MATH 295: ARITHMETIC DYNAMICS
SPRING 2021

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The goal of this course is to study a collection of Arithmetic Equidistribution Theorems. The basic idea of each is that, given a "good" *height function* $h : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ on the algebraic points of a projective variety defined over a number field, we can describe the geometry of the set of small-height points in $X(\mathbb{C})$. Already the case of $X = \mathbb{P}^1$ is rich in theory. Or, even simpler, take $X = \mathbb{A}^1$, where $X(\mathbb{C})$ is the complex plane $\mathbb{C}$. Our study will use tools from complex potential theory, algebraic number theory, the theory of Berkovich spaces, p-adic potential theory, dynamics and much more.

In studying arithmetic dynamical systems, by which I mean the iteration of rational maps or morphisms $f : X \to X$ defined over a given number field on a projective variety, we encounter a huge collection of examples of interesting height functions. The equidistribution theorems are playing a key role in current research. In this course, we will spend most of our time discussing examples from dynamical systems $f : \mathbb{P}^1 \to \mathbb{P}^1$.

Remarkably, these equidistribution theorems – though having their origin in the subject of dynamical systems and group actions – are not inherently dynamical. They have seen application in much more general arithmetic-geometric contexts and even settings that are not obviously arithmetic (for problems in complex analysis or geometry).

First Example. Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, with $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ for some $\tau \in \mathbb{H}$, with its zero the quotient of $0 \in \mathbb{C}$. The Lebesgue measure on $\mathbb{C}$ induces a Haar measure on E. We can normalize it to have total area 1. The *torsion* points on E are the quotients of points $x \in \mathbb{C}$ so that $nx \in \Lambda$ for some nonzero integer n. It is easy to see that the set of all torsion points is dense in E. Moreover, we see that the set of torsion points of order n (or $\leq n$) become uniformly distributed in E as $n \to \infty$.

Szpiro, Ullmo, and Zhang proved a strengthening of this geometric statement for elliptic curves E (and more generally, abelian varieties) defined over a number field K [SUZ]. In that setting, the torsion points of E are algebraic. They proved: given any non-repeating sequence $\{\alpha_n\} \subset E(\overline{\mathbb{Q}})$ of torsion points, the Galois orbits $\mathrm{Gal}(\overline{K}/K) \cdot \alpha_n$ are uniformly distributed with respect to the Haar measure on

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Date: February 9, 2026.

$E(\mathbb{C})$. Their theorem is really a statement about the Néron-Tate canonical height function $\hat{h} : E(\overline{\mathbb{Q}}) \to \mathbb{R}$; the torsion points are simply the zeroes of $\hat{h}$, and the same equidistribution holds if we only assume that $\hat{h}(\alpha_n) \to 0$ as $n \to \infty$. Their theorem was used in the proofs of the Bogomolov Conjecture [Ul, Zh], and it has seen many generalizations and further applications since then.

CONTENTS

1. BILU'S EQUIDISTRIBUTION THEOREM

In this section, we study an arithmetic equidistribution theorem of Yuri Bilu [Bi], following a proof by Robert Rumely [Ru1]. Pay close attention to the ingredients in the proof, as they will come up again in different contexts.

Recall that a complex number α is algebraic if it is the root of a polynomial with integer coefficients. The number α is an algebraic integer if it is the root of a monic polynomial with integer coefficients.

1.1. Logarithmic Weil height.

We think of height functions on $\overline{\mathbb{Q}}$ as measuring the "complexity" of an algebraic number. In the field of rational numbers $\mathbb{Q}$, a basic example is given by

$$h(a/b) = \log \max\{|a|, |b|\} \quad \text{when } (a,b) = 1,$$

which counts (roughly) the number of digits needed to write down the rational number. For algebraic numbers $\alpha \neq 0$, if $P \in \mathbb{Z}[x]$ is the minimal polynomial for α with lead coefficient a_0, then the function h can be extended to all of $\overline{\mathbb{Q}}$ by

$$h(\alpha) = (\deg P)^{-1} \left(\log |a_0| + \sum_{P(x)=0} \log^+ |x| \right) \tag{1.1}$$

where $\log^+ = \max\{\log, 0\}$. We put $h(0) = 0$.

Theorem 1.1 (Bilu, 1997)**.** *Suppose $\alpha_n \in \overline{\mathbb{Q}}$ is a non-repeating sequence of algebraic numbers with $h(\alpha_n) \to 0$. Then the Galois orbits of α_n are equidistributed with respect to Lebesgue measure on the unit circle S^1.*

More precisely, we let

$$\mu_n = \frac{1}{\deg \alpha_n} \sum_{P_n(z)=0} \delta_z$$

where P_n is the minimal polynomial for α_n, a discrete probability measure on $\mathbb{C}$ supported equally on α_n and its Galois conjugates. The theorem says that

$$\mu_n \longrightarrow \mu_{S_1} := m_{S^1}/2\pi$$

in the weak-* topology on $\mathbb{C}$, where m_{S^1} is the arclength (Lebesgue) measure on the unit circle. In other words, given any continuous function φ with compact support on $\mathbb{C}$, we have

$$\frac{1}{\deg \alpha_n} \sum_{P_n(z)=0} \varphi(z) = \int_{\mathbb{C}} \varphi \, d\mu_n \ \longrightarrow\ \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it})\, dt.$$

Rumely's proof of Theorem 1.1 is quite elementary, combining algebra with potential theory in a clever way. It also suggests how to generalize the statements to a large class of height functions, in particular to settings without group structure.

1.2. **Algebraic input.** Before we give Rumely's proof of Theorem 1.1, we need a few basic facts about number fields and their absolute values. A useful reference for this material is the beginning of Lang's *Fundamentals of Diophantine Geometry* [La] or Bombieri-Gubler's *Heights in Diophantine Geometry* [BG].

By definition, a number field is a finite extension of $\mathbb{Q}$. Each number field K comes equipped with a canonical collection M_K of inequivalent and nontrivial absolute values $|\cdot|_v$, with a finite number for each prime p, extending the usual p-adic valuation on $\mathbb{Q}$ given by $|p^r a/b|_p = p^{-r}$ when p does not divide a or b, and a finite number of archimedean absolute values coming from the embeddings of K into $\mathbb{C}$. For this collection M_K, we have the **product formula**,

$$\prod_{v \in M_K} |\alpha|_v^{N_v} = 1 \tag{1.2}$$

for all $\alpha \neq 0$ in K, where $N_v = [K_v : \mathbb{Q}_v]$; here, the subscript v on the field denotes the completion with respect to the given absolute value. The elements of M_K are often called the **places** of K. We let M_K^∞ denote the subset of archimedean places.

It is worth observing that these absolute values behave naturally with respect to field extensions. So, for example, if $\alpha \in K \subset K'$ and $v \in M_K$, then

$$|\alpha|_v^{[K':K]} = \prod_{v'|v} |\alpha|_{v'}^{[K'_{v'}:K_v]}, \tag{1.3}$$

taking the product over $v' \in M_{K'}$ such that v' extends v.

Proposition 1.2. *For α in number field K, the height $h(\alpha)$ satisfies*

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} N_v \log^+ |\alpha|_v,$$

Proof. For polynomials f, g with coefficients in a field K, and for each non-archimedean absolute value $|\cdot|_v$ on K, **Gauss's Lemma** states that

$$\|fg\|_v = \|f\|_v \|g\|_v$$

where $f(x) = c_0 x^d + \cdots + c_d$ has

$$\|f\|_v = \max_v |c_i|_v.$$

It follows that the minimal polynomial $P(x) = a_0 x^d + \cdots + a_d$ for $\alpha \in K$, with its coefficients in $\mathbb{Z}$, satisfies

$$1 = \|P\|_v = \left\| a_0 \prod_{i=1}^{d} (x - \alpha_i) \right\|_v = |a_0|_v \prod_i \max\{1, |\alpha_i|_v\}$$

for each non-archimedean $v \in M_K$. Combined with the product formula and with (1.3), we have

$$\begin{aligned} \log|a_0| = -\sum_p \log|a_0|_p &= -\frac{1}{[K:\mathbb{Q}]} \sum_p \sum_{v|p} N_v \log|a_0|_v \\ &= \frac{1}{[K:\mathbb{Q}]} \sum_p \sum_{v|p} N_v \sum_i \log^+ |\alpha_i|_v \\ &= \sum_p \sum_{v|p} N_v \log^+ |\alpha|_v, \end{aligned}$$

where the last equality follows because

$$\sum_i \log^+ |\alpha_i|_{v_0} = \sum_{v|p} N_v \log^+ |\alpha|_v$$

for each place v_0 of K that extends the p-adic absolute value on $\mathbb{Q}$. The same holds at the archimedean places, so we have

$$\sum_{P(x)=0} \log^+ |x| = \sum_{v \in M_K^\infty} N_v \log^+ |\alpha|_v.$$

Putting this together with the definition of h in (1.1), we have

$$\begin{aligned} h(\alpha) &= (\deg P)^{-1}\left(\log|a_0| + \sum_{P(x)=0} \log^+|x|\right) \\ &= \frac{1}{[K:\mathbb{Q}]}\left(\sum_p \sum_{v|p} N_v \log^+|\alpha|_v + \sum_{v|\infty} N_v \log^+|\alpha|\right) \\ &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K} N_v \log^+|\alpha|_v. \end{aligned}$$

□

From its definition (1.1) or from Proposition 1.2, we see that $h(\alpha) \geq 0$ for all $\alpha \in \overline{\mathbb{Q}}$.

Proposition 1.3 (Kronecker). *We have $h(\alpha) = 0$ if and only if $\alpha^m = 1$ for some integer m or $\alpha = 0$.*

Proof. Each root of unity ζ is the root of a monic polynomial with integer coefficients, and all of its Galois conjugates are in the unit circle. Thus, from the definition, we see that $h(\zeta) = 0$.

Now suppose that $\alpha \neq 0$ satisfies $h(\alpha) = 0$, and let $d = \deg \alpha$. Then, from the definition (1.1) of h, we must have $a_0 = 1$ (so that α is an algebraic integer) and all of the conjugates $\alpha_1, \ldots, \alpha_d$ of α lie in the closed unit disk. Moreover, all powers α^m, for integers $m \geq 1$, are algebraic integers of degree at most d over $\mathbb{Q}$.

For each positive integer m, consider the elementary symmetric functions

$$S_{i,m} := s_i(\alpha_1^m, \ldots, \alpha_d^m),$$

for $i = 1, \ldots, d$. As α is an algebraic integer, we have $S_{i,m} \in \mathbb{Z}$ for all i and m. Moreover, each $S_{i,m}$ is a sum of $\begin{pmatrix} d \\ i \end{pmatrix}$ terms, each of absolute value ≤ 1, and so

$$|S_{i,m}| \leq \begin{pmatrix} d \\ i \end{pmatrix}$$

But the $S_{i,m}$ are the coefficients of a monic, integer polynomial with root α^m. As there are only finitely many such choices for these polynomials, it follows that there exist $m > n > 0$ so that $\alpha^m = \alpha^n$. In other words, $\alpha^{m-n} = 1$, so α is a root of unity. □

Remark 1.4. Note that the points of height zero (namely, the roots of unity by Proposition 1.3) are clearly uniformly distributed on the unit circle, if you take all of the solutions to $z^n = 1$ and then let $n \to \infty$. So Theorem 1.1 is a generalization of this fact to Galois-invariant collections of points of height tending to 0.

