GEOMETRY OF PCF PARAMETERS IN SPACES OF QUADRATIC POLYNOMIALS

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Abstract. We study algebraic relations among postcritically finite (PCF) parameters in the family \( f_c(z) = z^2 + c \). In [GKNY], it was proved that an algebraic curve in \( \mathbb{C}^2 \) contains infinitely many PCF pairs \((c_1, c_2)\) if and only if the curve is special (i.e., the curve is a vertical or horizontal line through a PCF parameter, or the curve is the diagonal). Here we extend this result to subvarieties of \( \mathbb{C}^n \) for any \( n \geq 2 \). Consequently, we obtain uniform bounds on the number of PCF pairs on non-special curves in \( \mathbb{C}^2 \) and the number of PCF parameters in real algebraic curves in \( \mathbb{C} \), depending only on the degree of the curve. We also compute the optimal bound for the general curve of degree \( d \).

1. Introduction

For each \( c \in \mathbb{C} \), let \( f_c(z) = z^2 + c \). Recall that the polynomial \( f_c \) is postcritically finite (PCF) if the critical point at \( z_0 = 0 \) has a finite forward orbit. In this article, we study algebraic relations among the PCF parameters \( c \in \mathbb{C} \).

Figure 1.1. The Mandelbrot set with PCF parameters marked in yellow. There are only finitely many real lines in \( \mathbb{C} \) containing > 2 PCF parameters; see Theorem 1.8.

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Our starting point is the following theorem of Ghioca, Krieger, Nguyen, and Ye [GKNY]. Generalizations to algebraic curves in other polynomial families were obtained by Favre and Gauthier [FG].

**Theorem 1.1.** [GKNY] Let $C$ be an irreducible complex algebraic curve in $\mathbb{C}^2$. Then $C$ contains infinitely many PCF pairs $(c_1, c_2)$ if and only if $C$ is either

1. a vertical line $\{x = c_1\}$ for a PCF $f_{c_1}$; or
2. a horizontal line $\{y = c_2\}$ for a PCF $f_{c_2}$; or
3. the diagonal $\{x = y\}$.

Note that a real algebraic curve in $\mathbb{R}^2$ passing through a PCF parameter $c_0$ in the Mandelbrot set identifies $\mathbb{R}^2$ with $\mathbb{C}$ gives rise to a complex algebraic curve in $\mathbb{C}^2$ passing through the PCF pair $(c_0, \overline{c_0})$. So the above result also controls PCF points on real curves in $\mathbb{C}$. See Figure 1.1 and Section 5.

Theorem 1.1 was motivated by analogies between the PCF maps in the space of quadratic polynomials and the elliptic curves with complex multiplication (CM); see for example [Si, Ch. 6] [Jo, Conj. 3.11]. It was known that the only algebraic curves in $\mathbb{C}^2$ with infinitely many CM pairs are the modular curves (together with the infinite collection of vertical or horizontal lines through a CM point) [An], [Ed1].

Our first result is an extension of Theorem 1.1 to arbitrary dimensions, exactly analogously to the classification of special subvarieties in the CM case [Pi], [Ed2]:

**Theorem 1.2.** Let $n \geq 2$. Let $X$ be an irreducible complex algebraic subvariety in $\mathbb{C}^n$. There is a Zariski dense set of special points in $X$ if and only if $X$ is special.

By definition, a parameter $c \in \mathbb{C}$ is special if $f_c$ is PCF. For any positive integer $n$, a point $(c_1, \ldots, c_n) \in \mathbb{C}^n$ is special if $c_i$ is special for all $i = 1, \ldots, n$. We say that an irreducible curve $C \subset \mathbb{C}^2$ is special if it is one of the three types listed in Theorem 1.1.

The special subvarieties of $\mathbb{C}^n$ are the preimages of special curves from projections to $\mathbb{C}^2$, and their intersections. More precisely, an irreducible subvariety $Z$ of $\mathbb{C}^n$ is special if and only if there exist a partition $S_0 \cup \cdots \cup S_r$ of $\{1, \ldots, n\}$, where $r \geq 0$ and $S_k \neq \emptyset$ for each $k > 0$, and a collection of PCF parameters $c_i \in \mathbb{C}$ for $i \in S_0$ so that

$$Z = \left( \bigcap_{i \in S_0} \{x_i = c_i\} \right) \cap \left( \bigcap_{k=1}^r \bigcap_{j \in S_k} \{x_j = x_{i_k}\} \right)$$

where $(x_1, \ldots, x_n)$ are the coordinates of $\mathbb{C}^n$ and $i_k := \min S_k$ for each $k > 0$. Note that the dimension of $Z$ is equal to $r$.

Although Theorem 1.2 is worded the same as statements about modular curves, the proof methods are (necessarily) very different. As in the proof of Theorem 1.1, it is important that the PCF parameters are a set of algebraic numbers with bounded Weil height, which is not the case for singular moduli, and in fact of height 0 for
a dynamically-defined height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. This allows the use of certain arithmetic equidistribution theorems for points of small height; we rely on the recent equidistribution result of Yuan-Zhang [YZ] (though we could have used the older result of [Yu] as we explain in Remark 2.5). Focusing then on an archimedean place, and via the slicing of positive currents, we reduce the proof of Theorem 1.2 to Luo's theorem on the inhomogeneity of the Mandelbrot set [Lu].

As an application of Theorem 1.2, we obtain uniform versions of Theorem 1.1, in the spirit of Scanlon's automatic uniformity [Sc] (though we give a direct proof, not relying on [Sc, Theorem 2.4]).

**Theorem 1.3.** Fix $d \in \mathbb{N}$. There is a constant $M(d) < \infty$ such that

$$\#\{\text{special points in } C\} \leq M(d),$$

for all complex algebraic curves $C \subset \mathbb{C}^2$ of degree $d$ without special components.

It is natural to ask how many special points can lie on a non-special curve in $\mathbb{C}^2$. We obtain an explicit bound for the general curve of degree $d$:

**Theorem 1.4.** Fix $d \in \mathbb{N}$, and let $X_d$ denote the Chow variety of all plane curves with degree $\leq d$. There exists a Zariski-closed strict subvariety $V_d \subset X_d$ such that

$$\#\{\text{special points in } C\} \leq \frac{d(d+3)}{2},$$

for all curves $C \in X_d \setminus V_d$.

The upper bound in Theorem 1.4 is optimal:

**Theorem 1.5.** There is a Zariski-dense subset $S_d \subset X_d$ such that

$$\#\{\text{special points in } C\} = \frac{d(d+3)}{2},$$

for all curves $C \in S_d$.

Note that $d(d+3)/2$ is the dimension of the space $X_d$, and this is no accident. It is well known that there exists a curve of degree $d$ through any collection of $N_d = d(d+3)/2$ points in $\mathbb{C}^2$. Choosing those points to be special, we can build a Zariski-dense collection of curves in $X_d$ containing at least $N_d$ special points. The upper bound of Theorem 1.4 is obtained by showing there are no unexpected symmetries among general special-point configurations, as a consequence of Theorem 1.2 and the explicit description of the special subvarieties.

Our proof does not give a complete description of the exceptional variety $V_d$ in Theorem 1.4, though the methods can be used to classify its positive-dimensional components. For example, in the case of $d = 1$, we show:

**Theorem 1.6.** All but finitely many non-special lines in $\mathbb{C}^2$ contain at most 2 special points.
In other words, the subvariety $V_1$ of Theorem 1.4 can be taken to be the union of the 1-parameter families in $X_1$ of horizontal and vertical lines, together with a finite set of points in $X_1$. A more detailed result about lines in $\mathbb{C}^2$ is stated as Proposition 4.4.

Note that the finite set of non-special lines in $\mathbb{C}^2$ containing $\geq 3$ special points is not empty. For example, the line $\{y = -x\}$ passes through $(0,0), (i,-i)$, and $(-i,i)$; the line $\{y = ix\}$ passes through $(0,0), (-1,-i)$, and $(i,-1)$; and $\{y = -ix\}$ passes through $(0,0), (-1,i)$, and $(-i,-1)$.

**Question 1.7.** How many non-special lines in $\mathbb{C}^2$ pass through at least 3 distinct special points, and what is the optimal value of $M(1)$ in Theorem 1.3?

