

Spectra and Linearization

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A major technique in algebraic topology, in some sense the whole point, is to transform problems about spaces into problems in linear algebra. Linear algebra is a great place to be. We can add maps, we can take kernels and cokernels, and do all sorts of manipulations that don't make any sense with spaces and maps between them. On the other hand, extracting a vector space as an invariant usually loses an enormous amount of information and often hides geometry or deeper structure present in the original problem.

The goal today is to discuss a compromise, and try to answer the question: how can we linearize the category of spaces?

Our guiding prototype is the procedure in classical category theory which assigns, to a category \mathcal{C} with finite limits, the category $\mathbf{Ab}(\mathcal{C})$ of abelian group objects of \mathcal{C} , which is an abelian category.

(1) An abelian category satisfies the following properties (each one assuming the previous ones hold):

- It is **pointed**: there is an initial object \emptyset and a final object 1 and the map $\emptyset \rightarrow 1$ is an equivalence. We denote the initial/final object by 0 ; it's called a **zero object**.
- It is also **semiadditive**: \mathcal{C} admits finite coproducts and finite products and the map

$$X \amalg Y \xrightarrow{\begin{pmatrix} \text{id}_X & 0 \\ 0 & \text{id}_Y \end{pmatrix}} X \times Y$$

is an equivalence. The common value is denoted $X \oplus Y$ and called the **direct sum**. Note that we may now add maps: given $f, g : X \rightarrow Y$, we get

$$f + g : X \rightarrow X \times X \rightarrow Y \times Y \simeq Y \amalg Y \rightarrow Y.$$

- It is **additive**: the shearing map

$$X \oplus X \xrightarrow{(\pi_1 + \pi_2, \pi_2)} X \oplus X$$

is an equivalence.

- Finally, there is a condition on images which in particular implies: \mathcal{C} has kernels and cokernels and a diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{monic}} & B \\ \downarrow & & \downarrow \text{epi} \\ 0 & \longrightarrow & C \end{array}$$

is a pushout if and only if it's a pullback.

The first three definitions carry over verbatim to define pointed, semiadditive, and additive ∞ -categories. The last definition is sort of problematic, since the intuition for 'injective and surjective' is not very homotopy invariant. So we will need a different notion.

Definition 1.1. Let \mathcal{D} be a pointed ∞ -category. Suppose given a diagram $[1] \times [1] \rightarrow \mathcal{D}$, informally depicted by

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object of \mathcal{D} . We say that the diagram is a **cofiber sequence** if it is a pushout and a **fiber sequence** if it is a pullback.

Remark 1.2. Suppose every diagram $0 \leftarrow X \rightarrow Y$ admits a pushout, then in fact we can form a functor

$$\mathcal{D}^{[1]} \longrightarrow \mathcal{D}^{[1] \times [1]}$$

which takes an arrow $f : X \rightarrow Y$ and spits out a cofiber sequence with f at the top. One way to do stuff like this with ∞ -categories, where you have choices, is try to hope that you're building some diagram by iterated left and right Kan extensions and then appeal to the existence and uniqueness of Kan extensions. This is one of the cornerstones of the subject- and we briefly discussed it in class. In our case, we learn the following. First let $\mathcal{E}' \subseteq \text{Fun}(\Lambda_0^2, \mathcal{D})$ be the full subcategory of diagrams $0' \leftarrow X \rightarrow Y$ where $0'$ is a zero object. Then let $\mathcal{E}'' \subseteq \text{Fun}([1] \times [1], \mathcal{D})$ be the full subcategory of cofiber sequences. Existence and uniqueness of right Kan extensions ensures that $\mathcal{E}' \rightarrow \mathcal{D}^{[1]}$ is an equivalence and existence and uniqueness of left Kan extensions says that the restriction $\mathcal{E}'' \rightarrow \mathcal{E}'$ is an equivalence. There's a contractible space of choices of inverse to these equivalences, and composing them yields the desired functorial cofiber.

Definition 1.3. When they exist, we denote $\Sigma_{\mathcal{D}}X := \text{cofib}(X \rightarrow 0)$ and $\Omega_{\mathcal{D}}X := \text{fib}(0 \rightarrow X)$ which are well-defined up to the usual ambiguity, and can be made functorial as described above.

