fiber over 0 and $\mathcal{D} = \mathcal{M}_1$ be the (homotopy) fiber over 1. We can restrict the mapping space functor to define

$$\mathrm{Map}_{\mathcal{M}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathsf{Spaces},$$

or, equivalently, a functor

$$\hat{F} : \mathcal{C} \longrightarrow \mathsf{Fun}(\mathcal{D}, \mathsf{Spaces})^{\mathrm{op}}$$

That is *almost* a functor from $\mathcal{C}$ to $\mathcal{D}$, except that $\hat{F}(c)$ might not be representable. Asking that $\hat{F}(c)$ be representable means that there is an element $d \in \mathcal{D}$ and an element $u \in \hat{F}(c)(d) = \mathrm{Map}_{\mathcal{M}}(c, d)$ which exhibits $\hat{F}(c)$ as the functor corepresented by d. In other words: there is an arrow $u : c \to d$ which is initial amongst all arrows from c to something in $\mathcal{D}$. Or: giving a map $c \to d'$ is the same as giving a map $d \to d'$. This is some property of $\mathcal{M}$ and the functor to $[1]$ that may or may not hold. So we see there is a close relationship between functors $\mathcal{C} \to \mathcal{D}$ and certains special types of functors $\mathcal{M} \to [1]$ with fibers given by $\mathcal{C}$ and $\mathcal{D}$ respectively.

Definition 2.8. Let $p : E \to B$ be a functor.

- We say that an arrow $e \to e'$ in E **p-cocartesian** if, for all $e'' \in E$, the diagram

$$\begin{tikzcd}
\mathrm{Map}_E(e', e'') \arrow[r] \arrow[d] & \mathrm{Map}_E(e, e'') \arrow[d] \\
\mathrm{Map}_B(pe', pe'') \arrow[r] & \mathrm{Map}_B(pe, pe'')
\end{tikzcd}$$

  is a pullback in Spaces.

- We say that p is a **cocartesian fibration** if, for all arrows $f : b \to b'$ in B and $e \in E_b$ with chosen $\alpha : pe \simeq b$, there exists a p-cocartesian arrow $\tilde{f}$ in E with an equivalence $p\tilde{f} \simeq f$ compatible with α.

- We denote by

$$\mathsf{coCart}(B) \subseteq \mathsf{Cat}_{\infty/B}$$

  the subcategory with objects the cocartesian fibrations over B and morphisms those functors which preserve cocartesian edges.

Remark 2.9. In the model of quasicategories, if we require that $p : E \to B$ be an inner fibration at the outset (a mild condition), then we can require e to strictly lift b and $\tilde{f}$ to strictly lift f in the above definition. In this case the homotopy fibers over elements of B are equivalent to the literal fibers taken in simplicial sets.

Example 2.10. Let's revisit our example from earlier where we had a functor $p : \mathcal{M} \to [1]$. A good exercise is to check that p is a cocartesian fibration if and only if the corresponding functor $\mathcal{C} \to \mathsf{Fun}(\mathcal{D}, \mathsf{Spaces})^{\mathrm{op}}$ factors through the subcategory of corepresentable functors, equivalent to $\mathcal{D}$. We can also go the other way. Given $F : \mathcal{C} \to \mathcal{D}$, form

$$\mathsf{Cyl}(F) := \mathcal{C} \times [1] \amalg_{\mathcal{C} \times \{1\}} \mathcal{D} \longrightarrow [1].$$

One can check that this is a cocartesian fibration from which we may recover F. This procedure can be souped up to an equivalence

$$\mathsf{Cyl} : \mathsf{Fun}([1], \mathsf{Cat}_\infty) \longrightarrow \mathsf{coCart}([1]).$$

The following theorem of Lurie is one of the pillars of higher category theory:

Theorem 2.11 (Grothendieck construction/(Un)straightening equivalence)**.** *There is a natural equivalence*

$$\int_{(-)} : \mathsf{Fun}(-, \mathsf{Cat}_\infty) \xrightarrow{\simeq} \mathsf{coCart}(-)$$

which is essentially uniquely characterized by the requirement that $\int_{[0]}$ *is the tautological equivalence and* $\int_{[1]}$ *is equivalent to* Cyl.