The first observation in proving Bilu's equidistribution theorem is that the condition that $h(\alpha_n) \to 0$ implies that $\deg \alpha_n \to \infty$. This will follow from:

Proposition 1.5 (Northcott, 1950)**.** *For any bound $B > 0$ and any degree $D > 0$, the set*

$$\{\alpha \in \overline{\mathbb{Q}} : [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq D,\ h(\alpha) \leq B\}$$

is finite.

Proof. Suppose that $d = \deg \alpha$. Then the minimal polynomial $P(x) = a_0 x^d + \cdots + a_d \in \mathbb{Z}[x]$ for α must satisfy

$$\max_i |a_i| \leq \binom{d}{\lfloor d/2 \rfloor} |a_0| \prod_{P(x)=0} \max\{1, |x|\} \leq 2^d\, e^{d\, h(\alpha)}.$$

Consequently, the set of all α with $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq D$ and $h(\alpha) \leq B$ must be roots of polynomials with integer coefficients all bounded by $2^D e^{DB}$. This set is finite. □

1.3. **Analytic input.** A helpful reference for potential theory in $\mathbb{C}$ (or rather, the classical Newtonian potential theory in $\mathbb{R}^2$) is Ransford's *Potential Theory in the Complex Plane* [Ra]. Here I summarize a few key concepts from potential theory needed for this proof.

Fix a compact set K in $\mathbb{C}$ (or $\mathbb{R}^2$). Let μ be a probability measure on K. The **energy** of μ is defined by

$$E(\mu) = -\iint \log|z - w|\, d\mu(z)\, d\mu(w).$$

Note that this might be $+\infty$ but is never $-\infty$. The **capacity** of K is the value

$$\operatorname{cap} K = e^{-\inf_\mu E(\mu)} \geq 0,$$

taking the infimum over all probability measures μ with $\operatorname{supp} \mu \subset K$. A measure ν for which

$$E(\nu) = \inf_\mu E(\mu)$$

is called an **equilibrium measure** for the set K.

The following classical result is attributed to Frostman (circa 1930). The full statement of his theorem is given as Theorem 2.2.

Theorem 1.6 (Frostman's Theorem: version 1)**.** *Let K be a compact subset of $\mathbb{C}$. If $\operatorname{cap} K > 0$, then there exists a unique equilibrium measure μ_K for K. It satisfies $\operatorname{supp} \mu_K \subset \partial^o K$, the outer boundary of K.*

A proof of Theorem 1.6 appears in Sections 3.3 and 3.7 of [Ra], but an outline will be given here in Section 2.

Example 1.7. By uniqueness in Theorem 1.6, the equilibrium measure for a closed disk in the plane is the (normalized) Lebesgue measure on its circle boundary. (It has to be supported in the boundary of the disk and it must be rotation invariant.) We have

$$\operatorname{cap} D(a, R) = R$$

for any disk of radius R in $\mathbb{C}$.

1.4. **Proof of Theorem 1.1.** Let h be logarithmic Weil height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. Suppose α_n is a sequence with $h(\alpha_n) \to 0$. Put

$$\mu_n = \frac{1}{\deg \alpha_n} \sum_{P_n(\alpha)=0} \delta_\alpha$$

where P_n is the minimal polynomial (so we are summing over the Galois conjugates). From Proposition 1.5, we know that $\deg \alpha_n \to \infty$.

We begin with a basic observation; I leave the proof as an exercise.

Lemma 1.8. *Suppose that* $h(\alpha_n) \to 0$. *Then the sequence of probability measures* μ_n *is tight, and all subsequential limits* $\nu = \lim_{n_k \to \infty} \mu_{n_k}$ *are probability measures supported in the closed unit disk* $\overline{\mathbb{D}}$.

From Lemma 1.8, if $\mu_n \to \nu$, we have $E(\nu) \in [0, +\infty]$, since $E(\nu) \geq E(\mu_{S^1}) = 0$.

We define a modified energy that makes sense for atomic measures as

$$E^*(\mu_n) = -\iint_{\mathbb{C}\times\mathbb{C}\setminus \mathrm{Diag}} \log|z - w| \, d\mu_n(z) \, d\mu_n(w), \tag{1.4}$$

where Diag is the diagonal $\{z = w\}$.

Lemma 1.9. *Assume that* $\mu_n \to \nu$ *weakly. Then*

$$\liminf_{n\to\infty} E^*(\mu_n) \geq E(\nu).$$

Proof. This follows a fairly standard cut-off argument in analysis, but the supports of the measures μ_n are not necessarily uniformly bounded and the kernel $\log|z - w|$ is neither bounded nor compactly supported, so we have to be careful. Choose any large $M > 0$ and fix large open disks D_1 and D_2 so that

$$\overline{\mathbb{D}} \subset D_1 \subset \overline{D}_1 \subset D_2.$$

Choose any bump function φ which is $\equiv 1$ on D_1 and supported in D_2. Then

$$\begin{aligned}
E_M(\mu_n) &:= \iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}\, d\mu_n(z)\, d\mu_n(w) \\
&= \iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}\varphi(z)\varphi(w)\, d\mu_n(z)\, d\mu_n(w) \\
&\quad + 2\iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}\varphi(z)(1-\varphi(w))\, d\mu_n(z)\, d\mu_n(w) \\
&\quad + \iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}(1-\varphi(z))(1-\varphi(w))\, d\mu_n(z)\, d\mu_n(w) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

giving the names of I, II, and III to the three terms in the sum.

By weak convergence $\mu_n \to \nu$, with fixed M and φ, we have

$$\begin{aligned}
\mathrm{I} \longrightarrow & \iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}\varphi(z)\varphi(w)\, d\nu(z)\, d\nu(w) \\
&= \iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}\, d\nu(z)\, d\nu(w)
\end{aligned}$$

as $n \to \infty$. For II, we note that there is a uniform constant $L > 0$ so that

$$\log|z-w| \le \log^+|w| + L$$

for all $z \in D_2$ and all $w \in \mathbb{C}$, and therefore

$$\min\{M, -\log|z-w|\} \ge -\log^+|w| - L$$

with any $M > 0$, for any z in the support of φ. Therefore

$$\begin{aligned}
&\iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}\varphi(z)(1-\varphi(w))\, d\mu_n(z)\, d\mu_n(w) \\
&\qquad \ge \iint_{\mathbb{C}\times\mathbb{C}} (-\log^+|w| - L)\varphi(z)(1-\varphi(w))\, d\mu_n(z)\, d\mu_n(w) \\
&\qquad\qquad \ge \int_{\mathbb{C}\setminus D_1} (-\log^+|w| - L)\, d\mu_n(w).
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathrm{II} &\ge 2\int_{\mathbb{C}\setminus D_1} (-\log^+|w| - L)\, d\mu_n(w) \\
&\ge 2\left(-\frac{1}{\deg \alpha_n} \sum_{P_n(\alpha)=0} \log^+|\alpha| - L\,\mu_n(\mathbb{C}\setminus D_1)\right) \\
&\ge -2h(\alpha_n) - 2L\mu_n(\mathbb{C}\setminus D_1).
\end{aligned}$$

Both terms on the right hand side tend to 0 as $n \to \infty$, the first by assumption and the second by weak convergence of the measures.

For III, we can enlarge our constant L if needed to observe that

$$\log|z-w| \le \log^+|w| + \log^+|z| + L$$

for all $z, w \in \mathbb{C}$. This gives

$$\min\{M, -\log|z-w|\} \ge -\log^+|w| - \log^+|z| - L$$

with any $M > 0$, so that

$$\begin{aligned}\iint_{\mathbb{C}\times\mathbb{C}} &\min\{M, -\log|z-w|\}(1-\varphi(z))(1-\varphi(w))\,d\mu_n(z)\,d\mu_n(w)\\ &\ge \iint_{\mathbb{C}\times\mathbb{C}} (-\log^+|w| - \log^+|z| - L)(1-\varphi(z))(1-\varphi(w))\,d\mu_n(z)\,d\mu_n(w)\end{aligned}$$

and implies that

$$\mathrm{III} \ge -2h(\alpha_n) - L\mu_n(\mathbb{C}\setminus D_1)^2.$$

Again we see that the right-hand-side tends to 0 as $n \to \infty$.

It follows that

$$\liminf_{n\to\infty} E_M(\mu_n) \ge \iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}\,d\nu(z)\,d\nu(w)$$

for every $M > 0$ large. Letting $M \to \infty$, the limit of the right-hand-side is equal to $E(\nu)$ by the monotone convergence theorem. On the other hand, we have

$$\begin{aligned} E^*(\mu_n) &\ge \iint_{\mathbb{C}\times\mathbb{C}\setminus\mathrm{Diag}} \min\{M, -\log|z-w|\}\,d\mu_n(z)\,d\mu_n(w)\\ &= \iint_{\mathbb{C}\times\mathbb{C}} \min\{M, -\log|z-w|\}\,d\mu_n(z)\,d\mu_n(w) - \frac{M}{\deg\alpha_n}\\ &= E_M(\mu_n) - \frac{M}{\deg\alpha_n}.\end{aligned}$$

for every n and M. Thus,

$$\liminf_{n\to\infty} E^*(\mu_n) \ge \liminf_{n\to\infty} E_M(\mu_n)$$

for all $M > 0$. This completes the lemma. $\square$

Recall that $E(\nu) = 0$ if and only if $\mu = m_{S^1}/2\pi$ by Frostman's Theorem. Setting $d_n = \deg\alpha_n$,

$$E^*(\mu_n) = -\frac{1}{d_n^2}\log|D(\alpha_n)|$$

where

$$D(\alpha_n) = \prod_{i\ne j}(\alpha_n^i - \alpha_n^j) \in \mathbb{Q} \tag{1.5}$$

is (a rational multiple of) the discriminant of α_n, with α_n^i, $i = 1, \ldots, d_n$ the Galois conjugates of α_n.

For each absolute value $|\cdot|_v$ of $\mathbb{Q}$ (prime or infinite), choose an extension to $\overline{\mathbb{Q}}_v$, the algebraic closure of the completion $\mathbb{Q}_v$. (For $v = \infty$, we have the standard absolute value on $\overline{\mathbb{Q}}_\infty = \mathbb{C}$.) For any α in a number field K, we can define

$$\lambda_v(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \hookrightarrow \overline{\mathbb{Q}}_v} \log^+ |\sigma\alpha|_v$$

and note that

$$h(\alpha) = \sum_{v \in M_\mathbb{Q}} \lambda_v(\alpha)$$

from Proposition 1.2, independent of the choice of number field K.

Returning to our proof, note that

$$h(\alpha_n) \geq \sum_p \lambda_p(\alpha_n) \geq 0$$

when summing only over the primes p, so that $\sum_p \lambda_p(\alpha_n) \to 0$ as $n \to \infty$.