Definition 1.4. \mathcal{D} is **stable** if it's pointed, admits cofibers and fibers, and a square is a cofiber sequence if and only if it's a fiber sequence.

It turns out that this analogue of one of the last properties of an abelian category implies the others, as we'll come to see.

The following proposition is tricky to prove, but very useful (though note the implication that (i) implies both (ii) and (iii) is straightforward):

Proposition 1.5. *The following are equivalent for a pointed ∞ -category \mathcal{D} .*

- (i) \mathcal{D} is stable.
- (ii) \mathcal{D} admits finite colimits and $\Sigma_{\mathcal{D}}$ is an equivalence.
- (iii) \mathcal{D} admits finite limits and $\Omega_{\mathcal{D}}$ is an equivalence.

Example 1.6. $\text{Mod}_{\mathbb{F}_2}$ is stable.

Remark 1.7. Suppose \mathcal{D} is stable. Then:

$$\text{Map}_{\mathcal{D}}(X, Y) \simeq \text{Map}_{\mathcal{D}}(X, \Omega_{\mathcal{D}}^2 \Sigma_{\mathcal{D}}^2 Y) \simeq \Omega^2 \text{Map}_{\mathcal{D}}(X, \Sigma^2 Y).$$

So every mapping space is a double loop space. It follows that its set of connected components has the canonical structure of an abelian group, i.e. $[X, Y]_{\mathcal{D}}$ is an abelian group.

Remark 1.8. Some care is required to interpret the symbol $\Omega^2 \text{Map}_{\mathcal{D}}(X, Z)$; this symbol suggests that I have equipped each mapping space with a point. We'd like to say 'yeah, it's the composite $X \rightarrow 0 \rightarrow Z$ ' which is kinda true. Here's how it's true: we have a zig-zag

$$\text{Map}_{\mathcal{D}}(X, 0) \times \text{Map}_{\mathcal{D}}(0, Z) \leftarrow \{\sigma \in \text{Fun}([2], \mathcal{D}) : \sigma(0) = X, \sigma(1) = 0, \sigma(2) = Y\} \rightarrow \text{Map}_{\mathcal{D}}(X, Z)$$

Literally the whole point of ∞ -categories is that the left-hand arrow is an equivalence. Moreover, since 0 is both initial and final, we see that the first term, and hence the middle term, are contractible. So $\text{Map}_{\mathcal{D}}(X, Z)$ is canonically pointed¹ in the sense that it receives a map from a contractible space, which may as well be telling me a basepoint.

Remark 1.9. There is a little danger in representing diagrams informally like

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & B \end{array}$$

Indeed, this diagram produces an element $f : A \rightarrow \Omega B$, i.e. an element in the group $\pi_0 \Omega \text{Map}(A, B)$. If we flip the two vertices, then we get what looks like the *same* diagram

¹Ok, if you want me to not choose a zero object either, that's fine. One can prove that the subcategory of $\text{Fun}([2], \mathcal{D})$ with first vertex X , second vertex any zero object, and Z as the last vertex, with morphisms the identity on X and Z , is a contractible Kan complex.

but in fact yields $-f$! Why is that? Well, the data of a diagram is more than what I drew! We have a pushout square of simplicial sets:

$$\begin{array}{ccc} \partial\Delta^2 \amalg \partial\Delta^2 & \longrightarrow & \mathrm{sk}_1 N([1] \times [1]) \\ \downarrow & & \downarrow \\ \Delta^2 \times \Delta^2 & \longrightarrow & N([1] \times [1]) \end{array}$$

After making some choices, like a composition of $A \rightarrow 0 \rightarrow B$, a diagram as above amounts to just the data of:

- a map $g : A \rightarrow B$,
- a nullhomotopy $\alpha : g \rightarrow 0$,
- *another* nullhomotopy $\beta : 0 \rightarrow g$

Depending on some convention you picked in preschool, this corresponds to the loop $\alpha^{-1}\beta$ in $\Omega\mathrm{Map}(A, B)$. Now it's clear that reversing the diagram swaps the 2-cells, and hence the homotopies, spits out the opposite loop $\beta^{-1}\alpha = (\alpha^{-1}\beta)^{-1}$.