This allows one to break up the problem of defining a functor $B \to \mathsf{Cat}_\infty$ into a sequence of often easier problems, i.e. defining a category over B and then verifying certain properties of the structure map. Another useful consequence, as we will see below, is that there are things you can do with cocartesian fibrations that capture certain $(\infty, 2)$-categorical aspects of Cat_∞ which are hard to access from the functor category, e.g. considering maps between cocartesian fibrations which *don't* preserve cocartesian edges.

(3) Now we can revisit our definition of symmetric monoidal ∞-category.

Definition 3.12. A **symmetric monoidal ∞-category** is a cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Fin}_*$ such that, for any choice of unstraightening, the maps

$$\mathcal{C}^\otimes_{J_+} \to \mathcal{C}^\otimes_{J'_+} \times \mathcal{C}^\otimes_{J''_+}$$

are equivalences; and $\mathcal{C}^\otimes_{\varnothing_+} \simeq \bullet$.

Remark 3.13. One can reword this definition without reference to an unstraightening.

Definition 3.14. If $E, E' \to B$ are given, write $\mathsf{Fun}_{/B}(E, E')$ for the pullback in Cat_∞:

$$\begin{tikzcd}
\mathsf{Fun}_{/B}(E, E') \arrow[r] \arrow[d] & \mathsf{Fun}(E, E') \arrow[d] \\
\bullet \arrow[r] & \mathsf{Fun}(E, B)
\end{tikzcd}$$

If $\mathcal{C}^\otimes, \mathcal{D}^\otimes \to \mathrm{Fin}_*$ are symmetric monoidal ∞-categories, define

- the ∞-category of **symmetric monoidal functors** as the full subcategory

  $$\mathsf{Fun}^\otimes(\mathcal{C}, \mathcal{D}) \subseteq \mathsf{Fun}_{/\mathrm{Fin}_*}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$$

  consisting of those functors which preserve cocartesian arrows.

- the ∞-category of **lax symmetric monoidal functors** as the full subcategory

  $$\mathsf{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D}) \subseteq \mathsf{Fun}_{/\mathrm{Fin}_*}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$$

  consisting of those functors which preserve cocartesian lifts of inert morphisms in Fin_*.

Remark 3.15. Let's unpack this a bit to see why it deserves the name. Let's say I've got a functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. In particular, I get a functor $F : \mathcal{C} \to \mathcal{D}$, since this is map *over* Fin_*. If I'm given a couple objects $X, Y \in \mathcal{C}$, then there is an essentially unique object $(X, Y) \in \mathcal{C}(\{1,2\}_+)$ with 'components' X and Y. More explicitly: there is an essentially unique object (X, Y) equipped with cocartesian arrows

$$(X,Y) \to X,\ (X,Y) \to Y$$

covering the different collapse maps $\{1,2\}_+ \to \{1\}_+$ and $\{1,2\}_+ \to \{2\}_+$, respectively. Now, F sends $F(X,Y)$ somewhere in $\mathcal{D}^{\otimes}$, but the assumption (in either definition) assures us that the arrows

$$F(X,Y) \to FX,\ F(X,Y) \to FY$$

are *still* cocartesian lifts. So the condition that F preserve cocartesian lifts of inert morphisms just amounts to this: under the equivalences $\mathcal{C}^{\otimes}_{\{1,\dots,n\}_+} \simeq \mathcal{C}^{\times n}$ and $\mathcal{D}^{\otimes}_{\{1,\dots,n\}_+} \simeq \mathcal{D}^{\times n}$, the induced functor

$$\mathcal{C}^{\times n} \to \mathcal{D}^{\times n}$$

is just computed componentwise.