Now fix $\varepsilon > 0$. From the product formula on $\mathbb{Q}$, we have

$$\begin{aligned}
0 = \sum_{v \in M_\mathbb{Q}} \log|D(\alpha_n)|_v &= \log|D(\alpha_n)| + \sum_p \log|D(\alpha_n)|_p \\
&= \log|D(\alpha_n)| + \sum_p \sum_{i \neq j} \log|\alpha_n^i - \alpha_n^j|_p \\
&\leq \log|D(\alpha_n)| + \sum_p \sum_{i \neq j} \log\max\{|\alpha_n^i|_p, |\alpha_n^j|_p\} \\
&\leq \log|D(\alpha_n)| + 2d_n \sum_p \sum_i \log^+ |\alpha_n^i|_p \\
&\leq \log|D(\alpha_n)| + 2d_n^2 \sum_p \lambda_p(\alpha_n) \\
&= -d_n^2\, E^*(\mu_n) + 2d_n^2 \sum_p \lambda_p(\alpha_n) \\
&< d_n^2(-E(\nu) + \varepsilon) + 2d_n^2\,\varepsilon \;=\; -d_n^2\,(E(\nu) - 3\varepsilon)
\end{aligned}$$

for all n large, by Lemma 1.9 and the assumption on the height. (For $E(\nu) = \infty$, we replace the expression $-E(\nu) + \varepsilon$ with $-M$ for any fixed large $M > 0$.) So if $E(\nu)$ is positive, then we could choose ε small enough that the right hand side would be negative for all $n \gg 0$, giving a contradiction. Therefore $E(\nu) = 0$, so $\mu = m_{S^1}/2\pi$ by Theorem 1.6 and Example 1.7. This completes the proof of Theorem 1.1.

2. Potential theory in $\mathbb{C}$

Subharmonic functions are introduced in Chapter 2 of Ransford's book [Ra], and the Laplacian (in the sense of distributions) is treated in Section 3.7 of [Ra]. Capacity, as a function on sets in $\mathbb{C}$, is studied in his Chapter 5, and the equidistribution of Fekete points appears as an exercise in Section 5.5 of [Ra].

2.1. Subharmonic functions.

Recall from complex analysis that a *harmonic* function on a domain $\Omega \subset \mathbb{C}$ is a C^2 function $h : \Omega \to \mathbb{R}$ satisfying

$$\Delta h := \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

It is characterized by the *mean-value property* that

$$h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{it})\, dt$$

for every small disk $\overline{D(z_0, r)} \subset \Omega$. This is equivalent to saying the h is the real part of a holomorphic function on simply-connected open subsets of Ω, and so harmonic functions are always C^∞.

A *subharmonic function* $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is one which satisfies a *sub-mean-value property.* Specifically, we require that u be upper semicontinuous, so that

$$u(z_0) \geq \limsup_{z \to z_0} u(z)$$

at every point $z_0 \in \Omega$, and that

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt$$

on small disks $\overline{D(z_0, r)} \subset \Omega$. Important examples are the logarithm functions $\log|z|$ and $\log^+|z|$.

If u is a C^2 function, then u is subharmonic if and only if $\Delta u \geq 0$. For general upper semicontinuous functions, *u is subharmonic if and only if $\Delta u \geq 0$ in the sense of distributions.* In fact, the upper semicontinuous function u will be harmonic (and so C^∞) on open sets where $\Delta u = 0$ in the sense of distributions.

Example 2.1. For every $a \in \mathbb{C}$ the function $u_a(z) = \log|z - a|$ is subharmonic with

$$\frac{1}{2\pi} \Delta u_a = \delta_a$$

in the sense of distributions. For every $a \in \mathbb{C}$ and $r > 0$, the continuous function $u_{a,r} = \max\{\log|z - a|, \log r\}$ is subharmonic with

$$\frac{1}{2\pi} \Delta u_{a,r} = m_{a,r}$$

in the sense of distributions, where $m_{a,r}$ is the normalized Lebesgue measure on the circle $\{|z - a| = r\}$.

2.2. **Potential functions and Frostman's Theorem.** Recall from §1.3 that a set $E \subset \mathbb{C}$ has *capacity 0* if every probability measure supported on E has infinite energy. Such sets are also called *polar* in the literature. This is because, on one hand, the set $\{u = -\infty\}$ has capacity 0 for every subharmonic function $u \not\equiv -\infty$. And, on the other hand, every closed set E with capacity 0 is of the form $\{u = -\infty\}$ for some subharmonic function defined in a neighborhood of E. See Section 3.5 of [Ra].

The **potential function** associated to a compactly-supported probability measure μ is

$$V_\mu(z) = \int \log|z-w|\, d\mu(w)$$

and satisfies $\frac{1}{2\pi}\Delta V_\mu = \mu$ in the sense of distributions. A preliminary version of Frostman's Theorem was given as Theorem 1.6; here is the complete statement.

Theorem 2.2. (Frostman, ∼1930) *Let K be a compact subset of $\mathbb{C}$. If $\operatorname{cap} K > 0$, then there exists a unique equilibrium measure μ_K for K. It satisfies $\operatorname{supp} \mu_K \subset \partial^o K$, the outer boundary of K. Furthermore, $V_{\mu_K}(z) \geq -E(\mu_K)$ for all $z \in \mathbb{C}$, with $V_{\mu_K}(z) = -E(\mu_K)$ for all z in K except on a subset of capacity 0.*

2.3. **Main ideas in the proof of Theorem 2.2.** Let $K \subset \mathbb{C}$ be a compact set. We say a condition holds *nearly everywhere* if holds outside of a set of capacity 0.

Step 1: *Equilibrium measures exist for every compact $K \subset \mathbb{C}$.* If $\operatorname{cap} K = 0$, then we can choose any probability measure supported on K. If $\operatorname{cap} K > 0$, then let μ_n be a sequence of probablity measures with

$$E(\mu_n) \to \inf_\mu E(\mu)$$

on K. Because the measures μ_n are all supported on the compact set K, weak-$*$ compactness guarantees that there is a weakly convergent subsequence. Suppose that $\mu_n \to \nu$ weakly. As in Rumely's Lemma 1.9, we can show that

$$\liminf_{n\to\infty} E(\mu_n) \geq E(\nu). \tag{2.1}$$

In fact, the proof is easier than that of Lemma 1.9, because the measures are all supported in K, but the proof follows the same strategy of cutting the kernel $\log|z-w|$ off with $M > 0$. This shows that $E(\nu) = \inf_\mu E(\mu)$, so ν is an equilibrium measure.

Step 2: *Assume that $\operatorname{cap} K > 0$ and let ν be any equilibrium measure. Then $V_\nu \leq -E(\nu)$ nearly everywhere on K.* Suppose there is an $\varepsilon > 0$ so that the set

$$K_\varepsilon = \{z \in K : V_\nu \geq -E(\nu) + \varepsilon\}$$

has positive capacity. Then there is a probability measure μ_ε supported on K_ε with finite energy. On the other hand, since $-E(\nu) = \int V_\nu\, d\nu$, there must be a point

$z_0 \in \operatorname{supp} \nu$ where $V_\nu(z_0) \leq -E(\nu)$. By upper semicontinuity, there is $r > 0$ so that

$$V_\nu(z) \leq -E(\nu) + \frac{1}{2}\varepsilon$$

at all points $z \in D(z_0, r)$. Note that K_ε must then be disjoint from $D(z_0, r)$. Since $z_0 \in \operatorname{supp} \nu$ we have

$$m := \nu(D(z_0, r)) > 0.$$

Define a family of probability measures

$$\nu_t = \nu + t\,\mu_\varepsilon - \frac{t}{m}\,\nu|D(z_0, r).$$

for $0 \leq t \leq m$. Computing the energy of ν_t, we see that

$$\begin{aligned} E(\nu_t) = E(\nu) &+ t^2\, E\left(\mu_\varepsilon - \frac{1}{m}\,\nu|D(z_0, r)\right) \\ &- 2t \iint \log|z - w|\, d\nu(z)\, d\left(\mu_\varepsilon - \frac{1}{m}\,\nu|D(z_0, r)\right)(w) \end{aligned}$$

But

$$\begin{aligned} \iint \log|z - w|\, d\nu(z)\, d\left(\mu_\varepsilon - \frac{1}{m}\,\nu|D(z_0, r)\right)(w) &= \int V_\nu\, d\left(\mu_\varepsilon - \frac{1}{m}\,\nu|D(z_0, r)\right) \\ &\geq (-E(\nu) + \varepsilon) - (-E(\nu) + \frac{1}{2}\varepsilon) \\ &\geq \frac{1}{2}\varepsilon \ > \ 0 \end{aligned}$$

so the measure ν_t would have lower energy than ν for t small, a contradiction. So K_ε is a polar set, and the countable union $\bigcup_n K_{1/n}$ of polar sets is also polar, completing the proof.

Step 3: *We have* $V_\nu \geq -E(\nu)$ *at all points of* $\mathbb{C}$. We know from Step 2 that $V_\nu \leq -E(\nu)$ except on a polar set $S \subset K$. But polar sets must satisfy $\nu(S) = 0$ (because ν has finite energy). If there were a point $z_0 \in \operatorname{supp} \nu$ with $V_\nu(z_0) < -E(\nu)$, then the inequality would hold on a set of positive measure by upper semicontinuity. But then the ν-average of V_ν would be too small. So $V_\nu \geq -E(\nu)$ on $\operatorname{supp} \nu$. As V_ν is harmonic on each connected component of $\mathbb{C} \setminus \operatorname{supp} \nu$, the maximum principle applied to $-V_\nu$ gives the result. See, for example, [Ra, Theorem 3.6.9] for a version of the Maximum Principle that applies here.

Step 4: *Equilibrium measures satisfy* $\operatorname{supp} \nu \subset \partial^o K$*, the outer boundary of* K. We know that $V_\nu(z) = -E(\nu)$ nearly everywhere on K, and $V_\nu(z) \geq -E(\nu)$ on $\mathbb{C}$. Also, as $\frac{1}{2\pi}\Delta V_\nu = \nu$ in the sense of distributions, we have that V_ν is harmonic away from $\operatorname{supp} \nu$. So if $\mathbb{C} \setminus K$ has bounded components, then the maximum principle implies that $V_\nu \equiv -E(\nu)$ on these components. So $V_\nu = -E(\nu)$ nearly everywhere on all bounded components of $\mathbb{C} \setminus \partial^o(K)$, implying that it must be harmonic there.

Step 5: *Uniqueness.* Suppose that ν_1 and ν_2 are equilibrium measures on a set K of positive capacity. Then $V_{\nu_1} = V_{\nu_2}$ nearly everwhere on K. As both functions satisfy

$$V_{\nu_i} = \log |z| + o(1)$$

as $z \to \infty$, we have that $V_{\nu_1} - V_{\nu_2}$ is harmonic on $\mathbb{C} \setminus K$ and extending continuously to $\hat{\mathbb{C}} \setminus K$, and bounded by 0 on K nearly everywhere. Therefore $V_{\nu_1} - V_{\nu_2} \equiv 0$ on $\mathbb{C} \setminus K$ by the Maximum Principle. Thus, $V_{\nu_1} = V_{\nu_2}$ nearly everwhere on $\mathbb{C}$. Polar sets have Lebesgue measure 0, so the identity principle [Ra, Theorem 2.7.5] implies $V_{\nu_1} = V_{\nu_2}$ everywhere.

2.4. **Transfinite Diameter.** Let K be a compact set in $\mathbb{C}$, and set

$$\operatorname{diam}_n(K) = \left(\sup \prod_{i<j} |z_i - z_j| \right)^{2/n(n-1)}$$

where the supremum is taken over all collections of n points $z_1, \ldots, z_n \in K$. Note that $\operatorname{diam}_2(K)$ is the usual diameter of the set K.