As mentioned after Theorem 1.1, a real algebraic curve in $\mathbb{R}^2 = \mathbb{C}$ passing through given parameter $c$ in the Mandelbrot set will give rise to a complex algebraic curve in $\mathbb{C}^2$ passing through point $(c, \bar{c})$. Moreover, the subset of such curves is Zariski dense in $X_d$ for each degree $d \geq 1$. Theorems 1.3 and 1.4 therefore apply to bound PCF parameters on real algebraic curves of a given degree in $\mathbb{R}^2 = \mathbb{C}$. For example, we have:

**Theorem 1.8.** There is a uniform bound on the number of PCF parameters on any real algebraic curve in $\mathbb{R}^2 = \mathbb{C}$ depending only on the degree of the curve (as long as the curve does not contain the real axis). Moreover, there are only finitely many real lines in $\mathbb{C}$ that contain more than two PCF parameters.

**Remark 1.9.** Finiteness results analogous to Theorem 1.6, upon replacing “lines” with “curves of degree $d$” and the number 2 with $d(d + 3)/2$, do not hold for $d > 1$. For example, for algebraic curves of degree $d = 2$, we know there is a conic through any 5 given points in $\mathbb{C}^2$, and 5 is the optimal bound on special points in general conics (by Theorems 1.4 and 1.5), but there is a Zariski-dense set of curves in the 3-dimensional space of conics

$$x^2 + y^2 + Axy + B(x+y) + C = 0$$

in $\mathbb{C}^2$ containing at least 6 special points. Indeed, 3 given special points in $\mathbb{C}^2$ will (generally) determine the coefficients $A, B, C$, and the $(x,y) \mapsto (y,x)$ symmetry of the curve will (generally) guarantee an additional 3 special points. For real conics in $\mathbb{R}^2 = \mathbb{C}$, one can do the same with symmetry under complex conjugation; see Figure 1.2.

**Outline.** In Section 2, we prove Theorem 1.2. Section 3 provides a brief review of the Chow variety $X_d$ and basic results on families of curves passing through points. Section 4 contains the proofs of Theorems 1.3, 1.4, 1.5, and 1.6. Finally, in Section 5, we look at real algebraic curves passing through PCF parameters in the Mandelbrot set and prove Theorem 1.8.
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2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. In fact we prove a stronger result, showing that our classification theorem remains true if we treat small points in addition to the special points. Our notion of size is given by a height function

\[ h_{\text{crit}}(c_1, \ldots, c_n) := \sum_{i=1}^{n} \hat{h}_{f_{c_i}}(0) \geq 0 \]

for \((c_1, \ldots, c_n) \in \overline{\mathbb{Q}}^n\). Here \(\hat{h}_{f_{c_i}}\) is the canonical height associated to the quadratic polynomial \(z^2 + c_i\), introduced by Call and Silverman [CS]. We say that a sequence \(\{x_k\} \subset \overline{\mathbb{Q}}^n\) is small if \(h_{\text{crit}}(x_k) \to 0\) as \(k \to \infty\). Notice that our special points of \(\mathbb{C}^n\) are precisely the zeros of \(h_{\text{crit}}\).

Let \(Y \subset \mathbb{C}^n\) be a variety. A sequence \(\{x_k\} \subset Y\) is called generic if no subsequence lies in a proper subvariety of \(Y\).

**Theorem 2.1.** Let \(n \geq 2\). Let \(X\) be an irreducible algebraic subvariety in \(\mathbb{C}^n\) defined over \(\overline{\mathbb{Q}}\). Then, \(X\) contains a generic sequence of small points if and only if \(X\) is special.
The idea of considering points that are small with respect to some height function originates in Bogomolov’s work [Bo]; in a dynamical context, see for example [GHT, Conjecture 2.3] or [Zh].

Remark 2.2. In Theorem 2.1 we have assumed that $X$ is defined over $\mathbb{Q}$, which is not the case in Theorem 1.2. However, our special points are defined over $\mathbb{Q}$ so that a subvariety that contains a Zariski dense set of special points is automatically defined over $\mathbb{Q}$. Therefore, Theorem 1.2 follows from Theorem 2.1. Note here that the structure of the special subvarieties ensures that they contain a Zariski dense set of special points.

Remark 2.3. Assuming $X$ is a curve in $\mathbb{C}^2$, the conclusion of Theorem 2.1 is not contained in [GKNY] but follows immediately from the proof of Theorem B in [FG].

2.1. Arithmetic equidistribution. The first key ingredient in our proof is the following equidistribution theorem. Let $M$ denote the Mandelbrot set in $\mathbb{C}$. Let $\mu_M$ denote the bifurcation measure on $M$; that is, the harmonic measure supported on $\partial M$ for the domain $\hat{\mathbb{C}} \setminus M$, relative to the point at $\infty$. Note that $\mu_M$ has continuous potentials; see for example [De, §6].

Theorem 2.4. Let $n \geq 2$ and $H \subset \mathbb{C}^n$ be an irreducible hypersurface defined over a number field $K$. Assume that the projection $p_j : H \to \mathbb{C}^{n-1}$ omitting the $j$-th coordinate is dominant. Then, for any generic sequence $\{x_k\} \subset H(\overline{K})$ of small points, their $\text{Gal}(\overline{K}/K)$-orbits equidistribute to the probability measure

\[ \mu_j := c (\pi_1|_H)^* (\mu_M) \wedge \cdots \wedge (\pi_{j-1}|_H)^* (\mu_M) \wedge (\pi_j|_H)^* (\mu_M) \wedge \cdots \wedge (\pi_{n-1}|_H)^* (\mu_M) \]

on $H(\mathbb{C})$, where $\pi_i : \mathbb{C}^n \to \mathbb{C}$ is the projection to the $i$-th coordinate, $(\pi_i|_H)^* \mu_M$ is the pullback as a $(1,1)$-current, and $c > 0$ is a positive constant. That is, for any continuous function $\varphi$ on $H$ with compact support in the smooth part of $H$, we have

\[ \frac{1}{\# \text{Gal}(\overline{K}/K) \cdot x_k} \sum_{y \in \text{Gal}(\overline{K}/K) \cdot x_k} \varphi(y) \to \int \varphi \, d\mu_j \]

as $k \to \infty$.

To prove Theorem 2.4 we rely on the recent theory of Yuan and Zhang on adelic line bundles developed in [YZ]. We let $f : \mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ be the algebraic family of unicritical quadratic polynomials

\[ f(t, z) = (t, z^2 + t). \]

Let $L$ be the line bundle on $\mathbb{A}^1 \times \mathbb{P}^1$, isomorphic to $\mathcal{O}(1)$ on fibers $\mathbb{P}^1$ and such that $f^* L = 2L$. We denote by $\overline{L}_f$ the $f$-invariant extension of $L$ as defined in [YZ, Theorem 6.1.1]. Let $i : \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ be defined by $i(t) = (t, 0)$ and define

\[ \overline{L}_{\text{crit}} := i^* \overline{L}_f. \]
This is an adelic line bundle on $A^1$ as in [YZ, §6.2.1]. Furthermore, by [YZ, Lemma 6.2.1], the height associated to $L_{\text{crit}}$ is given by

\begin{equation}
(2.2) \quad h_{L_{\text{crit}}}(t) = \hat{h}_{f_\ell}(0) = h_{\text{crit}}(t),
\end{equation}

for each $t \in A^1(\mathbb{Q})$, where $h_{\text{crit}}$ is the height defined in (2.1) with $n = 1$.