Okay, great. But now what happened to the familiar things like 'lax structure maps' and all of that? Well, let's think about what happens to cocartesian lifts of *active* morphisms. In $\mathcal{C}^{\otimes}$ we have a cocartesian arrow

$$(X,Y) \to X \otimes Y$$

covering $\{1,2\} \to \{1\}$. This goes to some arrow

$$(FX, FY) \to F(X \otimes Y)$$

in $\mathcal{D}^{\otimes}$ covering the same map in Fin_*. But we are guaranteed a cocartesian lift, which is initial amongst such lifts, so we have an essentially unique factorization:

$$\begin{tikzcd}
(FX, FY) \arrow[r] \arrow[d] & F(X \otimes Y) \\
FX \otimes FY \arrow[ur, dashed] &
\end{tikzcd}$$

once we've chosen a tensor product for FX and FY. That is the lax structure map! It just falls out. If F preserves *all* cocartesian edges, then

$$(FX, FY) \to F(X \otimes Y)$$

already witnesses the target as a tensor product of FX and FY, so that lax structure map is guaranteed to be an equivalence by the essential uniqueness of cocartesian lifts.

Example 3.16. Often lax symmetric monoidal functors arise from the following important fact: right adjoints to symmetric monoidal functors are canonically lax symmetric monoidal.

One last definition before some examples:

Definition 3.17. If $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ is a symmetric monoidal ∞-category, we define the **∞-category of commutative algebra objects** in $\mathcal{C}$ as the full subcategory

$$\mathsf{CAlg}(\mathcal{C}) \subseteq \mathsf{Fun}_{/\mathrm{Fin}_*}(\mathrm{Fin}_*, \mathcal{C}^{\otimes})$$

consisting of those sections which send inert morphisms to cocartesian edges.

Example 3.18. The unit object in $\mathcal{C}$ has an essentially unique commutative algebra structure with the requirement that the unit map be the identity. (A quick way to construct this is to use the theory of *relative* left Kan extensions over Fin_*, which is a way to build sections analogous to the way Kan extensions build functors).

Remark 3.19. Observe that a lax symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $F : \mathsf{CAlg}(\mathcal{C}) \longrightarrow \mathsf{CAlg}(\mathcal{D})$, computed pointwise.

Example 3.20 ((Co)cartesian monoidal structures)**.** If $\mathcal{C}$ admits finite coproducts, one can form a symmetric monoidal ∞-category $\mathcal{C}^{\amalg} \to \mathrm{Fin}_*$ where the tensor product is given by $\amalg$ called the cocartesian monoidal structure. Similarly, if $\mathcal{C}$ admits products one can form a cartesian monoidal structure $\mathcal{C}^{\times} \to \mathrm{Fin}_*$. An important fact is that, if $\mathcal{C}$ is equipped with its cartesian monoidal structure, then there is a canonical equivalence

$$\mathsf{CAlg}(\mathcal{C}) \simeq \mathsf{CMon}(\mathcal{C}),$$

as one might hope.

Example 3.21 (Localizations)**.** If $\mathcal{C}$ is a symmetric monoidal ∞-category and W is a class of morphisms containing the equivalences, closed under composition, and is a tensor ideal (i.e. $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves arrows in W separately in each variable) then the localization arrow $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ has a symmetric monoidal enhancement which is essentially uniquely characterized by the property that, for every symmetric monoidal $\mathcal{D}$, composition with L gives equivalences

$$\mathsf{Fun}^{\mathrm{lax}}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\simeq} \mathsf{Fun}^{\mathrm{lax}}_{W}(\mathcal{C}, \mathcal{D}) \subseteq \mathsf{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D})$$

$$\mathsf{Fun}^{\otimes}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\simeq} \mathsf{Fun}^{\otimes}_{W}(\mathcal{C}, \mathcal{D}) \subseteq \mathsf{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

where Fun_W is the subcategory of functors which send arrows in W to equivalences in $\mathcal{D}$. As a special case, note that this produces a symmetric monoidal structure on $\mathsf{D}(\mathbb{F}_2) = \mathsf{Mod}_{\mathbb{F}_2} = \mathbf{Ch}(\mathbb{F}_2)[W^{-1}]$.