Proposition 2.3. *The sequence* $\{\operatorname{diam}_n(K)\}$ *is non-increasing and*

$$\operatorname{diam}_\infty(K) := \lim_{n\to\infty} \operatorname{diam}_n(K) = \operatorname{cap} K.$$

A proof is given in [Ra, Theorem 5.5.2]. Points that achieves the supremum in $\operatorname{diam}_n(K)$ for some n are called *Fekete points.*

Example 2.4. The n-th roots of unity are Fekete points for $K = \overline{\mathbb{D}}$, the closed unit disk. See Exercise 1 in [Ra, §5.5] for a useful hint.

Exercise 5 in [Ra, §5.5] asks you to prove the following:

Theorem 2.5 (Equidistribution of Fekete points)**.** *Suppose that compact set K satisfies* $\operatorname{cap} K > 0$. *For each $n \geq 2$, let F_n be a set of Fekete points. Then the discrete measures*

$$\mu_n = \frac{1}{n} \sum_{z \in F_n} \delta_z$$

converge weakly to the equilibrium measure μ_K.

3. Fekete-Szegö and Rumely's equidistribution theorem

3.1. **A theorem of Fekete and Szegö.** The following result might be the first that connected potential theory to algebraic number theory. It also shows that the value of capacity (not just its positivity) is meaningful.

Theorem 3.1 (Fekete 1930s, Fekete-Szegö 1950s))**.** *Let S be any compact set in $\mathbb{C}$ which is symmetric about the real axis. Then* $\operatorname{cap} S \geq 1$ *if and only if, for every neighborhood U of S, there exists a sequence α_n of algebraic integers so that α_n and all its Galois conjugates lie in U.*

Remark 3.2. Note that there are no conditions on the set S other than its symmetry and its capacity – for example it could be a circle of radius $r \geq 1$ centered at $x = \pi$, or it could be the Julia set of any monic, real polynomial. In particular, it is necessary to pass to a neighborhood of S to find the algebraic integers, as there might be no algebraic points in S at all.

Recall that capacity coincides with the transfinite diameter of S, introduced in §2.4. If α is an algebraic integer of degree n, with Galois conjugates given by $x_1, \ldots, x_n$ (including α itself), then the discriminant

$$D(\alpha) = \prod_{i \neq j} (x_i - x_j)$$

is a nonzero integer; compare line (1.5) above. If the set $\{x_1, \ldots, x_n\}$ is contained in a compact set K, then

$$\operatorname{diam}_n(K) = \left(\sup_{\{z_1,\ldots,z_n\} \subset K} \prod_{i<j} |z_i - z_j| \right)^{2/n(n-1)} \geq |D(\alpha)|^{1/n(n-1)}$$

Thus, one implication of Theorem 3.1 is easy. Fix an open neighborhood U of S. If the sequence of algebraic integers α_n exist, then the discriminants $D(\alpha_n)$ satisfy

$$\liminf_{n\to\infty} |D(\alpha_n)|^{1/d_n(d_n-1)} \geq 1,$$

where $d_n = \deg \alpha_n$. Note that we must have $d_n \to \infty$ from Proposition 1.5, because the heights of α_n are uniformly bounded. (The height of α_n is bounded by the radius of the smallest closed disk $D(0, R)$ containing $\overline{U}$.) It follows that

$$\operatorname{cap} \overline{U} = \lim_{n\to\infty} \operatorname{diam}_n(\overline{U}) \geq \limsup_{n\to\infty} |D(\alpha_n)|^{1/d_n(d_n-1)} \geq 1.$$

Letting U shrink, we conclude that $\operatorname{cap} S \geq 1$. This last step requires a lemma:

Lemma 3.3. *Suppose that $K_1 \supset K_2 \supset \cdots$ are nested compact sets in $\mathbb{C}$, with $K = \bigcap_j K_j$. Then*

$$\operatorname{cap} K = \lim_{j\to\infty} \operatorname{cap} K_j.$$

Proof. Since $K \subset K_j$ for each j, it is clear that $\operatorname{diam}_n(K) \leq \operatorname{diam}_n(K_j)$ for all n and j, so that $\operatorname{cap} K \leq \operatorname{cap} K_j$. If the sets K_j for $j \gg 0$ have capacity 0, then there is nothing further to show. Otherwise, let μ_j be the unique equilibrium measure on K_j

for each j. Pass to a subsequence so that $\mu_j \to \nu$ weakly. Note that $\operatorname{supp}\nu \subset K$. As in (2.1), we have $\liminf E(\mu_j) \geq E(\nu)$. Therefore,

$$\begin{aligned}\operatorname{cap}(K) = \exp\left(-\inf_{\operatorname{supp}\mu\subset K} E(\mu)\right) &\geq \exp(-E(\nu)) \\ &\geq \limsup_{j\to\infty} \exp(-E(\mu_j)) = \limsup_{j\to\infty} \operatorname{cap}(K_j).\end{aligned}$$

□

See [Ra, §5.1] for more information about capacity as a function on sets.

The reverse implication of Theorem 3.1 is much trickier. Here we give a sketch; details can be found in [Ru2, Chapter 6]. Assume that $\operatorname{cap} S \geq 1$ and S is symmetric about $\mathbb{R}$. We first replace the set S with a small neighborhood U of S, so that there are only finitely many bounded connected components of $\mathbb{C} \setminus \overline{U}$. Note that $\operatorname{cap} U > \operatorname{cap} S$. By cutting small slits through U, symmetrically across $\mathbb{R}$, we can find a compact set $K \subset U$ with $\operatorname{cap} K > \operatorname{cap} S$ and symmetric about $\mathbb{R}$ for which $\mathbb{C} \setminus K$ is connected. Let μ_K be the equilibrium measure on K. Then the potential function V_{μ_K} will be $\equiv -E(\mu_K)$ on K and > 0 at all points of the complement $\mathbb{C} \setminus K$.

By the symmetry of K, for each n even, we can find a configuration of Fekete points in K (or "nearly" Fekete points) that are symmetric across $\mathbb{R}$. Form the monic polynomial of degree n with these points as roots; by perturbing slightly we obtain a polynomial P_n which has coefficients in $\mathbb{Q}$. It turns out that the function $\frac{1}{n}\log|P_n(z)|$ will converge (locally uniformly) to V_{μ_K} on the complement of $\overline{U}$. In particular, by fixing a very large n, there is $R > 1$ so that $\{|P_n(z)| \leq R\}$ is contained in U and contains S. The strategy is then to cleverly construct a perturbation Q of a certain large power $(P_n)^M$ so that Q is monic with *integer* coefficients but so that there is still an $R' > 1$ so that $\{|Q(z)| \leq R\}$ is contained in U. In particular, the set $\{|Q(z)| \leq 1\}$ lies in U, and so the complete set of roots of $f_m(z) := Q(z)^m - 1$ are contained in U.

3.2. **Rumely's extension of the Bilu theorem.** Rumely's proof of the Bilu equidistribution theorem gives more. For any compact set in the plane with $\operatorname{cap} S > 0$, we can define a "height function" h_S on $\mathbb{P}^1(\overline{\mathbb{Q}})$ by replacing the function $\log^+|\cdot|$ at the archimedean places with the Green's function of S, defined by

$$G_S(z) = V_{\mu_S}(z) + E(\mu_S)$$

where V_{μ_S} is the potential for the equilibrium measure on S. In other words, we can put

$$\begin{aligned} h_S(\alpha) &= \frac{1}{\deg \alpha}\left(\log|a_0| + \sum_{P_\alpha(x)=0} G_S(x)\right) \\ &= \frac{1}{[K:\mathbb{Q}]}\left(\sum_p \sum_{\sigma: K\hookrightarrow \overline{\mathbb{Q}}_p} \log^+|\sigma\alpha|_p + \sum_{\sigma:K\hookrightarrow\mathbb{C}} G_S(\sigma\alpha)\right) \end{aligned}$$

where $P_\alpha(x) \in \mathbb{Z}[x]$ is the minimal polynomial with leading coefficient a_0, and K is any number field containing α.

Note that $V_{\mu_S} + E(\mu_S)$ is 0 on S (except on a subset of capacity 0). Since $G_S(z) - \log^+|z|$ is uniformly bounded on $\mathbb{C}$, there is a constant C (depending on the set S) so that

$$|h(\alpha) - h_S(\alpha)| \le C$$

for all $\alpha \in \overline{\mathbb{Q}}$. This function h_S should therefore qualify as a height in the sense of Weil.

Theorem 3.4. (Rumely, 1999) *Suppose S is any compact set in $\mathbb{C}$ which is symmetric about $\mathbb{R}$ with $\operatorname{cap} S = 1$. Suppose α_n is a sequence of algebraic numbers with $h_S(\alpha_n) \to 0$. Then the measures*

$$\mu_n = \frac{1}{\deg \alpha_n} \sum_{P_n(z)=0} \delta_z$$

converge weakly to the equilibrium measure μ_S.

Moreover, if h denotes the standard Weil height, then the sequence $\{h(\alpha_n)\}$ also converges, with limit given by

$$\lim_{n\to\infty} h(\alpha_n) = \int \log^+|z|\, d\mu_S(z) = \int G_S(z)\, d\mu_{S^1}.$$

A careful examination of our proof of Bilu's theorem shows that it applies to give a proof of Theorem 3.4 as well. The "Moreover" sentence follows from the weak convergence of measures. Indeed, we consider the continuous function

$$f(z) = G_S(z) - \log^+|z|$$

on $\mathbb{C}$. Since S has capacity 1, we know that $G_S = V_{\mu_S}$, so that this function satisfies $f(z) \to 0$ as $z \to \infty$. Let φ be a continuous bump function supported on a large disk in $\mathbb{C}$. We have

$$\begin{aligned} h_S(\alpha_n) - h(\alpha_n) &= \frac{1}{\deg \alpha_n} \sum_{P_n(x)=0} \left(G_S(x) - \log^+ |x|\right) \\ &\approx \int_{\mathbb{C}} f(z)\varphi(z)\, d\mu_n \\ &\longrightarrow \int_{\mathbb{C}} f(z)\varphi(z)\, d\mu_S \\ &= \int_{\mathbb{C}} f(z)\, d\mu_S \\ &= \int_{\mathbb{C}} \log^+ |z|\, d\mu_S(z) \end{aligned}$$

where the $\approx$ is where we need to take care of the support of μ_n which lies outside a large disk. (So there are some ε's and estimates that need to be introduced.) Finally note that

$$\begin{aligned} \int_{\mathbb{C}} \log^+ |z|\, d\mu_S(z) &= \iint \log|z-w|\, d\mu_{S^1}(w)\, d\mu_S(z) \\ &= \iint \log|z-w|\, d\mu_S(w)\, d\mu_{S^1}(w) = \int_{\mathbb{C}} G_S(z)\, d\mu_{S^1}(z) \end{aligned}$$

by Fubini.

Remark 3.5. Because S is symmetric with $\operatorname{cap} S = 1$, the theorem of Fekete-Szegö applies to show that sequences exist with $h_S(\alpha_n) \to 0$. The Galois orbits of those algebraic integers are therefore uniformly distributed with respect to μ_S.