Remark 1.10. In class I explained how to recover the octahedral axiom by starting with $X \rightarrow Y \rightarrow Z$ and iteratively right and left Kan extending to a big staircase, and then using various pasting lemma tricks to compute the values of the Kan extensions as cofibers and suspensions. See HA.1.1.2.14 for details.

Remark 1.11. One can use the abelian group structure on mapping spaces to construct inverses to the maps appearing in the definition of semiadditivity and additivity, thereby proving that stable ∞ -categories are also additive.

(2) Now: how do we *stabilize* and ∞ -category? By analogy with abelianization, we might hope that, in good cases, we will be supplied with a stable ∞ -category $\mathrm{Sp}(\mathcal{D})$ and a ‘forgetting-ish’ functor

$$\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \longrightarrow \mathcal{D}$$

which, in particular, ought to preserve limits. Whatever $\mathrm{Sp}(\mathcal{D})$ is, we can start to get a handle on it by considering the map

$$\mathrm{Sp}(\mathcal{D}) = \mathrm{Fun}(\bullet, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Fun}^{\mathrm{rex}}(\mathrm{Spaces}_*^{\mathrm{fin}}, \mathrm{Sp}(\mathcal{D})) \longrightarrow \mathrm{Fun}^{??}(\mathrm{Spaces}_*^{\mathrm{fin}}, \mathcal{D})$$

where $\mathrm{Spaces}_*^{\mathrm{fin}}$ is the smallest full subcategory of Spaces_* containing S^0 and closed under finite coproducts and pushouts, and $\mathrm{Fun}^{\mathrm{rex}}(-, -)$ denotes finite colimit-preserving functors. What kind of functors do we get to \mathcal{D} ? Well, pushouts in $\mathrm{Spaces}_*^{\mathrm{fin}}$ went to pushouts in $\mathrm{Sp}(\mathcal{D})$ *which are also pullbacks* and hence preserves by Ω^∞ (by assumption). So we are led to the following concept:

Definition 2.12. Suppose \mathcal{C} is pointed with finite² colimits, and \mathcal{D} has finite limits. Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor. We say that F is

- **reduced** if $F(0)$ is final in \mathcal{D} ;
- **excisive** if F takes pushout squares to pullback squares; and
- **linear** if it is reduced and excisive.

We denote by $\text{Lin}(\mathcal{C}, \mathcal{D})$ the full subcategory of linear functors.

Lemma 2.13. $\text{Lin}(\mathcal{C}, \mathcal{D})$ is stable.

Proof. The inclusion $\text{Lin}(\mathcal{C}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C}, \mathcal{D})$ is closed under limits, so the functor Ω is computed pointwise. Now observe that, if F is linear, then $F(X) \rightarrow \Omega_{\mathcal{D}}F(\Sigma_e X)$ is an equivalence. It follows that $F \mapsto F \circ \Sigma_{\mathcal{C}}$ gives an inverse to $\Omega_{\mathcal{D}}$, which is then an equivalence.³ \square

Lemma 2.14. \mathcal{D} is stable if and only if evaluation at S^0 gives an equivalence

$$\text{Lin}(\text{Spaces}_*^{\text{fin}}, \mathcal{D}) \longrightarrow \mathcal{D}.$$

Proof. If this is an equivalence, \mathcal{D} is stable by the previous lemma. If \mathcal{D} is stable then $\text{Lin}(\text{Spaces}_*^{\text{fin}}, \mathcal{D}) = \text{Fun}^{\text{rex}}(\text{Spaces}_*^{\text{fin}}, \mathcal{D})$ is equivalent to \mathcal{D} since $\text{Spaces}_*^{\text{fin}}$ is the free pointed ∞ -category with finite colimits on one object. \square

Definition 2.15. The **stabilization** of \mathcal{D} is defined to be $\text{Sp}(\mathcal{D}) := \text{Lin}(\text{Spaces}_*^{\text{fin}}, \mathcal{D})$. Denote by

$$\Omega^{\infty} : \text{Sp}(\mathcal{D}) \longrightarrow \mathcal{D}$$

evaluation at S^0 .