Example 3.22 (Tensor products of ∞-categories)**.** This is one of the 'secret weapons' of higher category theory. Let $\mathcal{K}$ be some class of diagrams and $\mathsf{Cat}_\infty(\mathcal{K})$ the ∞-category of ∞-categories which are $\mathcal{K}$-cocomplete and functors between them which preserve $\mathcal{K}$-colimits. Then this has a symmetric monoidal structure where the tensor product $\mathcal{C} \otimes \mathcal{D}$ corepresents functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which commutes with $\mathcal{K}$-colimits separately in each variable. Modulo size issues, this can be seen as a localization of the cartesian monoidal structure on Cat_∞ using the localization functor which takes $\mathcal{C}$ to the full subcategory $\mathsf{Psh}^{\mathcal{K}}(\mathcal{C}) \subseteq \mathsf{Psh}(\mathcal{C})$ generated by representables under $\mathcal{K}$-colimits.

If $\mathcal{K} = \mathcal{A}\mathrm{ll}$ is the class of all small diagrams, then the full subcategory $\mathsf{Pr}^L \subseteq \mathsf{Cat}_\infty(\mathcal{A}\mathrm{ll})$ of presentable ∞-categories is stable under the tensor product, and there is even a formula

$$\mathcal{C} \otimes \mathcal{D} \simeq \mathsf{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}).$$

For example, $\mathsf{Psh}(K) \otimes \mathcal{D}$ is equivalent to the ∞-category of presheaves on K valued in $\mathcal{D}$. Another important example is that

$$\mathsf{Sp} \otimes \mathcal{C} \simeq \mathsf{Sp}(\mathcal{C})$$

when $\mathcal{C}$ is presentable.

Example 3.23. The subcategory $\mathsf{Pr}^{L,\mathrm{st}} \subseteq \mathsf{Pr}^L$ of stable, presentable ∞-categories is a localization of Pr^L with localization functor

$$\mathsf{Pr}^L \longrightarrow \mathsf{Pr}^{L,\mathrm{st}}$$

given by $\mathcal{C} \mapsto \mathsf{Sp} \otimes \mathcal{C}$. Indeed, this follows from the example above, which, in particular, shows that $\mathsf{Sp} \otimes \mathsf{Sp} \simeq \mathsf{Sp}$. Note that the collection of arrows which map to an equivalence after tensoring with Sp is a tensor ideal, so $\mathsf{Pr}^{L,\mathrm{st}}$ gets a canonical symmetric monoidal enhancement. The unit object is Sp, and so it automatically has the structure of a commuative algebra object in $\mathsf{Pr}^{L,\mathrm{st}}$, indeed, it is the *initial* such. Unpacking this a bit reveals that Sp has an essentially unique symmetric monoidal structure with the following properties:

- The unit object is given by S^0,
- The tensor products commutes with colimits separately in each variable.

Note that this characterization tells us that there is an essentially unique symmetric monoidal enhancement of our previously consider homology functor:

$$C_* : \mathsf{Sp} \longrightarrow \mathsf{Mod}_{\mathbb{F}_2}$$

provided we verify that the tensor product in $\mathsf{Mod}_{\mathbb{F}_2}$ distributes over all colimits (which it does).

(4) We are now ready to complete the first part of our program, which is to make sense of taking tensor powers and remembering all of their symmetries. We want to have the assignment

$$X \mapsto X^{\otimes 2}$$

land in objects of $\mathcal{C}$ equipped with an action of Σ_2. This functor is the composite of the diagonal functor with the tensor product functor. We'll treat each in turn. First some notation:

Notation 4.24. If $\mathcal{C}$ is an ∞-category, then we denote a colimit for $X : \mathrm{B}\Sigma_2 \to \mathcal{C}$ by $X_{h\Sigma_2}$ and a limit by $X^{h\Sigma_2}$. Specializing to $\mathcal{C} = \mathsf{Cat}_\infty$, if $\mathcal{D} \in \mathsf{Cat}_\infty$ is viewed as a constant diagram on $\mathrm{B}\Sigma_2$, we have the calculation

$$\mathcal{D}^{h\Sigma_2} \simeq \mathsf{Fun}(\mathrm{B}\Sigma_2, \mathcal{D}).$$