Remark 3.6. If $\operatorname{cap} S < 1$, then there is a positive constant $c = c(S) > 0$ so that the set

$$\{\alpha \in \overline{\mathbb{Q}} : h_S(\alpha) < c\}$$

is *finite*. Rumely's equidistribution theorem would be vacuous in this case. This can be proved by repeating the arguments in Rumely's proof. Suppose there exists an infinite sequence $\{\alpha_n\}$ with $h_S(\alpha_n) \to 0$, and let μ_n be the discrete measure supported uniformly on the Galois conjugates of α_n. We can pass to a subsequence so that $\mu_n \to \nu$ weakly; then $\operatorname{supp} \nu \subset S$. As before, we have $\liminf_{n\to\infty} E^*(\mu_n) \geq E(\nu)$. The difference now is that $E(\nu) \geq E(\mu_S) > 0$. Following the chain of inequalities at the end of the proof of Theorem 1.1, we find that this positivity violates the product formula for the discriminant values $D(\alpha_n)$ when n is sufficiently large. (An alternative proof can be obtained by copying the proof of the easy implication of Theorem 3.1, but now we need to be careful about the leading coefficient of the minimal polynomial in the determinant expression.)

Remark 3.7. For $\operatorname{cap} S > 1$, small sequences exist for this h_S, but equidistribution will fail, in the sense that different sequences can have different limits. In fact, we can find compact subsets $S_1, S_2 \subset S$ that both have capacity 1 and are symmetric about $\mathbb{R}$, but satisfy $\mu_{S_1} \neq \mu_{S_2}$. Then we can apply the Fekete-Szegö theorem to find sequences $\{\alpha_n^1\}$ and $\{\alpha_n^2\}$ of algebraic integers near S_1 and S_2, respectively. Then $h_{S_i}(\alpha_n^i) \to 0$ and $h_S(\alpha_n^i) \to 0$ for $i = 1, 2$. Applying Rumely's theorem to each S_i, we determine that $\{\alpha_n^1\}$ is uniformly distributed with respect to μ_{S_1} while $\{\alpha_n^2\}$ is uniformly distributed with respect to μ_{S_2}.

Remark 3.8. The symmetry of S is no real restriction for equidistribution, though it was needed in the Fekete-Szegö theorem because Galois orbits over $\mathbb{Q}$ are symmetric. For height theory, we can simply work over the field $K = \mathbb{Q}(i)$ and modify the height function accordingly. More about this later.

4. Basics of complex dynamics on the Riemann sphere

Let $\hat{\mathbb{C}}$ denote the Riemann sphere. The holomorphic maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are rational functions, of the form $P(z)/Q(z)$ where P and Q are polynomials with complex coefficients having no common roots. The degree of f is given by

$$\deg f = \max\{\deg P, \deg Q\}.$$

We are interested in the behavior of the sequence $\{f^n\}$ of iterates of f, where $f^n = f \circ f^{n-1}$ and $f^1 = f$. For example, we might look at the structure of **orbits** of points, the sequences $\{f^n(z_0)\}$ for a given $z_0 \in \hat{\mathbb{C}}$, or we might study the measures μ on $\hat{\mathbb{C}}$ that are **invariant** for f, meaning that $f_*\mu = \mu$. A fantastic reference for complex dynamics in dimension 1 is Milnor's book [Mi1]. In this course, we will only examine a small part of this subject, and I will try to highlight the facts we need for our study of equidistribution theorems.

4.1. The Julia set.

The **Fatou set** of f is the largest open subset $\Omega \subset \hat{\mathbb{C}}$ on which the iterates $\{f^n\}$ form a normal family. That is, given any closed disk $\overline{D} \subset \Omega$ and any sequence f^{n_k} of iterates with $n_k \to \infty$, there exists a subsequence of $\{f^{n_k}\}$ that converges uniformly on $\overline{D}$ (and so, necessarily, to a holomorphic limit on D). The **Julia set** $J(f)$ is the complement in $\hat{\mathbb{C}}$ of the Fatou set. By definition, the Fatou set is open, so $J(f)$ is closed. I will say that an open $U \subset \hat{\mathbb{C}}$ "is Fatou" if it is a subset of the Fatou set.

Theorem 4.1. *Suppose that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has degree $d > 1$. Then its Julia set $J(f)$ is nonempty.*

Proof. If the Fatou set were all of $\hat{\mathbb{C}}$, then there must be a sequence of iterates $g_k = f^{n_k}$ that converges uniformly on all of $\hat{\mathbb{C}}$. But this would imply that the (topological) degrees of g_k must be eventually constant. However $\deg g_k = d^{n_k} \to \infty$, a contradiction. □

While $J(f)$ is never empty in degrees > 1, §4.4 provides a family of examples with $J(f) = \hat{\mathbb{C}}$.

Example 4.2. The easiest example in degree > 1 is $f(z) = z^2$. We have that the open unit disk $\mathbb{D}$ is Fatou, because all iterates converge locally uniformly to the constant function 0 on $\mathbb{D}$. Similarly, the disk $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is Fatou, because the iterates converge locally uniformly to the constant function ∞ there. But any open set that intersects S^1 cannot be Fatou, because the convergence cannot be uniform to both 0 and ∞. So $J(f) = S^1$.

4.2. **Equivalence.** We say that two holomorphic maps $f, g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are equivalent if they are conjugate by a Möbius transformation. That is, the group of conformal automorphisms is $\operatorname{Aut}\hat{\mathbb{C}} \simeq \mathrm{PSL}_2\mathbb{C}$, the group of linear fractional transformations, and the equivalence is defined by the existence of some $A \in \operatorname{Aut}\hat{\mathbb{C}}$ for which $g = A \circ f \circ A^{-1}$. Note that this implies that $g^n = A \circ f^n \circ A^{-1}$ for all $n \geq 1$.

Example 4.3. For maps of degree 1, there are only 4 types, up to conjugacy. Specifically, every map of degree 1 is conjugate to exactly one of the following:

1. the identity map $f(z) = z$;
2. the translation $f(z) = z + 1$;
3. a hyperbolic map of the form $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| > 1$;
4. a rotation of the form $f(z) = \lambda z$ for some $\lambda \in \mathbb{C} \setminus \{1\}$ with $|\lambda| = 1$.

I leave this as an exercise! Hint: how many fixed points does a Möbius transformation have? If one, use conjugation to put it at infinity; if two, put them at 0 and ∞. Follow-up question: *For each of the above types, what is the Julia set and what is the Fatou set?*

4.3. **Periodic and preperiodic points.** A point $z_0 \in \hat{\mathbb{C}}$ is **periodic** if there exists an $n > 0$ so that $f^n(z_0) = z_0$, and we say it has **period** n. If n is the smallest such positive integer, we say z_0 has **exact period** n. We say z_0 is **preperiodic** if its orbit is finite, so that there must be $n > m \geq 0$ for which $f^n(z_0) = f^m(z_0)$.

Assume that $\deg f \geq 2$. As $\mathbb{C}$ is algebraically closed, there must by $d^n + 1$ solutions to the equation

$$f^n(z) = z$$

counted with multiplicity, for each $n \geq 1$.

The derivative of f^n, at each point z_0 of period n, determines the local behavior the dynamics of f near z_0. We say that z_0 is

- **attracting** if $|(f^n)'(z_0)| < 1$;
- **repelling** if $|(f^n)'(z_0)| > 1$; and
- **neutral** otherwise.

Working with a Taylor series expansion of f^n near z_0, it is not hard to see that

- If z_0 is attracting, then there is an open disk D around z_0 so that $\overline{f^n(D)} \subset D$, and $f^{mn}(z) \to z_0$ as $m \to \infty$, for all $z \in D$.
- If z_0 is repelling, then f^n is locally invertible near z_0, and, denoting that local inverse by g, there is an open disk D around z_0 for which $\overline{g(D)} \subset D$, and $g^m(z) \to z_0$ as $m \to \infty$, for all $z \in D$.

The neutral case is complicated, and I will come back to that later.

Theorem 4.4. *Attracting periodic points are in the Fatou set. Repelling periodic points are in the Julia set.*

Proof. For attracting periodic points, we have seen that there is a neighborhood of the cycle on which orbits converge (uniformly) to that cycle. So this means that a small disk around each attracting periodic point must be Fatou. For repelling periodic points of period n, we have $|(f^n)'(z_0)| = C > 1$ so that $|(f^{nm})'(z_0)| = C^m \to \infty$ as $m \to \infty$. So there can be no subsequences of $\{f^{mn}\}$ that converge uniformly in a neighborhood of z_0. $\square$

An important result that I will not prove here is:

Theorem 4.5. *For every rational function f of degree $d > 1$, the Julia set $J(f)$ is the closure of the repelling periodic points.*

See [Mi1, Chapter 14] for a proof.

4.4. **Example: Lattès maps.** Let E be a smooth elliptic curve in $\mathbb{P}^2(\mathbb{C})$, given in Legendre form by the equation

$$y^2 = x(x-1)(x-t)$$

for some $t \in \mathbb{C}\setminus\{0,1\}$. In this form, a point $P = (x, y) \in E$ has additive inverse given by $(x, -y)$. The quotient $\pi(x, y) = x$ defines a branched cover $\pi : E \to \mathbb{P}^1$ of degree 2, with critical points at the four 2-torsion points of E. The multiplication-by-two map $[2]$ on E, or in other words $[2]P := P + P$, descends to a function of degree 4 on $\mathbb{P}^1 = \hat{\mathbb{C}}$ that is given by the explicit formula,

$$f_t(x) = \frac{(x^2 - t)^2}{4x(x-1)(x-t)} \tag{4.1}$$

and satisfies $\pi \circ [2] = f_t \circ \pi$. (Explicit formulas of this sort are worked out in, for example, [Si5, Chapter III §2].) This f_t has interesting dynamics! It is an example of what we call a **Lattès map**, named after Samuel Lattès; these arise more generally as quotients of maps on elliptic curves.

It turns out that $J(f_t) = \hat{\mathbb{C}}$ for every choice of t, and the proof is quite simple. Indeed, the periodic points of f_t are dense in $\hat{\mathbb{C}}$, as they are the quotients of a dense set of torsion points of E. (*Exercise: A point* $\pi(P) \in \hat{\mathbb{C}}$ *is preperiodic for* f_t *if and only if the sequence* P *is preperiodic for* $[2]$ *on* E *if and only if* P *is torsion on* E*. Which orders of torsion points on* E *descend to periodic points for* f_t*, and which orders are only preperiodic for* f_t*? Hint: use the fact that* $\pi \circ [2] = f_t \circ \pi$*.*) Moreover, the periodic points z_0 of period n (for all but finitely many n) will satisfy $|(f_t^n)'(z_0)| = 2^n$, and they are all repelling. (To see this last fact, it is best to work on the universal cover of E_t, expressing the elliptic curve as $E_t = \mathbb{C}/\Lambda_t$ for a choice of lattice $\Lambda_t \subset \mathbb{C}$; the endomorphism $[2]$ lifts to the linear transformation $z \mapsto 2z$ on $\mathbb{C}$. The only reason that the derivative might fail to be $\pm 2^n$ at a periodic point in $\mathbb{P}^1$ is if the point is a critical value of the quotient $E \to \mathbb{P}^1$.) This shows that a dense set of points in $\hat{\mathbb{C}}$ must be in $J(f_t)$, and therefore $J(f_t) = \hat{\mathbb{C}}$ (because it is a closed set). We will revisit these examples in §9.3.