The following result is now straightforward:

Theorem 2.16. Let \mathcal{C} be pointed with finite colimits and let \mathcal{D} have finite limits. Then $\text{Sp}(\mathcal{D})$ is characterized by the following universal property that Ω^{∞} gives an equivalence:

$$\text{Fun}^{\text{rex}}(\mathcal{C}, \text{Sp}(\mathcal{D})) \xrightarrow{\simeq} \text{Lin}(\mathcal{C}, \mathcal{D}).$$

Remark 2.17. If \mathcal{C} is stable, we can rewrite this as

$$\text{Fun}^{\text{ex}}(\mathcal{C}, \text{Sp}(\mathcal{D})) \simeq \text{Fun}^{\text{lex}}(\mathcal{C}, \mathcal{D}),$$

where an ‘exact’ functor between stable ∞ -categories is equivalently either a right or left exact functor.

²In the theory of ∞ -categories a diagram is **finite** if it has finitely many nondegenerate cells. That is, we consider the subcategory of simplicial sets containing Δ^0 and Δ^1 and closed under pushouts and finite coproducts.

³Unfortunately this argument is circular as it stands, because Proposition 1.5 is proved using this lemma as input. However, a weaker variant of that proposition is first proved and used to prove this lemma, and then later one proves the proposition. See HA.1.4.2.16 and follow the chain of cited lemmas for a proper proof.

(3) Let's play around with the universal property.

Corollary 3.18. *Evaluation at the spheres S^n gives an equivalence*

$$\mathbf{Sp}(\mathcal{D}) \simeq \lim(\cdots \rightarrow \mathcal{D}_* \xrightarrow{\Omega} \mathcal{D}_* \xrightarrow{\Omega} \mathcal{D}_*)$$

where $\mathcal{D}_* = \mathcal{D}_{1/}$ is the category of pointed objects of \mathcal{D} .

Proof. Immediate from the universal property (map in a test category \mathcal{C} and compare). \square

We'll talk a little later about how to think about these sorts of limits and colimits inside \mathbf{Cat}_∞ . For now, let's just push ahead and get more models and characterizations. We'll need a definition though:

Definition 3.19. An ∞ -category \mathcal{C} is **presentable** if it is given by a localization of $\mathbf{Psh}(K)$ at a set of morphisms, where K is a small ∞ -category.

The salient feature of presentable categories is that there is an adjoint functor theorem:

Theorem 3.20 (Adjoint functor theorem). *If \mathcal{C} and \mathcal{D} are presentable, then $h : \mathcal{C} \rightarrow \mathcal{D}$ is a left adjoint if and only if it preserves all colimits, and a right adjoint if and only if it preserves all limits and is accessible, i.e. preserves κ -filtered colimits for some regular cardinal κ .⁴*

Corollary 3.21. *Let \mathcal{C} and \mathcal{D} be presentable and suppose \mathcal{D} is stable. Then $\Omega^\infty : \mathbf{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$ admits a left adjoint, Σ_+^∞ , and precomposition yields an equivalence*

$$\mathbf{Fun}^{\mathrm{colim}}(\mathbf{Sp}(\mathcal{C}), \mathcal{D}) \simeq \mathbf{Fun}^{\mathrm{colim}}(\mathcal{C}, \mathcal{D}).$$

Proof. By the adjoint functor theorem, this reduces to checking that accessible, limit preserving functors $\mathcal{D} \rightarrow \mathbf{Sp}(\mathcal{C})$ correspond to accessible, limit preserving functors $\mathcal{D} \rightarrow \mathcal{C}$ after composing with Ω^∞ , which follows from the universal property of $\mathbf{Sp}(\mathcal{C})$. \square

Example 3.22. In the case $\mathcal{C} = \mathbf{Spaces}$, we can say more. The ∞ -category \mathbf{Spaces} is freely generated by $*$ under colimits, i.e. evaluation at the point gives an equivalence

$$\mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spaces}, \mathcal{D}) \simeq \mathcal{D}$$

for any cocomplete \mathcal{D} . It follows that, if \mathcal{D} is stable and presentable, then

$$\mathbf{Fun}^{\mathrm{colim}}(\mathbf{Sp}, \mathcal{D}) \simeq \mathcal{D}.$$

So defining colimit-preserving functors out of \mathbf{Sp} is easy. As an explicit example, the element $\mathbb{F}_2 \in \mathbf{Mod}_{\mathbb{F}_2}$ produces a functor

$$\mathbf{Sp} \rightarrow \mathbf{Mod}_{\mathbb{F}_2}$$

which you might either call tensoring with \mathbb{F}_2 or singular chains. This uniqueness result can be thought of as a generalization of the Eilenberg-Steenrod uniqueness theorem (that homology theories are determined by their behavior on a point).