Construction 4.25 (Symmetric diagonal)**.** Suppose $\mathcal{C}$ is any ∞-category, corresponding to a diagram $\bullet \to \mathsf{Cat}_\infty$. Consider the category $\mathrm{Orbit}(\Sigma_2)$ of transitive C_2-sets; i.e. we have the object $[\Sigma_2/\Sigma_2]$ and the object $[\Sigma_2]$. We may form the right Kan extension:

$$\begin{tikzcd}
\bullet \arrow[r, "\mathcal{C}"] \arrow[d, "{[\Sigma_2/\Sigma_2]}"'] & \mathsf{Cat}_\infty \\
\mathrm{Orbit}(\Sigma_2)^{\mathrm{op}} \arrow[ur, dashed] &
\end{tikzcd}$$

The value of this right Kan extension on $[\Sigma_2]$ is the product $\mathcal{C}\times\mathcal{C}$. Observe that $\mathrm{Orbit}(\Sigma_2)^{\mathrm{op}} \simeq \mathrm{B}\Sigma_2^{\triangleleft}$, so we may view the right Kan extension as giving a natural transformation of objects in $\mathsf{Fun}(\mathrm{B}\Sigma_2, \mathsf{Cat}_\infty)$

$$\mathcal{C} \xrightarrow{\mathrm{diag}} \mathcal{C} \times \mathcal{C}.$$

The right adjoint to the constant diagram functor is the homotopy limit functor, so we get a refinement of the diagonal as a functor:

$$\mathcal{C} \longrightarrow (\mathcal{C}^{\times 2})^{h\Sigma_2}.$$

Construction 4.26 (Symmetric tensor product)**.** Consider the functor $\mathrm{B}\Sigma_2^{\triangleright} \to \mathrm{Fin}_*$ taking the object in $\mathrm{B}\Sigma_2$ to $\{1,2\}_+$ with its automorphisms, the cone point to $\{1\}_+$, and the unique map to the cone point to the map coming from $\{1,2\} \to \{1\}$. If $\mathcal{C}$ is a symmetric monoidal ∞-category, then we can think of it as a commutative monoid object in Cat_∞. In particular, we get a composite

$$\mathrm{B}\Sigma_2^{\triangleright} \longrightarrow \mathrm{Fin}_* \longrightarrow \mathsf{Cat}_\infty$$

which again encodes a natural transformation in $\mathsf{Fun}(\mathrm{B}\Sigma_2, \mathsf{Cat}_\infty)$:

$$\mathcal{C}^{\times 2} \xrightarrow{\otimes} \mathcal{C}.$$

Taking homotopy limits over $\mathrm{B}\Sigma_2$, we extract:

$$(\mathcal{C}^{\times 2})^{h\Sigma_2} \longrightarrow \mathcal{C}^{h\Sigma_2} \simeq \mathsf{Fun}(\mathrm{B}\Sigma_2, \mathcal{C}).$$

Definition 4.27. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We denote by $(-)^{\otimes 2}$ the composite

$$\mathcal{C} \longrightarrow (\mathcal{C}^{\times 2})^{h\Sigma_2} \longrightarrow \mathcal{C}^{h\Sigma_2}.$$

If $\mathcal{C}$ admits limits and colimits for diagrams of shape $\mathrm{B}\Sigma_2$, then we define:

- The **symmetric square** functor by $X \mapsto \mathrm{Sym}^2(X) := X^{\otimes 2}_{h\Sigma_2}$.

- The **divided square** functor by $X \mapsto \Gamma_2(X) := (X^{\otimes 2})^{h\Sigma_2}$.

Remark 4.28. This generalizes/makes more precise the construction D_2 on chain complexes discussed earlier.

To go further, we need a good way to form the 'quotient' of Γ_2 by Sym^2. We'd also like to keep track of how these functors play with the monoidal structures.

(5) The point of forming the quotient $R^{\Sigma_2}/R_{\Sigma_2}$ in the beginning was to kill elements of R^{Σ_2} which were transfers. We begin by categorifying this procedure: how can we 'mod out' the 'transferred' elements in $\mathcal{C}^{h\Sigma_2} = \mathsf{Fun}(\mathrm{B}\Sigma_2, \mathcal{C})$?

For the remainder fo this lecture we assume that $\mathcal{C}$ is a stable, presentably symmetric monoidal ∞-category (i.e. commutative algebra object in $\mathsf{Pr}^{L,\mathrm{st}}$).