For more information about Lattès maps, see [Mi2].

4.5. **Local change of coordinates.** Suppose that h is a holomorphic function on a disk D centered at 0, and assume that $h(0) = 0$. Then we can express h as a power series on D as $h(z) = a_m z^m + a_{m+1} z^{m+1} + \cdots$ for some $m \geq 1$ with $a_m \neq 0$. In complex analysis, we learn that there exists a holomorphic and invertible function φ, defined on a small neighborhood of 0, with $\varphi(0) = 0$, so that $h(z) = a_m(\varphi(z))^m$. In other words, via a conformal coordinate change in the domain, given by $\zeta = \varphi(z)$, we can assume h is "equal to" the first term in its power series, as $h \circ \varphi^{-1}(\zeta) = a_m \zeta^m$. This is proved by writing

$$h(z) = a_m z^m \left(1 + \frac{a_{m+1}}{a_m} z + \cdots\right)$$

and observing that the term $1 + \frac{a_{m+1}}{a_m} z + \cdots$ is nonvanishing near 0, so we can take an m-th root holomorphically. Then set $\zeta = z\left(1 + \frac{a_{m+1}}{a_m} z + \cdots\right)^{1/m}$ for any choice of root.

It is much harder, however, to make a *dynamical* change of coordinates and find a conformal isomorphism φ defined near 0 so that

$$\varphi \circ h \circ \varphi^{-1}(\zeta) = a_m \zeta^m.$$

In fact, it is impossible in some cases, and it remains an open question to determine precise conditions on h for which such a φ exists.

Example 4.6. A holomorphic function $h(z) = z + \sum_{k \geq 2} a_k z^k$ near $z = 0$ can only be conformally conjugate to its leading term z if h itself is the identity. Indeed, any function conjugate to the identity must be the identity!

Improving on the description given in §4.3 of attracting and repelling periodic points, we can say this [Mi1, Chapter 8]:

Theorem 4.7 (Koenigs Linearization, 1884)**.** *Suppose that h is holomorphic on a disk D centered at 0, with $h(0) = 0$ and $h'(0) = \lambda$. Assume that $0 < |\lambda| < 1$ or that $|\lambda| > 1$. Then there exists a conformal φ defined near 0 so that $\varphi(0) = 0$ and $\varphi \circ h \circ \varphi^{-1}(\zeta) = \lambda\zeta$.*

Proof. First assume that $0 < |\lambda| < 1$. We will show that

$$\varphi(z) = \lim_{n\to\infty} \frac{1}{\lambda^n} h^n(z)$$

is well defined and conformal near $z = 0$. If the limit exists, note that $\varphi(h(z)) = \lim_{n\to\infty} \lambda^{-n} h^{n+1}(z) = \lambda\,\varphi(z)$, so it also gives the desired conjugacy.

Choose a real $c < 1$ so that $c^2 < |\lambda| < c$. From Taylor's theorem, we know that we can find a small disk of radius $r > 0$ around 0 and a constant $C > 0$ so that

$$|h(z)| \le c|z| \quad \text{and} \quad |h(z) - \lambda z| \le C|z|^2$$

for all $|z| < r$. It follows inductively that

$$|h^{n+1}(z) - \lambda h^n(z)| \le C|h^n(z)|^2 \le Cc^{2n}r^2,$$

so that

$$\left| \frac{h^{n+1}(z)}{\lambda^{n+1}} - \frac{h^n(z)}{\lambda^n} \right| \le \frac{Cr^2}{|\lambda|} \left(\frac{c^2}{|\lambda|} \right)^n$$

This tells us that the functions defining φ will converge uniformly on the disk of radius r. Moreover, the function φ must be holomorphic with $\varphi'(0) = 1$, so that φ is a conformal isomorphism on a neighborhood of $z = 0$.

For the case $|\lambda| > 1$, we apply the same argument to a locally-defined inverse function to h. □

4.6. **Neutral points in the Julia set.** Recall that Theorem 4.4 says nothing about the neutral periodic points. Nor does Theorem 4.7. Neutral points are sometimes in the Fatou set and sometimes in the Julia set. For example, if z_0 has period n and $(f^n)'(z_0) = \lambda$ is a root of unity, then z_0 must be in the Julia set. See [Mi1, Chapter 10] for details. If λ is not a root of unity, and if there exists a conformal change of coordinates so that f^n is conjugate to $z \mapsto \lambda z$ in a neighborhood of z_0 (as in the conclusion of Theorem 4.7), then the iterates $\{f^n\}$ will form a normal family on that neighborhood. So this z_0 will be in the Fatou set. When such a conjugacy exists near a point z_0 of period n, we say that f^n is **linearizable** near z_0. In general, we can say the following:

Theorem 4.8. *If f is a rational function of degree $d > 1$, and if z_0 is a neutral periodic point of period n, then z_0 is in the Fatou set of f if and only if f^n is linearizable near z_0.*

See [Mi1, Lemma 11.1] for a proof of this statement. Chapter 11 of [Mi1] provides a glimpse into the delicate study of existence of linearizations near fixed points.

4.7. **Most periodic points are repelling.** We conclude this section with another fundamental fact about rational functions over $\mathbb{C}$. We do not go into the proof here.

Theorem 4.9. *If f is a rational function of degree $d > 1$, then there are at most $2d-2$ non-repelling periodic cycles.*

In Chapter 13 of [Mi1], Milnor presents a proof that there are at most $6d-6$ non-repelling periodic cycles. Theorem 4.9, with the bound of $2d-2$, was first proved by Shishikura [Sh]. The book [BF] gives lots of helpful information about the methods of quasiconformal surgery in complex dynamics.

5. Polynomial dynamics in $\mathbb{C}$

Here we follow some of Chapter 9 of [Mi1].

5.1. **Ramification at infinity.** Observe that each polynomial f of degree $d \geq 2$ has a fixed point at ∞ that is maximally ramified. That is, in local coordinates w near ∞, we have $f(w) = \frac{1}{a_0} w^d + O(w^{d+1})$ where a_0 is the leading coefficient of f. In fact, this property characterizes the polynomials:

Proposition 5.1. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational function with a maximally ramified fixed point, then f is conjugate to a polynomial.*

Proof. By a Möbius change of coordinates, we can assume that the maximally ramified fixed point is at ∞. Because it is maximally ramified, there can be no other preimages of ∞, as every point has exactly d preimages counted with multiplicity. Thus, the function has no poles other than ∞ and so must be a polynomial. $\square$

5.2. **Polynomial Julia sets.** Fix $d \geq 2$, and let

$$f(z) = a_0 z^d + a_1 z^{d-1} + \cdots + a_d$$

be a polynomial of degree d. The **filled Julia set** of f is the set

$$K(f) = \left\{ z \in \mathbb{C} : \sup_n |f^n(z)| < \infty \right\}.$$

Theorem 5.2. *For each polynomial f of degree $d \geq 2$, the filled Julia set $K(f)$ is a compact subset of $\mathbb{C}$, the complement $B(f) = \hat{\mathbb{C}} \setminus K(f)$ is connected, and the Julia set is*

$$J(f) = \partial K(f).$$

Proof. First note that for any polynomial $f(z) = a_0 z^d + a_1 z^{d-1} + \cdots + a_d$, there is a radius $R > 0$ so that $|z| > R$ implies that $|f(z)| \geq 2|z|$ because the leading term $a_0 z^d$ will dominate the absolute value. In particular, this implies that $f^n(z) \to \infty$ whenever $|z| > R$. Set $U_R = \{\infty\} \cup \{|z| > R\}$, and put

$$B(f) = \bigcup_{n \geq 0} f^{-n}(U_R). \tag{5.1}$$

Then $B(f)$ is open, and is, by definition, the complement of $K(f)$ in $\hat{\mathbb{C}}$. This shows that $K(f)$ is a compact subset of $\mathbb{C}$. Suppose $B(f)$ has a connected component U that is bounded in $\mathbb{C}$; then, since the boundary of U must lie in $K(f)$, the Maximum Principle implies that the iterates $\{f^n\}$ are uniformly bounded on U, contradicting their being in $B(f)$. This shows that $B(f)$ is connected.

Now observe that the iterates $\{f^n\}$ converge uniformly to the constant function $\{\infty\}$ on U_R, so U_R is a subset of the Fatou set. Furthermore, for any disk D so that $\overline{D} \subset B(f)$, it takes only finitely many iterates to reach U_R, so D must also be Fatou. In other words, all of $B(f)$ lies in the Fatou set. On the other hand, on any open subset of $K(f)$, the iterates $\{f^n\}$ will be uniformly bounded. It follows from Montel's Theorem (Theorem 5.3) that the interior of $K(f)$ must then be Fatou. Finally, if U is an open set that intersects $\partial K(f)$, then there is an open subset of U on which $\{f^n\}$ converges uniformly to the constant ∞, but there are points in U on which the iterates are bounded; so this U cannot be Fatou. We conclude that $J(f) = \partial K(f)$. ☐

The set $B(f)$ defined by (5.1) is called the **basin of infinity** of f.

5.3. **Montel's Theorem (easy version).**

Theorem 5.3. *Suppose that $\mathcal{F}$ is a family of holomorphic functions on a domain $U \subset \mathbb{C}$. Assume there exists $M > 0$ so that $|f| \leq M$ on U for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family.*

Proof. Let D be a disk of radius r with closure contained in U. Let γ denote the boundary of D, taken counterclockwise. By the Cauchy Integral Formula, for any z, w within $r/2$ of the center of D, and any $f \in \mathcal{F}$, we have

$$\begin{aligned} |f(z) - f(w)| &= \frac{1}{2\pi} \left| \int_\gamma \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) d\zeta \right| \\ &= \frac{1}{2\pi} \left| \int_\gamma \frac{f(\zeta)(z-w)}{(\zeta - z)(\zeta - w)} \, d\zeta \right| \\ &\leq \frac{1}{2\pi} \frac{2\pi r M |z - w|}{(r/2)^2} \\ &= \frac{4M|z-w|}{r}. \end{aligned}$$

Now let K be any compact subset of U, and fix $\epsilon > 0$. Choose r smaller than the distance K to the boundary of U. Choose

$$\delta < \min\left\{\frac{r}{4}, \frac{\epsilon r}{4M}\right\}.$$

Thus for each $z, w \in K$ with $|z - w| < \delta$, the above estimate implies that

$$|f(z) - f(w)| \leq \frac{4M|z-w|}{r} < \frac{4M\delta}{r} < \epsilon$$

for all $f \in \mathcal{F}$. So $\mathcal{F}$ is equicontinuous on compact sets, and hence by the Arzela-Ascoli theorem is a normal family. $\square$

5.4. **Connected Julia sets.**

Theorem 5.4. *Let f be a polynomial of degree $d \geq 2$. The filled Julia set $K(f)$ and Julia set $J(f)$ are connected if and only if all of the critical points of f have bounded orbit. If $K(f)$ is disconnected, then it has infinitely many connected components.*

Proof. This is a consequence of the Riemann-Hurwitz formula from topology. For each $R > 0$, recall that U_R denotes the open disk $\hat{\mathbb{C}} \setminus \{|z| \leq R\}$ around ∞. For R large enough, we know that $U_R \subset B(f)$, and in fact $B(f) = \bigcup_n f^{-n}(U_R)$.