**Proof of Theorem 2.4.** Fix $j$ as in the statement of the theorem. As $\mu_{\mathcal{M}}$ has continuous potentials, we deduce that $\mu_j$ does not put mass on the singular locus $H_{\text{sing}}$ of $H$, so we may replace $H$ with $H \setminus H_{\text{sing}}$ and assume that $H$ is smooth. We define a metrized line bundle on the smooth, quasiprojective hypersurface $H$ by

\begin{equation}
(2.3) \quad \mathcal{M}_j = \otimes_{i \neq j} (\pi_i|_H)^*(L_{\text{crit}}).
\end{equation}

This defines an adelic line bundle on $H$, so that $\mathcal{M}_j \in \widehat{\text{Pic}}(H)_\mathbb{Q}$ in the notation of [YZ]. By [YZ, Theorem 6.1.1] we know that $L_f$ is nef in the sense of [YZ] and by the functoriality of nefness we also have that $\mathcal{M}_j$ is nef; see [YZ, page 8]. As in [YZ, Lemma 6.3.7], the curvature form associated to $L_{\text{crit}}$ at an archimedean place is precisely the bifurcation measure associated to $f$; that is, $c_1(L_{\text{crit}}) = \mu_{\mathcal{M}}$. Since $c_1(L_{\text{crit}})^{\wedge 2} \equiv 0$, we thus see that

\begin{equation}
(2.4) \quad c_1(\mathcal{M}_j)^{(n-1)} = (n-1)! (\pi_1|_H)^*(\mu_{\mathcal{M}}) \wedge \cdots \wedge (\pi_j|_H)^*(\mu_{\mathcal{M}}) \wedge \cdots \wedge (\pi_n|_H)^*(\mu_{\mathcal{M}}),
\end{equation}

where the $\wedge$ means the $j$th term is omitted and the pullbacks are defined in the sense of currents. Our assumption that the projection $p_j$ is dominant ensures that this measure is non-trivial. By [YZ, Lemma 5.4.4], we infer that $\mathcal{M}_j$ is non-degenerate as defined in [YZ, §6.2.2]. In other words, the adelic line bundle $\mathcal{M}_j$ satisfies all the assumptions of [YZ, Theorem 5.4.3]. Thus, if $\{y_k\} \subset H(\mathbb{Q})$ is a generic sequence with $h_{\mathcal{M}_j}(y_k) \to h_{\mathcal{M}_j}(H)$, then its Galois conjugates equidistribute with respect to the probability measure associated to $c_1(\mathcal{M}_j)^{(n-1)}$.

Now let $\{x_k\} \subset H(\mathbb{Q})$ be a generic sequence of small points, as in the statement of the theorem. Note that

\begin{equation}
(2.5) \quad h_{\mathcal{M}_j}(x) = \sum_{i \neq j} h_{\text{crit}}(\pi_i(x)),
\end{equation}

for $x \in H(\mathbb{Q})$ so that by (2.2) we have

\begin{equation}
\lim_{k \to \infty} h_{\mathcal{M}_j}(x_k) = 0.
\end{equation}

Therefore by the number field case of the fundamental inequality [YZ, Theorem 5.3.3], we have that

\begin{equation}
(2.6) \quad h_{\mathcal{M}_j}(H) \leq 0.
\end{equation}
Note that here we have used the fact that $\overline{M}_j$ is nef and non-degenerate. By the nefness of $\overline{M}_j$ we also have that $h_{\overline{M}_j}(H) \geq 0$ by [YZ, Proposition 4.1.1]. Thus, $h_{\overline{M}_j}(H) = 0$.

Therefore, the result follows by the equidistribution theorem [YZ, Theorem 5.4.3]. □

**Remark 2.5.** To prove Theorem 2.4 we used the recent equidistribution theory in quasiprojective varieties developed in [YZ]. However, it is known that $L_{\text{bif}}$ extends to define an adelic metrized line bundle on the projective line $\mathbb{P}^1$; see, for example, [BD]. Arguments similar to the ones in [MSW] would allow us to use the equidistribution result established by Yuan [Yu] instead.

2.2. **Inhomogeneity of $\mathcal{M}$.** To deduce Theorem 2.1, we will combine Theorem 2.4 with the following result of Luo that the Mandelbrot set has no local symmetries.

**Theorem 2.6.** [Lu] Let $U$ be an open set in $\mathbb{C}$ with $U \cap \partial \mathcal{M} \neq \emptyset$. Suppose $\varphi : U \to V$ is a conformal isomorphism such that $\varphi(U \cap \partial \mathcal{M}) = V \cap \partial \mathcal{M}$. Then $\varphi$ is the identity.

We begin by proving Theorem 2.1 for a certain class of hypersurfaces.

**Proposition 2.7.** Let $n \geq 2$. Assume that $H \subset \mathbb{C}^n$ is an irreducible hypersurface, defined over $\overline{\mathbb{Q}}$, which projects dominantly on each collection of $n-1$ coordinates and which contains a generic sequence of small points. Then $n = 2$ and $H \subset \mathbb{C}^2$ is the diagonal line.

**Proof.** Let $H$ be a hypersurface as in the statement. In particular $H$ contains a generic small sequence. Since $H$ projects dominantly on each collection of $n-1$ coordinates, we may apply Theorem 2.4 to deduce that the Galois orbits of our sequence equidistribute with respect to $\mu_j$ for all $j$ (for the $\mu_j$ in the statement of Theorem 2.4). In particular

$$T \wedge (\pi_{n-1}|_H)^*(\mu_\mathcal{M}) = \alpha \cdot T \wedge (\pi_n|_H)^*(\mu_\mathcal{M})$$

for some constant $\alpha > 0$, where $T = (\pi_1|_H)^*(\mu_\mathcal{M}) \wedge \cdots \wedge (\pi_{n-2}|_H)^*(\mu_\mathcal{M})$ is an $(n-2, n-2)$-current on $H$.

For $n = 2$, equation (2.6) means that $(\pi_1|_H)^*(\mu_\mathcal{M}) = \alpha \cdot (\pi_2|_H)^*(\mu_\mathcal{M})$ on the curve $H$ in $\mathbb{C}^2$. But the projections are locally invertible away from finitely many points, so the measure equality induces a local isomorphism between a neighborhood of a point in $\partial \mathcal{M} \subset \mathbb{C}$ and its image. Theorem 2.6 then implies that this local isomorphism is the identity. That is, the curve $H$ must be the diagonal line in $\mathbb{C}^2$ as claimed.

Assume now that $n \geq 3$. Let $\pi : H \to \mathbb{C}^{n-2}$ be the projection to the first $n-2$ coordinates. By our assumption, $\pi$ is dominant and the fiber-dimension theorem yields that the fibers $H_z := H \cap \{x_1 = z_1, \ldots, x_{n-2} = z_{n-2}\} \subset H$ are curves for $z = (z_1, \ldots, z_n)$ in a Zariski open and dense subset of $\mathbb{C}^{n-2}$. Note that each $(1, 1)$-current $\pi_j^* \mu_\mathcal{M}$ has continuous potentials on $\mathbb{C}^n$, so the measure $\pi_* T$ does not charge
pluripolar sets. Thus the fiber $H_z$ is a curve for $(\pi_*T)$-almost every $z$, and, by the characterization of slicing of currents as in [BB, Proposition 4.3], equation (2.6) implies that
\[
\int_{\mathbb{C}^{n-2}} \left( \int_{H_z} \varphi \, d\pi_{n-1}|^*_H(\mu_M)(x) \right) \, d\pi_*T(z) = \alpha \int_{\mathbb{C}^{n-2}} \left( \int_{H_z} \varphi \, d\pi_{n-1}|^*_H(\mu_M)(x) \right) \, d\pi_*T(z),
\]
for every continuous and compactly supported function $\varphi$ on $H$. Thus, for almost all points $z := (z_1, \ldots, z_{n-2})$ with respect to $\pi_*T$, we have equality of measures:
\[
\pi_{n-1}|^*_H(\mu_M) = \alpha \cdot \pi_{n-1}|^*_H(\mu_M).
\]
In detail, suppose there exists a point $z_0$ in the support of $\pi_*T$ where $H_{z_0}$ is a curve and such that
\[
\pi_{n-1}|^*_H(\mu_M) \neq \alpha \cdot \pi_{n-1}|^*_H(\mu_M).
\]
Then we can find a continuous function $\psi_{z_0}$ on $H_{z_0}$ so that
\[
\int_{H_{z_0}} \psi_{z_0} \pi_{n-1}|^*_H(\mu_M) \neq \alpha \int_{H_{z_0}} \psi_{z_0} \pi_{n-1}|^*_H(\mu_M).
\]
Note that the measures $\pi_i|^*_H(\mu_M)$ vary continuously as functions on $z$ on a neighborhood of $z_0$, by the continuity of the potentials. We thus infer that
\[
\int_{H_{z_0}} \psi_{z_0} \pi_{n-1}|^*_H(\mu_M) \neq \alpha \int_{H_{z_0}} \psi_{z_0} \pi_{n-1}|^*_H(\mu_M),
\]
for all $z$ in a neighborhood $U$ of $z_0$ with $\pi_*T(U) > 0$. We can therefore find $\varphi = h \cdot \psi_{z_0}$ where $h$ is a continuous function supported on $U$ and for which the equality (2.7) fails.