First we should say what transferred elements look like. We have a functor

$$\Sigma_2 \otimes (-) : \mathcal{C} \longrightarrow \mathcal{C}^{h\Sigma_2}$$

given by left Kan extension along $\bullet \to \mathrm{B}\Sigma_2$. The underlying object of $\Sigma_2 \otimes X$ is indeed $X \amalg X$. Denote by

$$\mathcal{C}^{h\Sigma_2}_{\mathrm{ind}} \subseteq \mathcal{C}^{h\Sigma_2}$$

the smallest stable subcategory containing all objects of the form $\Sigma_2 \otimes X$ for $X \in \mathcal{C}$. We will denote by

$$\mathsf{Fun}_{\mathrm{ind-acyclic}}(\mathcal{C}^{h\Sigma_2}, \mathcal{D}) \subseteq \mathsf{Fun}(\mathcal{C}^{h\Sigma_2}, \mathcal{D})$$

the full subcategory of functors which send objects of $\mathcal{C}^{h\Sigma_2}_{\mathrm{ind}}$ to zero objects in $\mathcal{D}$.

Theorem 5.29 (Nikolaus-Scholze)**.** *Let $\mathcal{D}$ be stable and presentably symmetric monoidal. Then the inclusion*[1]

$$\mathsf{Fun}^{\mathrm{lax,ex}}_{\mathrm{ind-acyclic}}(\mathcal{C}^{h\Sigma_2}, \mathcal{D}) \subseteq \mathsf{Fun}^{\mathrm{lax,ex}}(\mathcal{C}^{h\Sigma_2}, \mathcal{D})$$

admits a left adjoint L which can be computed by the formula:

$$LF(X) \simeq \operatorname*{colim}_{Y \in \mathcal{C}^{h\Sigma_2}_{\mathrm{ind}} \downarrow X} F(\mathrm{cofib}(Y \to X)).$$

Example 5.30 (Tate construction)**.** Define the **Tate fixed points** by

$$(-)^{t\Sigma_2} := L((-)^{h\Sigma_2}).$$

Since $(-)^{h\Sigma_2}$ is right adjoint to the symmetric monoidal diagonal functor, it has a canonical lax symmetric monoidal structure, so this makes sense. The Tate fixed point functor comes equipped with a natural transformation $(-)^{h\Sigma_2} \to (-)^{t\Sigma_2}$ of lax symmetric monoidal functors which is initial amongst natural transformations of lax symmetric monoidal functors to exact

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[1] Technically I need to assume that the functors are *accessible* and not just exact, which means they commute with κ-filtered colimits for sufficiently large κ, but I'll ignore this sorta thing for the purposes of this lecture.

functors which annihilate objects in $\mathcal{C}_{\text{ind}}^{h\Sigma_2}$. It is possible to compute that the fiber of the map $(-)^{h\Sigma_2} \to (-)^{t\Sigma_2}$ is given by $(-)_{h\Sigma_2}$ so that we have a cofiber sequence[2]:

$$X_{h\Sigma_2} \xrightarrow{\text{trace}} X^{h\Sigma_2} \longrightarrow X^{t\Sigma_2}.$$

In particular, we may define

$$T_2(X) := (X^{\otimes 2})^{t\Sigma_2},$$

which is then symmetric monoidal since it is a composite of symmetric monoidal functors. Note that this is, a priori, not exact, but it turns out to be. As a sanity check, one can compute:

$$(X \oplus Y)^{\otimes 2} \simeq X^{\otimes 2} \oplus Y^{\otimes 2} \oplus \Sigma_2 \otimes (X \otimes Y)$$

and the 'error term' dies after taking the Tate construction. The case of a general cofiber sequence turns out to follow from this. Notice how this is an exact categorification of the computation we made much earlier for the diagonal $r \mapsto r \otimes r$.

(6) We are finally ready to build a Frobenius First we need a natural transformation

$$X \to T_2(X).$$

Unfortunately, in general there aren't any interesting ones. For example, any such natural transformation in $\mathsf{Mod}_{\mathbb{F}_2}$ is pointwise zero. This is a categorification of the observation that the map

$$r \mapsto r \otimes r$$

to the Tate fixed points is *additive* but not, in general, compatible with scalar multiplication. We might expect this problem to be fixed if all of our scalars came from 'addition', i.e. if we worked over the *initial* ring. Classically, that's $\mathbb{Z}$, but in higher algebra it's S^0, modules over which form the ∞-catgeory Sp.