Assume first that all $d-1$ of the critical points of f lie in $K(f)$, so the map $f : B(f) \to B(f)$ only ramifies at ∞. Then $f : f^{-1}(U_R) \to U_R$ is a proper holomoprhic map (and so a branched cover) of degree d with exactly $d-1$ critical points at ∞, so that $\chi(f^{-1}(U_R)) = d\,\chi(U_R) - (d-1) = 1$. In other words, $f^{-1}(U_R)$ is connected and simply connected. Continuing inductively, we express $B(f)$ as an increasing union of open disks in $\hat{\mathbb{C}}$, so that $B(f)$ is also simply connected. This implies that $K(f)$ is connected. Therefore $J(f)$ is also connected, because $B(f)$ has no bounded connected components.

Now suppose there exists a critical point of f in $B(f)\setminus\{\infty\}$. Then at some minimal step n_0 we have that $f : f^{-n_0}(U_R) \to f^{-n_0+1}(U_R)$ is a branched cover of degree d with $> d-1$ ramification points. This implies that $\chi(f^{-n_0}(U_R)) < d\,\chi(f^{-n_0+1}(U_R)) - (d-1) = 1$. The Euler characteristic $\chi(f^{-n}(U_R))$ will continue to decrease as n grows. $\square$

5.5. **Böttcher's Theorem.**

Theorem 5.5. *Suppose h is a holomorphic function on a disk around 0 with power series expansion of the form*

$$h(z) = a_m z^m + a_{m+1} z^{m+1} + \cdots$$

with $a_m \neq 0$ and $m \geq 2$. Then there exists a conformal isomorphism φ from a small neighborhood U of 0 to a round disk around 0 such that

$$\varphi(h(z)) = \varphi(z)^m.$$

The function φ is unique up to multiplication by an $(m-1)$-st root of unity.

We call this φ the **Böttcher isomorphism** for h at 0.

Proof. Here is a sketch. First let $\varphi_0(z) = \zeta z$ for some ζ satisfying $\zeta^{m-1} = a_m$. Then

$$g(w) = \varphi_0 \circ h \circ \varphi_0^{-1}(w) = w^m + O(w^{m+1})$$

for w near 0. Then we compute inductively that

$$g^n(w) = w^{m^n} + \cdots = w^{m^n}(1 + O(z))$$

so that there is a well-defined branch of the m^n-th root satisfying

$$\psi_n(z) := (g^n(z))^{1/m^n} = z(1 + O(z)).$$

It can be shown that ψ_n converges to a univalent ψ on a neighborhood of the origin. Then set $\varphi = \psi \circ \varphi_0$. Note that

$$\psi_n(g(z)) = (g^{n+1}(z))^{1/m^n} = \left((g^{n+1}(z))^{1/m^{n+1}}\right)^m = \psi_{n+1}(z)^m$$

so this φ provides the desired conjugacy.

For uniqueness, we assume that φ_1 and φ_2 both satisfy $\varphi_i(h(z)) = \varphi_i(z)^m$. Then $\varphi_2 \circ \varphi_1^{-1}(w^m) = (\varphi_2 \circ \varphi_1^{-1}(w))^m$ for all w on a small disk around $w = 0$. Let $g = \varphi_2 \circ \varphi_1^{-1}$ and write it as a power series $g(w) = \sum_{n \geq 1} c_n w^n$. Then

$$g(w^m) = \sum_{n \geq 1} c_n w^{mn} = g(w)^m = (c_1)^m z^m + m c_1^{m-1} c_2 z^{m+1} + \cdots$$

Equating the coefficients, we see that $c_1^{m-1} = 1$ and all other coefficients will be 0. ☐

5.6. **Böttcher coordinates at ∞.** For a polynomial f of degree $d \geq 2$ with leading coefficient a_0, we can apply Theorem 5.5 to the holomorphic function

$$h(z) = 1/f(1/z) = \frac{1}{a_0} z^d + O(z^{d+1})$$

to show that f is conformally conjugate to z^d in a neighborhood of ∞. We let φ_f denote a Böttcher isomorphism that satisfies $\varphi_f(\infty) = \infty$.

In some cases, this conformal isomorphism near ∞ extends to the entire basin of ∞:

Theorem 5.6. *If $K(f)$ is connected, then the Böttcher isomorphism for f at ∞ extends to a conformal isomorphism $\varphi_f : B(f) \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ so that $\varphi_f(f(z)) = \varphi_f(z)^d$ for all $z \in B(f)$.*

In other words, the Böttcher isomorphism extends to give a Riemann map from (the simply-connected) $B(f)$ to a disk.

Proof ingredients. The key observation is the following: from Theorem 5.4, we know that the filled Julia set $K(f)$ is connected if and only if there are no critical points of f in $B(f) \setminus \{\infty\}$. Then, as Milnor proves in [Mi1, Theorem 9.3], the isomorphism φ extends to all of $B(f)$. This result is stated explicitly as [Mi1, Theorem 9.5]. ☐

But even when $K(f)$ is not connected, we have:

Theorem 5.7. *For any polynomial f of degree $d \geq 2$, the function*

$$G_f(z) = \log|\varphi_f(z)|$$

extends to a harmonic function on all of the basin $B(f) \setminus \{\infty\}$ and satisfies

$$G_f(f(z)) = d\, G_f(z).$$

The proof of Theorem 5.7 follows from the construction of φ_f. Indeed, we can define

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log|f^n(z)| \tag{5.2}$$

for all $z \in B(f)$. Note that the convergence is locally uniform on $U_R = \{|z| > R\}$ for some $R > 0$, and the limit must coincide with $\log|\varphi_f(z)|$ by the construction of φ_f. But, for every $z \in B(f)$, there is a small disk D and some iterate N so that $f^N(D) \subset U_R$, so the convergence is locally uniform on all of $B(f)$. Because each of the functions $\log|f^n(z)|$ is harmonic where $f^n(z) \neq 0$, we conclude that G_f is well-defined and harmonic on all of $B(f)$. The relation $G_f(f(z)) = d\, G_f(z)$ follows from the definition as a limit in (5.2).

6. Potential theory and polynomial dynamics

Let f be a polynomial of degree $d \geq 2$ with complex coefficients. Recall that we defined its filled Julia set by

$$K(f) = \{z \in \mathbb{C} : \sup_n |f^n(z)| < \infty\}$$

so that $J(f) = \partial K(f)$ is the Julia set. The basin of infinity is the open (and connected) set $B(f) = \hat{\mathbb{C}} \setminus K(f)$.

6.1. Capacity of the filled Julia set.

Let f be a polynomial of degree $d \geq 2$ with leading coefficient a_0. Fekete proved that, for any compact set $K \subset \mathbb{C}$, we have

$$\operatorname{cap} f^{-1}(K) = \frac{1}{|a_0|^{1/d}} \operatorname{cap}(K)^{1/d}.$$

See, for example, [Ra, Theorem 5.2.5]. Recall that, from the definition of the basin of infinity (5.1), we can express the filled Julia set as the nested intersection

$$K(f) = \bigcap_{n\geq 1} f^{-n}\left(\overline{D(0,R)}\right)$$

of preimages of a closed disk, for any sufficiently large R. Observe that the leading coefficent of f^n is $b_n = a_0^{1+d+d^2+\cdots+d^{n-1}}$. Since capacity of a nested intersection is the limit of the capacities of the sets (Lemma 3.3), we have that

$$\operatorname{cap} K(f) = \lim_{n\to\infty} \operatorname{cap}\left(f^{-n}\left(\overline{D(0,R)}\right)\right) = \lim_{n\to\infty} \frac{1}{|b_n|^{1/d^n}} R^{1/d^n} = |a_0|^{-1/(d-1)}. \tag{6.1}$$

In particular, a filled Julia set will always have positive capacity, and the capacity is 1 for monic polynomials f.

6.2. **Escape rate.** The escape-rate function for a polynomial f of degree $d \geq 2$ is defined by

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)| \tag{6.2}$$

where $\log^+ = \max\{\log, 0\}$. Note that this extends the definition given in (5.2) to all of $\mathbb{C}$.

Theorem 6.1. *Let f be any polynomial of degree $d \geq 2$. The escape-rate function G_f, defined by* (6.2)*, is continuous and subharmonic on $\mathbb{C}$, and it satisfies $G_f(z) \geq 0$ with $G_f(z) = 0$ if and only if $z \in K(f)$. Moreover, it is a potential function for the equilibrium measure μ_f on $K(f)$ and* $\operatorname{supp} \mu_f = J(f)$.

Proof. For each $n \geq 1$, the function $g_n(z) = d^{-n} \log^+ |f^n(z)|$ is continuous and subharmonic on $\mathbb{C}$, being the maximum of two subharmonic functions $d^{-n} \log |f^n(z)|$ and 0. (In fact, you may notice that the function g_n is a potential for the equilibirum measure on the lemniscate $K_n = \{z : |f^n(z)| \leq 1\}$, since it is 0 on K_n and positive and harmonic outside of K_n, growing like $\log |z|$ near ∞.)

These functions g_n converge uniformly to G_f on $\mathbb{C}$. Indeed, we first observe that there is a radius R and a constant $C > 1$ so that

$$C^{-1}|z|^d \leq |f(z)| \leq C|z|^d$$

for all $z \in \mathbb{C}$ with $|z| \geq R$. Therefore, we can find another constant C so that

$$\left|\log^+ |f(z)| - d \log^+ |z|\right| \leq C$$

for all $z \in \mathbb{C}$. Write

$$g_n(z) = \log^+ |z| + \sum_{j=1}^{n} \frac{1}{d^j} \left(\log^+ |f^j(z)| - d \log^+ |f^{j-1}(z)|\right)$$

for each $n \geq 1$. Then, for any $m > n$, we have

$$|g_m(z) - g_n(z)| = \left| \sum_{j=n+1}^{m} \frac{1}{d^j} \left(\log^+ |f^j(z)| - d \log^+ |f^{j-1}(z)|\right) \right| \leq \sum_{j=n+1}^{m} \frac{C}{d^j}.$$

This upper bound can be made as small as desired, taking m and n large, so we obtain the desired convergence.

It follows that G_f is continuous and subharmonic on $\mathbb{C}$, being a uniform limit of such functions. Note that G_f is, by definition, $\equiv 0$ on the set $K(f)$ of points with bounded orbits. And it is ≥ 0 at all points of $\mathbb{C}$. On the other hand, we know that G_f is harmonic on $B(f)$, as observed in Theorem 5.7. So, by the Maximum Principle (applied to the harmonic function $-G_f$ on $B(f) \setminus \{\infty\}$), the function G_f must be everywhere positive on $B(f)$.

Since we know that $\operatorname{cap} K(f) > 0$ from (6.1), there exists a unique equilibrium measure μ_f on the filled Julia set $K(f)$. Let

$$V_f(z) = \int \log|z - w| \, d\mu_f(w)$$

denote its potential function. Recall that, by Theorem 2.2, we know that V_f is constant nearly everywhere on $K(f)$. Consider the function $G_f - V_f$ on $\mathbb{C}$. It is bounded on all of $\mathbb{C}$ and harmonic away from $J(f) = \partial K(f)$. We know that it is constant on the interior of $K(f)$ and nearly everwhere on $J(f)$. Furthermore, the values of $G_f - V_f$ tend to a constant nearly everywhere as $z \to J(f)$ from within $B(f)$. It follows from the Maximum Principle that the function itself must be constant on $B(f)$ and is therefore constant (nearly everywhere) on all of $\mathbb{C}$. But as both G_f and V_f are subharmonic, we may conclude that $G_f - V_f$ is constant on $\mathbb{C}$. In particular, this implies that

$$\frac{1}{2\pi} \Delta G_f = \mu_f$$

in the sense of distributions.