Again by Theorem 2.6, equation (2.8) yields that $H_z$ is special for $\pi_*T$-almost all $z$. Since $T$ does not charge pluripolar sets and since $H$ projects dominantly on each $n-1$ coordinates, we infer that $H \subset \pi^{-1}_{(n-1,n)}(\Delta)$, where $\pi_{(i,j)} : \mathbb{C}^n \to \mathbb{C}^2$ is the projection to the $i$ and $j$-th coordinates. Repeating the argument using the equalities of all measures $\mu_j$, we get
\[
H \subset \bigcap_{i \neq j \in \{1, \ldots, n\}} \pi^{-1}_{(i,j)}(\Delta).
\]
But since $H$ has dimension $n-1$ and $n \geq 3$ this is impossible. This completes our proof.

2.3. Proof of Theorem 2.1. We can now complete the proof of Theorem 2.1 (and so also of Theorem 1.2) by reducing it to Proposition 2.7. This argument is inspired by [GNY].

First we will show that if Theorem 2.1 holds for hypersurfaces $X$, then it holds in general. So assume that the theorem is true when $X$ is a hypersurface and let $X$ be an irreducible subvariety of $\mathbb{C}^n$ with dimension $d < n-1$ which contains a generic sequence of small points. Permuting the coordinates if necessary, we may assume that
Now let \( \pi_{(j)} : \mathbb{C}^n \to \mathbb{C}^{d+1} \) denote the projection to the first \( d \) and the \( j \)th coordinates. Let \( X_j \) denote the Zariski closure of \( \pi_{(j)}^{-1}(\pi_{(j)}(X)) \) in \( \mathbb{C}^n \). Each \( X_j \) is a hypersurface in \( \mathbb{C}^n \) and contains a generic sequence of small points. Therefore by our assumption \( X_j \) must be special. If \( X \subset X_j \) is special, then our claim follows. Otherwise \( X_j = \{ (x_1, \ldots, x_n) \in \mathbb{C}^n : x_j = c_j \} \) for a special point \( c_j \) or \( X_j = \{ (x_1, \ldots, x_n) \in \mathbb{C}^n : x_j = x_k \} \) for some \( k \in \{1, \ldots, d\} \). From the precise form of each \( X_j \), it is easy to see that \( \cap_{j=d+1}^n X_j \) has dimension \( d = \dim X \). But \( X \subset \cap_{j=d+1}^n X_j \), so we must have \( X = \cap_{j=d+1}^n X_j \) and our claim follows.

Therefore, it suffices to prove Theorem 2.1 for hypersurfaces \( X \). Arguing by induction on \( n \), we may further assume that \( X \) projects dominantly on each \( n-1 \) coordinates. Indeed, if \( n = 2 \) and the curve \( X \) is vertical or horizontal then since it contains a generic sequence of small points, it must be special. Assume now that \( n \geq 3 \), and that \( X \) does not project dominantly on, say, the last \( n-1 \) coordinates. Then it has the form \( X = \mathbb{C} \times X_0 \) for a hypersurface \( X_0 \subset \mathbb{C}^{n-1} \) (see e.g. [MSW, Lemma 3.1]). By induction, Theorem 2.1 follows by Proposition 2.7. As explained in Remark 2.2, Theorem 1.2 also holds.

### 3. Maximal variation and the lower bound

In this section, we provide some basic background on the Chow variety \( X_d \) of curves of degree \( \leq d \) in \( \mathbb{C}^2 \), and we prove lower bounds on the number of special points in families of curves.

#### 3.1. Chow and maximal variation.

Fix integer \( d \geq 1 \). We will work with the Chow variety \( X_d \) of algebraic curves in the plane \( \mathbb{C}^2 \), defined over the field \( \mathbb{C} \) of complex numbers, of degree \( \leq d \). This is simply the complement of a single point in a projective space \( \mathbb{P}^{N_d}_\mathbb{C} \), where

\[
N_d = \binom{d+2}{2} - 1 = \frac{d(d+3)}{2}.
\]

Indeed, each curve is the vanishing locus of a nonzero homogeneous polynomial \( F(x, y, z) \) of degree \( d \), uniquely determined up to scale, and evaluated at points of the form \( (x, y, 1) \) for \( (x, y) \in \mathbb{C}^2 \). We exclude the polynomial \( F(x, y, z) = z^d \).

Let \( \mathcal{C} \to V \) be a family of plane algebraic curves, parameterized by an algebraic variety \( V \) defined over \( \mathbb{C} \). There is an induced map from \( V \) to \( X_d \) for some degree \( d \). We say that the family \( \mathcal{C} \to V \) is **maximally varying** if the induced map \( V \to X_d \) is finite. For each integer \( m \geq 1 \), we let

\[
\mathcal{C}^m_V := \mathcal{C} \times_V \cdots \times_V \mathcal{C}
\]

de note the \( m \)-th fiber power of \( \mathcal{C} \) over \( V \). There is a natural map

\[
\rho_m : \mathcal{C}^m_V \to \mathbb{C}^{2m}
\]
defined by sending a tuple of $m$ points $x_1, \ldots, x_m$ on a curve $C \in V$ to the $m$-tuple $(x_1, \ldots, x_m)$ in $(\mathbb{C}^2)^m$.

**Proposition 3.1.** Suppose that $V$ is an irreducible quasiprojective complex algebraic variety of dimension $\ell \geq 1$. If $\mathcal{C} \to V$ is a maximally varying family of curves in $\mathbb{C}^2$ of degree $\leq d$, then the natural map $\rho_m$ is dominant for all $m \leq \ell$ and generically finite for $m = \ell$.

**Proof.** The result is clear for $m = 1$, because the image of $\mathcal{C}$ in $\mathbb{C}^2$ cannot be contained in a single algebraic curve if the image of $V$ in $X_d$ is not a point. For $m > 1$, it suffices to show that the image of $\rho_m$ contains the union of subvarieties of the form $\{(z_1, \ldots, z_{m-1})\} \times U(z_1, \ldots, z_{m-1})$, where $U(z_1, \ldots, z_{m-1})$ is Zariski open in $\mathbb{C}^2$, over a Zariski dense and open subset of points $(z_1, \ldots, z_{m-1}) \in (\mathbb{C}^2)^{m-1}$. Indeed, the dominance follows because the maps are algebraic, and the generic finiteness for $m = \ell$ follows because $\dim C_V^m = \ell + m$.

We proceed by induction. We have already seen that the result holds for $m = 1$ and any $\ell \geq 1$. Now assume $\ell > 1$, and fix $1 < m \leq \ell$. Assume the result holds for $\rho_{m-1}$. Then, as the smooth part $V^{sm}$ of $V$ is Zariski open and dense, the image of $\rho_{m-1}$ restricted to $C_V^{m-1}$ over $V^{sm}$ contains a Zariski open set $U \subset \mathbb{C}^{2(m-1)}$. Choose any point $(z_1, \ldots, z_{m-1})$ in $U$. Suppose that $\lambda_0 \in V^{sm}$ is a parameter for which $C_{\lambda_0}$ contains the points $z_1, \ldots, z_{m-1}$. There is a subvariety $V_1$ of $V$ containing $\lambda_0$ and with codimension $\leq m - 1$ consisting of curves $C_\lambda$ that persistently contain the points $z_1, \ldots, z_{m-1}$. In particular, the dimension of $V_1$ is at least 1. Maximal variation implies that the image of $\rho_1$ on $\mathcal{C}$ over $V_1$ is dominant to $\mathbb{C}^2$. It follows that $\rho_m$ on $C_V^m$ is dominant to $\{(z_1, \ldots, z_{m-1})\} \times \mathbb{C}^2$. Letting the point $(z_1, \ldots, z_{m-1})$ vary over the image of $\rho_m$, we use the induction hypothesis to see that $\rho_m$ is dominant from $C_V^m$ to $\mathbb{C}^{2m}$. \qed

### 3.2. A lower bound on the number of special points.

For a family of plane algebraic curves $\mathcal{C} \to V$ parameterized by $V$, recall the definition of $C_V^m$ in (3.1).