⁴A diagram J is κ -filtered if every map $K \rightarrow J$ from a simplicial set K with less than κ many nondegenerate cells extends over K^\triangleright .

Corollary 3.23. *If \mathcal{C} is presentable then the functors $\Sigma^{-n}\Sigma_+^\infty$ exhibit $\mathbf{Sp}(\mathcal{C})$ as the colimit computed in the ∞ -category \mathbf{Pr}^L of presentable ∞ -categories and colimit-preserving functors between them:*

$$\mathbf{Sp}(\mathcal{C}) \simeq \operatorname{colim}_{\mathbb{Z}_{\geq 0} \rightarrow \mathbf{Pr}^L} \left(\mathcal{C}_* \xrightarrow{\Sigma} \mathcal{C}_* \xrightarrow{\Sigma} \dots \right).$$

Proof. By the adjoint functor theorem, one can show that colimits in \mathbf{Pr}^L are computed as the limits of the corresponding diagrams of right adjoints, so the result follows from Corollary 3.18. \square

Finally, recall that if \mathcal{C} is a small ∞ -category, then $\mathbf{Ind}(\mathcal{C}) \subseteq \mathbf{Psh}(\mathcal{C})$ is the subcategory generated by the image of the Yoneda embedding under (finitely) filtered colimits. In the case where \mathcal{C} admits finite colimits, $\mathbf{Ind}(\mathcal{C})$ is presentable, and one can show that the functor

$$\mathbf{Ind} : \mathbf{Cat}_\infty^{\operatorname{rex}} \longrightarrow \mathbf{Pr}^L$$

preserves colimits.

Corollary 3.24. *Let \mathcal{C}^ω be a small, finite cocomplete ∞ -category, and define the **Spanier-Whitehead** category of \mathcal{C}^ω to be the colimit in \mathbf{Cat}_∞ :*

$$\mathbf{SW}(\mathcal{C}^\omega) = \operatorname{colim}(\mathcal{C}_*^\omega \xrightarrow{\Sigma} \mathcal{C}_*^\omega \xrightarrow{\Sigma} \dots).$$

Let $\mathcal{C} = \mathbf{Ind}(\mathcal{C}^\omega)$. Then

$$\mathbf{Sp}(\mathcal{C}) \simeq \mathbf{Ind}(\mathbf{SW}(\mathcal{C}^\omega)).$$

(4) Before we examine in more detail what these various models of spectra tell us, let's pause and explain how to compute sequential limits and colimits of ∞ -categories.

The key observation is that the following diagram is a homotopy pushout of quasicategories:

$$\begin{array}{ccc} \coprod\{i, i+1\} & \longrightarrow & \coprod[i, i+1] \\ \downarrow & & \downarrow \\ \coprod\{i\} & \longrightarrow & N(\mathbb{Z}_{\geq 0}) \end{array}$$

It follows that colimits of a functor $F : \mathbb{Z}_{\geq 0} \rightarrow \mathcal{C}$ sit as pushouts:

$$\begin{array}{ccc} \coprod(F(i) \amalg F(i+1)) & \longrightarrow & \coprod F(i+1) \\ \downarrow & & \downarrow \\ \coprod F(i) & \longrightarrow & \operatorname{colim} F \end{array}$$

and limits of $F : \mathbb{Z}_{\geq 0}^{\operatorname{op}} \rightarrow \mathcal{C}$ sit as pullbacks

$$\begin{array}{ccc} \lim F & \longrightarrow & \prod F(i) \\ \downarrow & & \downarrow \\ \prod F(i) & \longrightarrow & \prod(F(i) \times F(i+1)) \end{array}$$

where I leave you to work out the maps. In particular: we find that the category $\mathbf{Sp}(\mathcal{D})$ consists of tuples (X_i) equipped with *equivalences* $X_i \rightarrow \Omega X_{i+1}$. Observe that this naturally sits in the ‘right lax limit’ which consists of tuples (X_i) equipped with just maps $X_i \rightarrow \Omega X_{i+1}$, or equivalently $\Sigma X_i \rightarrow X_{i+1}$ in the case where \mathcal{D} has colimits. This right lax limit is called the ∞ -category of **prespectra**, denoted $\mathbf{PSp}(\mathcal{D})$.