Theorem 6.31 (Nikolaus)**.** *Let $\mathcal{C}$ be presentably symmetric monoidal, and denote by $U : \mathcal{C} \longrightarrow \mathsf{Sp}$ the essentially unique functor with $\Omega^\infty U = \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, -)$. Then U is initial amongst lax symmetric monoidal, exact functors to Sp.*

Corollary 6.32. *There is an essentially unique natural transformation of lax symmetric monoidal functors*

$$\Delta_2 : U \to UT_2.$$

We call this the ***Tate diagonal****.*

Example 6.33. When $\mathcal{C} = \mathsf{Mod}_{\mathbb{F}_2}$, the functor U is the forgetful functor to spectra. So we learn that every $\mathbb{F}_2$-module X admits a natural map

$$X \to (X^{\otimes 2})^{t\Sigma_2}$$

of spectra. (So in this case we often suppress U).

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[2] The first map is commonly called the 'norm' but it feels more like *adding* over orbits rather than *multiplying* over orbits, so I prefer 'trace'

Definition 6.34. Suppose $X \in \mathcal{C}$ is equipped with a **symmetric multiplication**, i.e. a map $\mathrm{Sym}^2(X) \to X$ or, equivalently, a map $m : X^{\otimes 2} \to X$ in $\mathcal{C}^{h\Sigma_2}$. We denote by $\mathrm{Frob}_2 : UX \to UX^{t\Sigma_2}$ the composite

$$UX \xrightarrow{\Delta_2} UT_2(X) = U(X^{\otimes 2})^{t\Sigma_2} \xrightarrow{m} UX^{t\Sigma_2}.$$

(7) Now we specialize to $\mathcal{C} = \mathsf{Mod}_{\mathbb{F}_2}$. If M is an $\mathbb{F}_2$-module, one can generalize the computation of the cohomology of $\mathbb{R}P^\infty$ to give a canonical equivalence:

$$\pi_* M^{t\Sigma_2} \simeq M_*(\!(t)\!)$$

Definition 7.35. Let $R \in \mathsf{Mod}_{\mathbb{F}_2}$ be equipped with a symmetric multiplication. Then we define a **total power operation** $Q(t)$ by the effect on homotopy of $\mathrm{Frob}_2 : R \to R^{t\Sigma_2}$. Under the identification of the coefficients as Laurent series, we write:

$$Q(t)(x) = \sum_{i \in \mathbb{Z}} Q^i(x) t^i.$$

Keeping track of degrees, we see that taking the coefficient of t^i gives a map

$$Q^i : R_* \to R_{*+i}.$$

Example 7.36. In the case where $R = C^*(X)$ is the cochains on a space, it turns out that

$$Q^i(x) = \mathrm{Sq}^{-i} x$$

so these power operations recover the Steenrod operations.

Remark 7.37. It follows from the uniqueness of the Tate diagonal that we actually could have just viewed R as a *spectrum*, and used

$$R \to (R^{\wedge 2})^{t\Sigma_2} \to R^{t\Sigma_2}$$

and gotten the *same* Frobenius. So the Frobenius doesn't really care about the $\mathbb{F}_2$-module structure: that structure enters in only to give a *preferred* identification

$$R_*^{t\Sigma_2} \simeq R_*(\!(t)\!).$$

Now, $\mathbb{F}_2$ is a commutative algebra object in Sp[3] and hence so are the various objects

$$\mathbb{F}_2^{h\Sigma_2}, \mathbb{F}_2 \wedge \mathbb{F}_2, (\mathbb{F}_2 \wedge \mathbb{F}_2)^{h\Sigma_2}$$

---

[3] Indeed, we have a unique symmetric monoidal left adjoint $\mathsf{Sp} \to \mathsf{Mod}_{\mathbb{F}_2}$ with right adjoint the forgetful functor, which is thus lax symmetric monoidal. The unit object in $\mathsf{Mod}_{\mathbb{F}_2}$ is $\mathbb{F}_2$ then maps to a commutative algebra in Sp.