**Proposition 3.2.** Suppose that $\mathcal{C} \to V$ is a maximally varying family of irreducible, complex algebraic curves in $\mathbb{C}^2$, over an irreducible quasiprojective complex algebraic variety $V$ of dimension $\ell \geq 1$. Then the preimage $\rho_\ell^{-1}(S)$ of the set of special points $S \subset \mathbb{C}^{2\ell}$ is Zariski dense in $C_V^\ell$. In particular, there is a Zariski dense set of curves $\lambda \in V$ for which the fiber $C_\lambda$ contains at least $\ell$ distinct special points of $\mathbb{C}^2$.

**Proof.** By maximal variation, we know that the fiber product $C_V^\ell$ maps generically finitely and dominantly by $\rho_\ell$ to $\mathbb{C}^{2\ell}$. The special points are Zariski dense in the image. This implies that the set of points $P = (\lambda, x_1, \ldots, x_\ell) \in \rho_\ell^{-1}(S) \subset C_V^\ell$, where $\lambda \in V$ and $\{x_1, \ldots, x_\ell\}$ is a collection of special points on the fiber $C_\lambda$ of $\mathcal{C}$ over $\lambda$, is Zariski dense in $C_V^\ell$. In particular, the $x_i$’s must be generally all distinct. \qed
In this section we prove Theorems 1.3, 1.4, 1.5, and 1.6. For each integer $d \geq 1$, let $X_d$ denote the Chow variety of all algebraic curves in $\mathbb{C}^2$ of degree $\leq d$ defined over $\mathbb{C}$.

4.1. The uniform bound of Theorem 1.3. Let $\mathcal{C} \to V$ denote a family of algebraic curves in $\mathbb{C}^2$, parameterized by an irreducible, quasiprojective variety $V$ over $\mathbb{C}$ of dimension $\ell \geq 1$, for which the general curve in the family is irreducible. As introduced in §3.1, there is a natural map

$$\rho_1 : \mathcal{C} \to \mathbb{C}^2,$$

sending each curve to its image in $\mathbb{C}^2$. Recall the definitions of $\mathcal{C}_V^m$ and

$$\rho_m : \mathcal{C}_V^m \to \mathbb{C}^{2m}$$

given there, for each integer $m \geq 1$. Recall also that the family is maximally varying if the induced map $V \to X_d$ is finite. From Proposition 3.1, we know that maximal variation implies that the maps $\rho_m$ are dominant for all $m \leq \dim V$.

**Proposition 4.1.** Suppose that $\mathcal{C} \to V$ is a maximally varying family of curves in $\mathbb{C}^2$ with $\ell = \dim V > 0$. Assume the general curve in the family is irreducible. Then the Zariski-closure of the image in $\mathbb{C}^{2(\ell+1)}$ of the fiber power $\mathcal{C}_V^{\ell+1}$ by $\rho_{\ell+1}$ is not special, unless $\ell = 1$ and $\mathcal{C}$ is a family of horizontal or vertical lines in $\mathbb{C}^2$.

As a consequence we have:

**Theorem 4.2.** For any family $\mathcal{C} \to V$ of curves in $\mathbb{C}^2$ – irreducible or not, maximally varying or not – there is a uniform upper bound $M = M(\mathcal{C})$ on the number of special points on $C_\lambda$, for all $\lambda \in V$ over which the fiber $C_\lambda$ of $\mathcal{C}$ has no special irreducible components.

**Proof of Theorem 4.2.** Assume Proposition 4.1. If $V$ or the generic curve is reducible, we work with irreducible components. If the family fails to be maximally varying, it is convenient to factor through the image of $V$ in the Chow variety $X_d$ for some degree $d$. So we now assume that $V$ is an irreducible subvariety in $X_d$ of dimension $\geq 1$ and the associated curve family $\mathcal{C} \to V$ consists of generally irreducible curves and not exclusively of horizontal or vertical curves.

Consider the fiber powers $\mathcal{C}_V^m \to V$ for each $m \geq 1$. Suppose there is a generic sequence of points $C_n \in V$ for which the number of special points of the curves $C_n \subset \mathbb{C}^2$ is larger than $n$, for each $n \in \mathbb{N}$. This implies that the special points of $\mathbb{C}^{2m}$ are Zariski dense in the images $\rho_m(\mathcal{C}_V^m)$ for every $m \geq 1$. Indeed, this is clear for $m = 1$ because $\rho_1$ maps $\mathcal{C}$ dominantly to $\mathbb{C}^2$. For each positive integer $m \geq 2$, the set of special points in $\rho_m(\mathcal{C}_V^m)$ includes the $m$-tuples formed from the $n$ distinct special points on $C_n$; note that this set of $m$-tuples in $\mathbb{C}^{2m}$ is symmetric under permutation.
of the $m$ copies of $\mathbb{C}^2$. Let $Z_m$ be the Zariski closure of these special points within $\rho_m(C^m_V)$; note that $Z_m$ will also be symmetric under permutation of the $m$ copies of $\mathbb{C}^2$. Because $\{C_n\}$ is a generic sequence in $V$, note also that $\rho_m^{-1}(Z_m)$ must project dominantly to $V$. If $Z_m$ is not equal to all of $\rho_m(C^m_V)$, then $\rho_m^{-1}(Z_m)$ is contained in a subvariety $H \subset C^m_V$ which is a family of hypersurfaces over $V$ that are symmetric with respect to permutation of the components in each fiber $C \times \cdots \times C$. Now consider the projections from $C^m_V \to C^m_V-1$ forgetting one factor, restricted to the hypersurface $H$. By the symmetry of $H$, each of these projections is generically finite and of the same degree, say $r(m)$. So over a Zariski open subset of $V$, this bounds the number of points on a given curve $C$; in particular this contradicts the assumption on the sequence of curves $C_n$. So the special points of $\mathbb{C}^{2m}$ must be Zariski dense in $\rho_m(C^m_V)$ for every $m \geq 1$.

From Theorem 1.2, the density of special points in $\rho_m(C^m_V)$ implies that (the Zariski closure) of $\rho_m(C^m_V)$ is special. But taking $m = \dim V + 1$, this contradicts Proposition 4.1.

So there is a uniform bound on the number of special points in the curve $C \subset \mathbb{C}^2$ for all curves $C$ in a Zariski open subset $U$ of $V$. We then repeat the argument on each of the finite set of irreducible components of $V \setminus U$. We continue until we are left with families of vertical or horizontal lines. □

Proof of Proposition 4.1. From Proposition 3.1, we know that the map $\rho_m : C^m_V \to \mathbb{C}^{2m}$ is dominant for all $m \leq \ell$ and generically finite for $m = \ell$. Note that $\dim C^{\ell+1}_V = 2\ell + 1 < 2\ell + 2$, so the map

$$\rho_{\ell+1} : C^{\ell+1}_V \to \mathbb{C}^{2(\ell+1)}$$

cannot be dominant. Consider the projections $\pi_{ij}$ from $C^{\ell+1}_V$ to $\mathbb{C}^2$ defined by composing $\rho_{\ell+1}$ with

$$(x_1, \ldots, x_{2\ell+2}) \mapsto (x_i, x_j)$$

for each pair $1 \leq i < j \leq 2\ell + 2$.

Assume $\ell > 1$. The projections $\pi_{ij}$ are dominant for all pairs $i < j$, because they factor through the dominant maps $C^{\ell+1}_V \to C^\ell_V \to \mathbb{C}^{2\ell}$, where the first arrow forgets the $k$-th factor of $C$ over $V$ for some choice of indices $\{2k - 1, 2k\}$ not containing $i$ or $j$, and the second arrow is $\rho_\ell$. In view of the structure of special subvarieties from Theorem 1.2, we see immediately that $\rho_{\ell+1}(C^{\ell+1}_V)$ cannot be special in $\mathbb{C}^{2\ell+2}$.