(5) Now let’s study our various models of spectra.

First, we’ve got what is almost Boardman’s model:

$$\mathbf{Sp} \simeq \mathbf{Ind}(\mathbf{SW}).$$

We’ll deal with it one piece at a time. The objects $[0]$ and $[1]$ in \mathbf{Cat}_∞ are compact, so any object and morphisms in \mathbf{SW} arises at some finite stage of the colimit. In other words: objects amount to just pairs (X, n) where X is a finite pointed complex and $n \geq 0$ is an integer. We will suggestively denote this pair by the symbol

$$\Sigma^{-n} \Sigma^\infty X$$

Maps are then computed as colimits:

$$\mathrm{Map}_{\mathbf{SW}}(\Sigma^{-n} \Sigma^\infty X, \Sigma^{-m} \Sigma^\infty Y) \simeq \mathrm{colim}_{N \gg 0} \mathrm{Map}(\Sigma^{N-n} X, \Sigma^{N-m} Y).$$

Any object of $\mathbf{Ind}(\mathbf{SW})$ can be written as a filtered colimit of objects in \mathbf{SW} , and mapping out of objects in \mathbf{SW} commutes with filtered colimits. So a spectrum is a formal gadget like

$$\mathrm{colim}_\alpha \Sigma^{-n_\alpha} \Sigma^\infty X_\alpha$$

and maps are computed by the formula:

$$\mathrm{Map}_{\mathbf{Sp}}(\mathrm{colim}_\alpha \Sigma^{-n_\alpha} \Sigma^\infty X_\alpha, \mathrm{colim}_\beta \Sigma^{-n_\beta} \Sigma^\infty Y_\beta) \simeq \lim_\alpha \mathrm{colim}_{\beta, N} \mathrm{Map}(\Sigma^{N-n_\alpha} X_\alpha, \Sigma^{N-m_\beta} Y_\beta).$$

Informally, we can think of $\Sigma^{-n_\alpha} \Sigma^\infty X_\alpha$ as trying to fill out our spectrum by finite *stable* subcomplexes (se we are allowed to have negative dimensional cells). This formula for maps can be interpreted as follows: in order to define a map $X \rightarrow Y$, each cell of X must map to some eventual skeleton of Y , and the different maps on overlapping cells must be eventually compatible after passing to a larger subcomplex of Y . Working this out precisely is essentially what Adams does in the ‘blue book’.

Next up we have the formula

$$\mathbf{Sp} = \lim(\cdots \xrightarrow{\Omega} \mathbf{Spaces}_* \xrightarrow{\Omega} \mathbf{Spaces}_*)$$

which is telling us to think of spectra as tuples (X_i) of pointed spaces equipped with equivalences $X_i \rightarrow \Omega X_{i+1}$. A map is then specified by giving maps $X_i \rightarrow Y_i$ together with witnesses for the homotopy commutativity of each diagram

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ \Omega X_i & \longrightarrow & \Omega Y_i \end{array}$$

(no further data is required because of the simple description of the ∞ -category $\mathbb{Z}_{\geq 0}$ discussed earlier). This is the point of view that spectra are **infinite loop spaces**. An example of this paradigm is the sequence of Eilenberg-MacLane spaces $K(A, n)$ for an abelian group A , which present a spectrum denoted, in this class, simply by A again.