By the remark above, we have a commutative diagram:

$$\begin{tikzcd}
\mathbb{F}_2^{h\Sigma_2} \arrow[r, "\mathrm{Frob}_2"] \arrow[d, "\psi_R"'] & (\mathbb{F}_2^{h\Sigma_2})^{t\Sigma_2} \arrow[d, "\psi_R"] \\
(\mathbb{F}_2 \wedge \mathbb{F}_2)^{h\Sigma_2} \arrow[r, "\mathrm{Frob}_2"'] & ((\mathbb{F}_2 \wedge \mathbb{F}_2)^{h\Sigma_2})^{t\Sigma_2}
\end{tikzcd}$$

where ψ_R arises from the map $\mathbb{F}_2 = \mathbb{F}_2 \wedge S^0 \to \mathbb{F}_2 \wedge \mathbb{F}_2$. Writing

$$\pi_* \mathbb{F}_2^{h\Sigma_2} = \mathbb{F}_2[\![s]\!],$$

$$\pi_* (\mathbb{F}_2 \wedge \mathbb{F}_2)^{h\Sigma_2} \simeq \mathcal{A}_*[\![s]\!]$$

we get a formula of the form:

$$\psi_R(Q(t)s) = Q(t)\psi_R(s) \in \mathcal{A}_*[\![s]\!](\!(t)\!).$$

Now we need two important computations:

- The computation of the action of the Steenrod operations on $H^*(\mathbb{R}P^\infty)$ translates to:

  $$Q(t)s = s + s^2 t^{-1}$$

- The Steenrod coaction on $H^*(\mathbb{R}P^\infty) = \pi_* \mathbb{F}_2^{h\Sigma_2}$ turns out to be:

  $$\psi_R(s) = \zeta(s) = \sum_{i \geqslant 0} \zeta_i s^{2^i} \in \mathcal{A}_*[\![s]\!].$$

  This is actually a formal consequence of the fact that we defined the generators ζ_i to come from the duals of s^{2^i} in the homology of $\mathbb{R}P^\infty = K(\mathbb{F}_2, 1)$.

Note that the computation also forces a computation for $\psi_R(t)$, the coaction on the generator in the Tate construction, since ψ_R is a ring map.

So the left hand side simplifies to:

$$\psi_R(Q(t)s) = \psi_R(s + s^2 t^{-1}) = \zeta(s) + \zeta(s)^2 \zeta(t)^{-1}$$

and the right hand side simplifies to:

$$Q(t)\psi_R(s) = Q(t)(\sum \zeta_i s^{2^i}) = \sum Q(t)(\zeta_i)(Q(t)(s))^{2^i} = \sum_i Q(t)(\zeta_i)(s^{2^i} + s^{2^{i+1}} t^{-2^i}).$$

Theorem 7.38. *The action of power operations on $\mathbb{F}_2 \wedge \mathbb{F}_2$ is determined implicitly by the identity:*

$$\zeta(s) + \zeta(s)^2 \zeta(t)^{-1} = \sum_i Q(t)(\zeta_i)(s^{2^i} + s^{2^{i+1}} t^{-2^i}).$$

Corollary 7.39. $Q^{2^i}\zeta_i \equiv \zeta_{i+1}$ *mod decomposables.*

Proof. Compare coefficients of $t^0 s^{2^{i+1}}$ on both sides. (Here you have to use a fact which is not immediately obvious from what we've done so far, that $Q^j(x) = 0$ when $j < |x|$.) ☐

Remark 7.40. It turns out the operation Q^{2^i} on a class in dimension $2^i - 1$ actually only depends on a small part of the symmetric multiplication, namely the restriction

$$R^{\otimes 2} \wedge_{\Sigma_2} S^1 \to R^{\otimes 2}_{\Sigma_2} \to R.$$

It turns out that this piece of structure is, for example, available in any $\mathbb{E}_2$-algebra, which is a less-commutative variant of commutative algebra objects that will be discussed later. One consequence is that, if $R \to \mathbb{F}_2$ is a map of objects equipped with a 'weak symmetric multiplication' as above, and which hits ζ_1 on homology, then the map on homology is automatically surjective.