Now suppose that $\ell = 1$. If $C$ is not a family of vertical or horizontal curves, then a general curve $C$ in the family projects dominantly to both coordinates in $\mathbb{C}^2$. It follows that $\rho_2(C^2_V)$ cannot be contained in the hyperplane $\{x_i = c_i\}$ for a special parameter $c_i$ and any $i \in \{1, 2, 3, 4\}$. Because the curves over $V$ are not all equal to the diagonal curve in $\mathbb{C}^2$, the space $\rho_2(C^2_V)$ also cannot be contained in the special hypersurfaces defined by $\{x_k = x_{k+1}\}$ for $k \in \{1, 3\}$. It remains to check that $\rho_2(C^2_V)$
does not lie in the hypersurface \( \{ x_k = x_{k+2} \} \) for any \( k \in \{1, 2\} \) nor in \( \{ x_2 = x_3 \} \). But for the general curve \( C \) in the family, we know that the product \( C \times C \subset \mathbb{C}^4 \) maps dominantly to the space of pairs \( (x_1, x_3) \) or \( (x_2, x_4) \) or \( (x_2, x_3) \).

Finally suppose that \( C \) is a family of vertical or horizontal lines. For concreteness, we can take \( V = \mathbb{C}^2 \) and \( \lambda \in V \) corresponding to the vertical line \( \{ x = \lambda \} \) for \( \lambda \in \mathbb{V} \). Then the image of \( C^2 V \) in \( \mathbb{C}^4 \) is the set of all 4-tuples \( (\lambda, x_2, \lambda, x_4) \) for any \( (\lambda, x_2, x_4) \in \mathbb{C}^3 \). In other words, the image of \( C^2 V \) is the special hypersurface defined by \( \{ x_1 = x_3 \} \). Similarly for families of horizontal lines. \( \square \)

**Proof of Theorem 1.3.** The theorem is an immediate consequence of Theorem 4.2, taking \( V = X_d \). \( \square \)

### 4.2. Optimal general bound over Chow; proof of Theorems 1.4 and 1.5.

Let \( V = X_d \) be the Chow variety of all affine curves of degree \( \leq d \) in \( \mathbb{C}^2 \) and \( C \to V \) the universal family of such curves. Recall from §3.1 that

\[
N_d := \dim V = \frac{d(d + 3)}{2}.
\]

Consider the fiber power \( C^{N_d+1} \to V \) and its image under the natural map

\[
\rho := \rho_{N_d+1} : C^{N_d+1} \to \mathbb{C}^{2(N_d+1)}.
\]

Suppose that \( S \) is a special subvariety of \( \mathbb{C}^{2(N_d+1)} \) that is contained in the Zariski closure of the image \( \rho(C^{N_d+1}) \) for which \( \rho^{-1}(S) \) projects dominantly to \( V \). We aim to show that \( S \) must lie in the union of special diagonals

\[
\Delta_{i,j} := \{ (x_1, \ldots, x_{2N_d+2}) \in \mathbb{C}^{2N_d+2} : (x_i, x_{i+1}) = (x_j, x_{j+1}) \}
\]

for odd integers \( i \) and \( j \) satisfying \( 1 \leq i < j \leq 2N_d + 1 \).

This classification will imply the two theorems. Indeed, if there were a generic sequence of points \( C_n \in V \) for which the curve \( C_n \subset \mathbb{C}^2 \) contains at least \( N_d + 1 \) distinct special points of \( \mathbb{C}^2 \), then the \( (N_d + 1) \)-tuples of such points will be special in \( \mathbb{C}^{2(N_d+1)} \) and will lie outside of the special diagonals \( \Delta_{i,j} \). From Theorem 1.2, each irreducible component \( Z \) of the Zariski closure of these special points in \( \mathbb{C}^{2(N_d+1)} \) is itself a special subvariety, and by construction is contained in the closure of \( \rho(C^{N_d+1}) \).

As \( \{ C_n \} \) is a generic sequence of points in \( V \), the preimage \( \rho^{-1}(Z) \) of each component will project dominantly to \( V \). In other words, this \( Z \) is a special subvariety of the type described, but not contained in the special diagonals \( \Delta_{i,j} \), leading to a contradiction. This will prove Theorem 1.4. The equality of Theorem 1.5 is then a consequence of the lower bound in Proposition 3.2.

For the proof, suppose that \( S \) is a special subvariety of \( \mathbb{C}^{2(N_d+1)} \) that is contained in the Zariski closure of the image \( \rho(C^{N_d+1}) \) for which \( \rho^{-1}(S) \) projects dominantly to \( V \). Note that \( \dim \rho^{-1}(S) \geq \dim S \), so the dominance of the projection to \( V \) implies
that a general fiber of this projection has dimension $\geq \dim S - N_d$. In other words, the intersection of $S$ with $C \times \cdots \times C$ in $\mathbb{C}^{2(N_d+1)}$ has dimension at least
$$\dim S - N_d = N_d + 2 - \text{codim } S,$$
for a general curve $C$ in $V$.

We begin by working case by case through some examples of special subvarieties, as classified in Theorem 1.2. As the image $\rho(C_{V, \Delta}^{N_d+1})$ is not itself special in $\mathbb{C}^{2(N_d+1)}$ by Proposition 4.1, the codimension of $S$ in $\mathbb{C}^{2(N_d+1)}$ will be at least 2.

- $S = \{x_1 = c_1 \text{ and } x_2 = c_2\}$: a general curve $C \in V$ does not pass through the point $(c_1, c_2) \in \mathbb{C}^2$, so $\rho^{-1}(S)$ cannot project dominantly to $V$.
- $S = \{x_1 = c_1 \text{ and } x_3 = c_3\}$: A general curve in $\mathbb{C}^2$ of degree $d$ projects dominantly to its 1st coordinate, so $\rho^{-1}(S)$ does project dominantly to $V$ in this case. However, the intersection of $S$ with a general fiber $C \times \cdots \times C \subset \mathbb{C}^{2N_d+2}$ will have dimension only $N_d - 1$, so this $S$ could not have been contained in the closure of $\rho(C_{V, \Delta}^{N_d+1})$.
- $S = \{x_1 = c_1 \text{ and } x_2 = x_3\}$: Again this $S$ has codimension 2, while the intersection with $C \times \cdots \times C$ for a general curve $C \in V$ has dimension only $N_d - 1$.
- $\Delta_{1,3} = \{x_1 = x_3 \text{ and } x_2 = x_4\}$: These relations again impose conditions on two of the $N_d + 1$ components of $C \times \cdots \times C$. But note that any collection of $N_d + 1$ points $(x_1, x_2), \ldots, (x_{2N_d+1}, x_{2N_d+2})$ in $\mathbb{C}^2$ satisfying $(x_1, x_2) = (x_3, x_4)$ will lie on some curve of degree $d$, because at most $N_d$ of the points are distinct. So all of $\Delta_{1,3}$ is contained in $\rho(C_{V, \Delta}^{N_d+1})$. The intersection of $\Delta_{1,3}$ with a general $C \times \cdots \times C$ has dimension $N_d$, which is the expected dimension.
- $S = \{x_1 = c_1 \text{ and } x_2 = x_3 = x_4\}$: Again we impose relations on only two of the $N_d + 1$ components of $C \times \cdots \times C$, but there are too many relations; a general curve will not intersect both $(c_1, y)$ and $(y, y)$ for any choice of $y \in \mathbb{C}$. Consequently, the preimage $\rho^{-1}(S)$ in $\mathbb{C}_{V, \Delta}^{N_d+1}$ will not project dominantly to $V$.
- $S = \{x_1 = c_1 \text{ and } x_2 = x_3 \text{ and } x_4 = c_4\}$: These three relations are imposed upon only two of the $N_d + 1$ components of $C \times \cdots \times C$. As in the previous example, a general curve $C$ will not intersect both $(c_1, y)$ and $(y, c_4)$ for any choice of $y \in \mathbb{C}$. The preimage $\rho^{-1}(S)$ in $\mathbb{C}_{V, \Delta}^{N_d+1}$ will not project dominantly to $V$.

In general, we see that if we define $S$ by imposing up to $N_d + 1$ special relations on the coordinates of the $N_d + 1$ components of a general product $C \times \cdots \times C$ in $\mathbb{C}^{2(N_d+1)}$, as long as no one point is constant (as in the first example) nor that there are three relations on coordinates of two components (as in the last two examples), nor that two of the coordinates are required to agree (so as to be a subvariety of a special diagonal $\Delta_{i,j}$), then the preimage $\rho^{-1}(S)$ in $\mathbb{C}_{V, \Delta}^{N_d+1}$ would project dominantly.
to \(V\), but the general intersection of \(S\) with \(C \times \cdots \times C\) will not have sufficiently large dimension. That is, this \(S\) will not lie in \(\rho(C^{N_d+1}_V)\) in \(\mathbb{C}^{2(N_d+1)}\). If we specify that one of the components is a fixed special point, or if there are at least three relations imposed upon a pair of components of \(C \times \cdots \times C\) (as will be the case if \(\text{codim } S > N_d + 1\)), then the preimage \(\rho^{-1}(S)\) will not project dominantly to \(V\). This completes the proofs of Theorems 1.4 and 1.5.