Recall that we had an inclusion

$$\mathbf{Sp} \subseteq \mathbf{PSp}$$

of spectra in prespectra; where the latter drops the requirement that the map $X_i \rightarrow \Omega X_{i+1}$ be an equivalence. This inclusion admits a left adjoint \mathbf{Q} for formal reasons, and we can actually compute it; it's given by:

$$\mathbf{Q}((X_i))_n = \operatorname{colim} \Omega^k X_{n+k}.$$

So we can also think of \mathbf{Sp} as the localization of \mathbf{PSp} at those maps which become an equivalence after applying \mathbf{Q} , i.e. the stable equivalences. Since \mathbf{Q} preserves colimits, we can think of a prespectrum as *presenting* for us the spectrum:

$$\operatorname{colim} \Sigma^{-n} \Sigma^\infty X_n,$$

where this colimit is computed in \mathbf{Sp} . This point of view is also the one that comes from thinking about \mathbf{Sp} as a colimit in \mathbf{Pr}^L .

Finally, we have the ‘Goodwillie’ point of view on spectra, the definition we used first:

$$\mathbf{Sp} = \operatorname{Lin}(\operatorname{Spaces}_*^{\operatorname{fin}}, \operatorname{Spaces}).$$

So a spectrum is a kind of invariant of finite pointed spaces. If $A \rightarrow B \rightarrow C$ is a cofiber sequence, and F is a linear functor, we get a fiber sequence $F(A) \rightarrow F(B) \rightarrow F(C)$ and hence a long exact sequence on homotopy groups. It follows that $A \mapsto \pi_* F(A)$ is a homology theory, which is another point of view on what spectra are supposed to ‘be’. To relate this to the previous definition, note that $\{F(S^n)\}$ is the corresponding sequence of spaces delooping $F(S^0)$. From this point of view the homology theory associated to a finite complex X wants to be $A \mapsto A \wedge X$, except this isn't linear. This brings us to the next topic.

(6) Suppose \mathcal{C} and \mathcal{D} are as in the definition of linear functors. If \mathcal{D} admits sequential colimits which commute with finite limits, then it turns out that the subcategory of excisive functors

$$\operatorname{Exc}(\mathcal{C}, \mathcal{D}) \subseteq \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$

admits a left adjoint called P_1 (this is a wonderful theorem of Goodwillie). It can be described by the formula

$$P_1 F = \operatorname{colim} T_1^n F$$

where $(T_1 F)(X)$ is the pullback of $F(0) \rightarrow F(\Sigma X) \leftarrow F(0)$. In particular, if F is reduced then

$$P_1 F = \operatorname{colim} \Omega_{\mathcal{D}}^n F \Sigma_{\mathcal{C}}^n.$$

Example 6.25. Consider the identity functor $\text{id} : \mathbf{Spaces}_* \rightarrow \mathbf{Spaces}_*$. On the one hand, playing around with adjoints shows that $P_1(\text{id})$ must be $\Omega^\infty \Sigma^\infty$. On the other hand, the formula above recovers the familiar fact

$$\Omega^\infty \Sigma^\infty = \text{colim } \Omega^n \Sigma^n.$$

More generally, we learn that the linear functor associated to $\Sigma^\infty X$ is given on finite pointed complexes by

$$A \mapsto \text{colim } \Omega^n (\Sigma^n A \wedge X).$$

(It's also possible to recover these facts directly from, say, the description of spectra as a localization of prespectra).

Remark 6.26. From the equivalence

$$\text{Fun}^L(\mathbf{Sp}, \mathbf{Sp}) \simeq \mathbf{Sp}$$

we see that every spectrum X gives rise to a colimit preserving functor denoted $X \otimes (-) : \mathbf{Sp} \rightarrow \mathbf{Sp}$ (or more classically as $X \wedge (-)$). In fact, the left hand side is evidently a monoid object under composition in quasicategories, so this equivalence endows \mathbf{Sp} with some kind of 'weak monoid' structure. We'll explore that more later, but for now note that we may at least extract functors

$$X \mapsto X^{\otimes k}.$$

Example 6.27. Now let's consider functors from spectra to spectra. If $k \geq 2$ then

$$P_1(X \mapsto X^{\otimes k})$$

vanishes, as might be expected. More generally this holds for the 'monomials' $X \mapsto A \wedge X^{\otimes k}$ for any spectrum A and $k \geq 2$. We leave this as an exercise (hint: if each map in a sequential colimit is null, the colimit vanishes).