4.3. Lines and the proof of Theorem 1.6. The classification of special subvarieties (Theorem 1.2) and the proof strategy in §4.2 for Theorem 1.4 suggest that the general curve in a “generically chosen” maximally-varying family \(C \to V\) of curves in \(\mathbb{C}^2\) with dimension \(\ell = \dim V\), in any degree, will intersect at most \(\ell\) distinct special points. Here we show this is indeed the case in degree \(d = 1\). Before doing so, we give an example where this expectation fails.

Example 4.3. [An exceptional family of lines] Consider the pencil of lines in \(\mathbb{C}^2\) passing through a given special point \(P\), parameterized by \(V \simeq \mathbb{P}^1\); for example take \(P = (-1, -2)\). Because we can connect any special point in \(\mathbb{C}^2\) to \(P\) with a line, there are infinitely many lines in this family containing at least 2 special points, though \(\dim V = 1\).

Less obvious is the fact that the general bound on special points for the family of lines in Example 4.3 is also 2. That is, there can be at most finitely many lines in \(\mathbb{C}^2\) through the special point \(P\) containing more than 2 distinct special points. On the other hand, if we consider the pencil of lines in \(\mathbb{C}^2\) passing through a non-special point such as \(P = (1, 1)\), then all but finitely many lines in the family will have at most 1 special point. These facts are contained in the following proposition:

Proposition 4.4. Let \(V \subset X_1\) be an irreducible curve in the Chow variety of lines in \(\mathbb{C}^2\), not consisting exclusively of vertical or horizontal lines. Then, outside of finitely many parameters \(\lambda \in V\), the lines of the family will intersect at most 1 special point in \(\mathbb{C}^2\), unless \(V\) is

1. the family of all lines through a special point \(P = (p_1, p_2) \in \mathbb{C}^2\);
2. the family of lines defined by \(L_\lambda = \{(x, y) \in \mathbb{C}^2 : x + y = \lambda\}\), for \(\lambda \in \mathbb{C}\);
3. the family of lines \(L_\lambda\) containing \((c_1, \lambda)\) and \((\lambda, c_2)\) for special parameters \(c_1 \neq c_2\), for \(\lambda \in \mathbb{C}\).

In each of these 3 cases, outside of a finite set of parameters \(\lambda\), there will be at most 2 special points on the line \(L_\lambda\); moreover, there are infinitely many parameters \(\lambda \in V\) for which the line \(L_\lambda\) contains exactly 2 special points of \(\mathbb{C}^2\).

Proof. Let \(V \subset X_1\) be an irreducible algebraic curve defined over \(\mathbb{C}\), not consisting of vertical or horizontal lines. Let \(C \to V\) denote this family of lines over \(V\). Consider the special points in \(\rho_2(C^2_V)\) in \(\mathbb{C}^4\), for the natural map defined in §3.1. Note that
their Zariski closure must contain the diagonal surface
\[ \Delta := \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : (x_1, x_2) = (x_3, x_4)\}, \]
because \(\rho_1\) is dominant to \(\mathbb{C}^2\) (in which special points are Zariski dense). The general bound on the number of special points on lines \(L \in V\) will be 1 unless either

(S) there is a special surface in \(\rho_2(C^2_V)\) other than the diagonal \(\Delta\), intersecting \(L \times L\) in a curve for general \(L \in V\) and whose preimage by \(\rho_2\) projects dominantly to the base \(V\); or

(C) there is a special curve in \(\rho_2(C^2_V)\), not contained in \(\Delta\), intersecting \(L \times L\) in a (nonempty) finite set for general \(L \in V\) and whose preimage by \(\rho_2\) projects dominantly to the base \(V\).

Indeed, if an infinite collection of lines \(L\) had at least 2 special points, then the Zariski closure of those pairs of points would lie outside of \(\Delta\) and form a special subvariety of the image \(\rho_2(C^2_V)\) as described; this special subvariety will have dimension 1 or 2 because the hypersurface \(\rho_2(C^2_V)\) in \(\mathbb{C}^4\) cannot be special by Proposition 4.1.

We will see that cases (1) and (2) of the proposition correspond to the existence of special surfaces (S) and case (3) of the proposition gives rise to a special curve as in (C). We work case by case, considering each type of special surface or curve in \(\mathbb{C}^4\):

(S1) \(\{x \in \mathbb{C}^4 : x_i = c_i \text{ and } x_j = c_j\}\) for \(i < j\) in \(\{1, 2, 3, 4\}\): If \(\{i, j\}\) is \(\{1, 2\}\) or \(\{3, 4\}\), then this special surface is contained in \(\rho_2(C^2_V)\) if and only if \(V\) is the pencil of lines through the special point \(P = (c_i, c_j)\), and we denote these surfaces by
\[ S_{P,1} := \{(p_1, p_2)\} \times \mathbb{C}^2 \quad \text{and} \quad S_{P,2} = \mathbb{C}^2 \times \{(p_1, p_2)\}. \]

In this case, an infinite set of such lines will pass through at least two distinct special points. If \(\{i, j\}\) is \(\{1, 3\}\), \(\{2, 4\}\), \(\{1, 4\}\), or \(\{2, 3\}\), then the intersection with \(L \times L\) is finite for general \(L \in V\) so this surface cannot be contained in \(\rho_2(C^2_V)\) with preimage projecting dominant to \(V\).

(S2) \(\{x \in \mathbb{C}^4 : x_i = c_i \text{ and } x_j = x_k\}\) for three distinct indices \(i, j, k\): If \(\{j, k\}\) is \(\{1, 2\}\) or \(\{3, 4\}\), then the surface is not contained in \(\rho_2(C^2_V)\) because the intersections of \(L\) with \(\{x = y\}\) and \(\{x = c_i\}\) are finite for general \(L \in V\). If \(\{i, j\}\) is \(\{1, 2\}\), the intersection of the special surface with \(L \times L\) is again generally finite; other cases are similar.

(S3) \(\{x_i = x_j \text{ and } x_k = x_m\}\) with disjoint pairs of indices \(\{i, j\}\) and \(\{k, m\}\) in \(\{1, 2, 3, 4\}\): If the pairs are \(\{1, 2\}\) and \(\{3, 4\}\), then the intersection with \(L \times L\) is finite for general \(L \in V\). If the pairs are \(\{1, 3\}\) and \(\{2, 4\}\), then the special subvariety is
\[ \Delta = \{(x_1, x_2) = (x_3, x_4)\}. \]
If they are \(\{1, 4\}\) and \(\{2, 3\}\), then the the surface lies in \(\rho_2(C^2_V)\) if and only if the points on \(L\) come in symmetric pairs (so \((x, y) \in L\) if and only if \((y, x) \in L\) for
general $L \in V$. In other words, the family is of the form

$$x + y = b$$

for a nonconstant function $b$ on $V$, and we will denote this special surface by

$$D := \{x_1 = x_4 \text{ and } x_2 = x_3\}.$$  

For this family of lines, a point $(x, y)$ on the line will be special if and only if $(y, x)$ is special, so there are infinitely many such lines with at least two distinct special points.

(S4) $\{x_i = x_j = x_k\}$ for three distinct indices $i, j, k$: There is at least one pair of indices which is either $\{1, 2\}$ or $\{3, 4\}$. But then the intersection with a $L \times L$ is finite for general $L \in V$.

We now consider the existence of special curves $C \subset \mathbb{C}^4$ that are contained in $\rho_2(C_V^2)$ and whose preimage $\rho^{-1}(C)$ dominates $V$, but do not lie in the diagonal $\Delta$. We work case by case again.

(C1) $\{x \in \mathbb{C}^4 : x_i = c_i, x_j = c_j, x_k = c_k\}$ for $i < j < k$ in $\{1, 2, 3, 4\}$: There is a pair of indices equal to $\{1, 2\}$ or $\{3, 4\}$. So the product $L \times L$ intersects this curve for a general $L \in V$ if and only if the family of lines persistently contains a special point $P$. In particular, this curve lies in one of the surfaces of case (S1) and was already treated.

(C2) $\{x_i = c_i, x_j = c_j, x_k = x_m\}$ for disjoint pairs $\{i, j\}$ and $\{k, m\}$: If $\{i, j\} = \{1, 2\}$ then the general intersection with $L \times L$ is empty unless the lines contain $(c_1, c_2)$ for all $L \in V$, and this special curve lies in the special surface of case (S1). Similarly for $\{i, j\} = \{3, 4\}$. If $\{i, j\} = \{1, 3\}$, then the equality $x_2 = x_4$ in $L \times L$ would imply that $c_1 = c_3$ because $L$ is degree 1 and not horizontal for all $L$, making this special curve lie in the diagonal $\Delta$. Similarly for $\{i, j\} = \{2, 4\}$. For $\{i, j\} = \{1, 4\}$ with $c_1 \neq c_4$, the relation $x_2 = x_3$ means that lines of this family contain both $(c_1, \lambda)$ and $(\lambda, c_4)$ for some $\lambda \in \mathbb{C}$. This determines the line uniquely. Note that infinitely many such lines will contain at least two distinct special points. We let

$$C_{a,b,1} = \{x_1 = a \text{ and } x_4 = b \text{ and } x_2 = x_3\}$$

and

$$C_{a,b,2} = \{x_2 = a \text{ and } x_3 = b \text{ and } x_1 = x_4\}$$

denote these special curves in $\mathbb{C}^4$. For $c_1 = c_4$, this curve lies in $\rho_2(C_V^2)$ if the lines are of the form $x + y = b$ as in case (S3), and the special curve will be contained in the surface $D$. Similarly for $\{i, j\} = \{3, 2\}$.

(C3) $\{x_i = c_i, x_j = x_k = x_m\}$: Assume first that $i = 1$. The relations determine a curve in $\rho_2(C_V^2)$ dominating $V$ if the lines of the family contain both $(c_1, y)$ and
For some \( y \in \mathbb{C} \). But this means that the family consists of horizontal lines. Similarly for \( i = 2, 3, 4 \).

\[(C4) \{x_1 = x_2 = x_3 = x_4\}: \text{This curve is contained in the diagonal surface } \Delta.\]

The case-by-case analysis shows that the three types of families of lines listed in the proposition are the only exceptions: any other 1-parameter maximally-varying family of lines will have at most 1 special point on the general line in the family.

To see that the bound is at most 2 on the three types of exceptions, we look again at the cases and the structure of the closure of the special points in \( C_2^{C_2} \). Suppose we are in case (1), and assume there are at least 3 distinct special points on infinitely many lines. Then, considering pairs of special points that are not equal to the given \( P \in \mathbb{C}^2 \), we build a component of the Zariski closure of special points in \( \rho_2(C_2^{C_2}) \) that is neither in \( \Delta \) nor in \( S_{P,1} \) or \( S_{P,2} \). Similarly for case (2), choosing pairs of distinct special points that are not symmetric (as \((x, y)\) and \((y, x)\)) leads to a component of the Zariski closure of special points in \( \rho_2(C_2^{C_2}) \) that is neither in \( \Delta \) nor in \( D \). And finally, for case (3), the existence of pairs of points that are distinct and not equal to the pair \((c_1, \lambda)\) and \((\lambda, c_4)\) as described in \((C2)\) leads to a special component not contained in \( \Delta \) nor in the curves \( C_{c_1, c_4, 1} \) or \( C_{c_1, c_4, 2} \).

We have already observed that the bound 2 is optimal for those families, as there are infinitely many lines in each family containing at least 2 special points.

\[\Box\]

**Proof of Theorem 1.6.** The Chow variety of lines \( X_1 \) has dimension 2. From Theorem 1.4, we know that there is a finite union \( V_1 \) of irreducible curves and points in \( X_1 \) so that there are at most 2 special points on each line \( L \notin V_1 \). Now fix an irreducible curve \( C \subset V_1 \), and assume it is not the family of vertical or horizontal lines in \( \mathbb{C}^2 \). From Proposition 4.4, there is again a bound of 2 on the number of special points for all but finitely many \( L \in C \). This completes the proof.

\[\Box\]

### 5. Real algebraic curves in \( \mathbb{R}^2 \)

In this section, we observe that Theorems 1.3, 1.4, 1.5, and 1.6 apply to real algebraic curves in \( \mathbb{R}^2 \) passing through PCF parameters in the Mandelbrot set, in particular providing a proof of Theorem 1.8.

Suppose \( P(x, y) \in \mathbb{R}[x, y] \) is a polynomial with degree \( d \geq 1 \). Writing \( x = \frac{1}{2}(c + \bar{c}) \) and \( y = \frac{1}{2i}(c - \bar{c}) \), we obtain a polynomial of \( c \) and \( \bar{c} \) of degree \( d \) with complex coefficients. In this way, any real algebraic curve in \( \mathbb{R}^2 \) passing through a collection \( \{c_1, \ldots, c_m\} \) of PCF parameters in \( \mathbb{C} \) gives rise to a complex algebraic curve in \( \mathbb{C}^2 \) passing through special points \( \{(c_1, \bar{c}_1), \ldots, (c_m, \bar{c}_m)\} \). (Recall that the set of special parameters is symmetric under complex conjugation.)

For example, if we begin with the line in \( \mathbb{R}^2 \) defined by

\[\{(x, y) \in \mathbb{R}^2 : ax + by = r\}\]
with $a, b, r \in \mathbb{R}$, then this line contains $c \in \mathbb{C}$ if and only if the complex line

$$\left\{ (x, y) \in \mathbb{C}^2 : \frac{1}{2} (a - ib) x + \frac{1}{2} (a + ib) y = r \right\}$$

contains the point $(c, \bar{c})$ in $\mathbb{C}^2$. In particular, taking $b = 1$ and $a = r = 0$ shows that the real axis in $\mathbb{C}$ corresponds to the diagonal line $x = y$ in $\mathbb{C}^2$. Note that the vertical and horizontal lines in $\mathbb{C}^2$ cannot arise by this construction.

**Example 5.1.** The imaginary axis in $\mathbb{C}$ contains the three PCF parameters $\{i, 0, -i\}$ and corresponds to the complex line $y = -x$ in $\mathbb{C}^2$. Other than the real and imaginary axes in $\mathbb{C}$, we do not know any examples of real lines with more than two PCF parameters.

We see immediately that Theorem 1.3 applies to real algebraic curves in each degree $d \geq 1$, implying there is a uniform bound on the number of PCF parameters on any such curve, depending only on the degree. And so does Theorem 1.6, as the real axis in $\mathbb{C}$ is the only line containing infinitely many PCF parameters, implying that there are only finitely many real lines in $\mathbb{C}$ passing through more than 2 PCF parameters. This completes the proof of Theorem 1.8.

Note that the set of all complex algebraic curves of degree $d$ built from real curves in the above way is Zariski-dense in the Chow variety $X_d$ of all complex curves of degree $\leq d$ in $\mathbb{C}^2$. Therefore, Theorem 1.4 also holds for real algebraic curves. Finally, observing that there always exists a real algebraic curve of degree $d$ through any collection of $d(d+3)/2$ points in $\mathbb{C}$, we see that Theorem 1.5 also holds by choosing the curves to pass through collections of PCF parameters, so the bound of $d(d+3)/2$ is optimal for a general real curve in degree $d$.

**Example 5.2.** Holly Krieger pointed out to us that Theorem 1.6 also implies there are only finitely many horizontal real lines in $\mathbb{C}$ that contain more than 1 PCF parameter. Indeed, if two distinct PCF parameters, say $c_1$ and $c_2$, have the same nonzero imaginary part, then $c_2 - c_1 = \bar{c}_2 - \bar{c}_1 \in \mathbb{R}$, and the four pairs $(c_1, \bar{c}_2)$, $(\bar{c}_2, c_1)$, $(c_2, \bar{c}_1)$, and $(\bar{c}_1, c_2)$ are on the complex line

$$x + y = \alpha := c_1 + \bar{c}_2$$

in $\mathbb{C}^2$. We do not know of any examples of horizontal lines, other than the real axis in $\mathbb{C}$, containing more than one PCF parameter.
REFERENCES


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