Finally, recall that $\operatorname{supp} \mu_f \subset J(f) = \partial K(f)$, because μ_f is the equilibrium measure. But, in fact, we have that $\operatorname{supp} \mu_f = J(f)$ because G_f is $\equiv 0$ on $K(f)$ and > 0 on $B(f)$. Indeed, if G_f were harmonic in a neighborhood of some point $x \in J(f)$, then the fact that $G_f \geq 0$ everywhere and $G_f(x) = 0$ would imply by the Maximum Principle that $G_f \equiv 0$ on that neighborhood of x, which is nonsense. $\square$

7. Equidistribution for complex polynomials

Let f be a polynomial of degree $d \geq 2$. First a definition: we say that a point z_0 is **exceptional** for f if the set of iterated preimages $\bigcup_{n\geq 1} f^{-n}(z_0)$ of z_0 is finite. Note that ∞ is exceptional for every polynomial, because $f^{-1}(\infty) = \infty$. It turns out that there is at most one exceptional point for f in $\mathbb{C}$. And if such a point exists, then f is conjugate to z^d with the exceptional point at $z_0 = 0$. *Exercise: prove this. Hint: see the proof of* [Mi1, Lemma 4.9].

In the 1960s, Brolin proved a fundamental equidistribution result for complex polynomials:

Theorem 7.1. [Br] *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a polynomial of degree* $d \geq 2$*. Suppose that* z_0 *is not exceptional for* f*. Then the preimages* $f^{-n}(z_0)$ *are uniformly distributed with respect to the equilibrium measure* μ_f *on* $K(f)$ *as* $n \to \infty$*. That is, the discrete measures*

$$\mu_n = \frac{1}{d^n} \sum_{f^n(z)=z_0} \delta_z$$

converge weakly to μ_f *on* $\mathbb{C}$ *as* $n \to \infty$*.*

In the early 1980s, this result was strengthened (and a version was proved for all rational maps on $\hat{\mathbb{C}}$). A probability measure μ is **exceptional** for f if $\mu(\{z_0\}) > 0$ for some exceptional point z_0 of f.

Theorem 7.2. [Ly, FLM] *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a polynomial of degree* $d \geq 2$*. Let* μ *be any probability measure on* $\hat{\mathbb{C}}$ *that is not exceptional for* f*. Then the sequence of measures*

$$\frac{1}{d^n}(f^n)^*\mu$$

converges weakly to μ_f*.*

In particular, taking $\mu = \delta_{z_0}$ for a non-exceptional z_0 in Theorem 7.2, we recover Theorem 7.1.

Now we must be careful: measures can't usually be pulled back by a map! But here, as f is proper and of degree d, we can make sense of $f^*\mu$. Indeed, any holomorphic $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d > 0$ defines a push-forward operator on continuous functions on $\hat{\mathbb{C}}$ by

$$f_*\varphi(y) := \sum_{f(x)=y} \varphi(x)$$

where the preimages of y are counted with multiplicity. The action on measures is its dual:

$$\int_{\hat{\mathbb{C}}} \varphi(x)\, d(f^*\mu)(x) := \int_{\hat{\mathbb{C}}} f_*\varphi(y)\, d\mu(y).$$

Note that the measure $f^*\mu$ has total mass $d\,\mu(\hat{\mathbb{C}})$. *Exercise: Smooth measures* μ *can also be expressed as smooth differential forms, and this* $f^*\mu$ *coincides with the usual pullback of forms. Hint: see §3 of* [HP] *and §6 of the textbook of Bott and Tu.*

Remark 7.3. Recall from Theorem 4.5 that the repelling periodic points are dense in the Julia set. In fact, they are also equidistributed. In [Ly], Lyubich proves that, for any rational function R (which is not constant equal to an exceptional point of f, the solutions to equations

$$f^n(z) = R(z)$$

become uniformly distributed with respect to μ_f as $n \to \infty$. Taking $R(z) = z$ gives the equidistribution of periodic points.

7.1. **Brolin's proof of Theorem 7.1.** You will see parallels between this proof and Rumely's proof of arithmetic equidistribution. A key ingredient is Frostman's theorem, giving uniqueness of the energy-minimizing measure on the filled Julia set $K(f)$.

Without loss of generality, we may assume that f is a *monic* polynomial of degree $d \geq 2$. Then $\operatorname{cap} K(f) = 1$ and $E(\mu_f) = 0$, and the potential function V_{μ_f} coincides with the escape rate G_f defined in (6.2).

Fix any non-exceptional point $z_0 \in \mathbb{C}$. We define

$$\mu_n = \frac{1}{d^n} \sum_{f^n(z)=z_0} \delta_z.$$

Pass to a subsequence μ_{n_k} so that

$$\mu_{n_k} \to \nu$$

weakly. Since all points $z \notin K(f)$ tend to ∞ under iteration, it must be that their backward orbits tend to $K(f)$. In other words, the support of ν must lie in $K(f)$.

Brolin shows two things about this limiting measure ν:

(1) The potential function satisfies $V_\nu(z) \geq 0$ at all points $z \in \operatorname{supp} \mu_f = \partial K(f) = J(f)$, and
(2) we have $\operatorname{supp} \nu \subset J(f)$.

Assuming these two points, we can compute that

$$E(\nu) = -\int V_\nu \, d\nu \leq 0 = E(\mu_f).$$

By Frostman's Theorem (Theorem 2.2), we find that $\nu = \mu_f$. If (1) and (2) hold for every convergent subsequence of $\{\mu_n\}$, then we may conclude that $\mu_n \to \mu_f$.

To prove (1), consider the potential functions

$$V_k(z) = \int_{\mathbb{C}} \log|z-w| \, d\mu_{n_k}(w) = \frac{1}{d^{n_k}} \log|f^{n_k}(z) - z_0|$$

of μ_{n_k}. Since the orbit $\{f^n(z)\}$ is bounded for all $z \in K(f)$, we have

$$\limsup_{k\to\infty} V_k(z) \leq 0$$

for all $z \in K(f)$. Taking the μ_f-average of V_k, we have

$$\limsup \int V_k \, d\mu_f \leq \int \limsup V_k \, d\mu_f \leq 0$$

by Fatou's Lemma. On the other hand, since ν is supported in $K(f)$, we also have

$$\limsup \int V_k \, d\mu_f = \limsup \int V_{\mu_f} \, d\mu_{n_k} = 0$$

by Fubini. Therefore,

$$\limsup_{k\to\infty} V_k(z) = 0$$

at μ_f-a.e. point z of $K(f)$.

Observe that

$$V_\nu(z) = \int \log|z-w|\,d\nu(w) = \lim_{M\to\infty} \int \max\{\log|z-w|, -M\}\,d\nu(w)$$

by Monotone Convergence, and

$$\begin{aligned} V_k(z) = \int \log|z-w|\,d\mu_{n_k}(w) &\leq \int \max\{\log|z-w|, -M\}\,d\mu_{n_k}(w) \\ &\longrightarrow \int \max\{\log|z-w|, -M\}\,d\nu(w) \end{aligned}$$

as $k\to\infty$ by weak convergence, since the supports of the measures μ_n are uniformly bounded. We deduce that

$$V_\nu(z) \geq 0$$

at μ_f-a.e. point $z \in K(f)$. Since V_ν is subharmonic, it must be upper-semi-continuous, and therefore

$$V_\nu(z) \geq 0$$

on all of $\operatorname{supp}\mu_f = \partial K(f) = J(f)$. This proves (1).

So the proof hinges on showing (2), that $\operatorname{supp}\nu \subset J(f)$ for every convergent subsequence of $\{\mu_n\}$. Brolin appeals to the dynamics of f to show this; in modern language, we would say that he uses the classification of Fatou components to know that the measures ν cannot put positive mass on the interior of $K(f)$. I omit this argument, though it is important to know that Brolin's proof uses something more than just the Frostman Theorem.

Remark 7.4. From the potential theory alone, we can prove Theorem 7.1 for starting points z_0 in $J(f)$. Then, since $f^{-1}(J(f)) \subset J(f)$, any limit of the measures μ_n must be supported in $J(f)$. Ransford gives this proof with this extra hypothesis in [Ra, Theorem 6.5.8].

7.2. **Modern, more robust proof.**

Although I want to emphasize the potential-theoretic input in the proofs of Theorems 7.1 and 7.2, it is not necessary to use Frostman's theorem, if we work directly with the construction of G_f, and if we simply define μ_f by $\mu_f = \frac{1}{2\pi}\Delta G_f$. On the other hand, it is exactly the potential theory that allows us to deduce that μ_f *is* the equilibrium measure for $K(f)$.

Here, I sketch the proof given in [HP, §4]. We need two important facts from analysis.

Fact 7.5. *The operator Δ is continuous. More precisely, convergence of potentials in L^1_{loc} implies weak convergence of the measures.*

Fact 7.6. *The operator Δ commutes with f^*. That is, for the definition of the pullback of a measure given above, and for bounded subharmonic functions u, we have $f^*(\Delta u) = \Delta(f^* u)$ where $f^* u = u \circ f$.*

First assume that μ has bounded potentials. That is, there is a constant C so that the potential function

$$V_\mu(z) = \int_{\mathbb{C}} \log|z - w| \, d\mu(w)$$

satisfies

$$\left| V_\mu(z) - \log^+ |z| \right| \leq C$$

for all $z \in \mathbb{C}$. Set

$$b(z) := V_\mu(z) - \log^+ |z|.$$

For every $n \geq 1$, we have

$$\frac{1}{d^n} V_\mu(f^n(z)) = \frac{1}{d^n} \log^+ |f^n(z)| + \frac{1}{d^n} b(f^n(z)).$$

We know that the first term on the right-hand-side converges uniformly to the potential G_f for the equilibrium measure. The second term on the right-hand-side converges uniformly to 0. Combined with Fact 7.5, this tells us that

$$\frac{1}{2\pi} \Delta \left(\frac{1}{d^n} V_\mu \circ f^n \right) \longrightarrow \frac{1}{2\pi} \Delta G_f = \mu_f.$$

On the other hand, Fact 7.6 implies that

$$\frac{1}{2\pi} \Delta \left(\frac{1}{d^n} V_\mu \circ f^n \right) = \frac{1}{d^n} (f^n)^* \left(\frac{1}{2\pi} \Delta V_\mu \right) = \frac{1}{d^n} (f^n)^* \mu.$$

This completes the proof for μ with bounded potential.

To treat general measures, it suffices to consider $\mu = \delta_{z_0}$ for non-exceptional points z_0; its potential function is $V_\mu(z) = \log|z - z_0|$. (The potential for a general measure is the convolution of $\log|z|$ with μ.)

Let ν_k be a sequence of smooth probability measures supported on shrinking disks D_k around z_0, so that $\nu_k \to \delta_{z_0}$. Since each ν_k has bounded (in fact smooth) potential, we have

$$\frac{1}{d^n} (f^n)^* \nu_k \longrightarrow \mu_f$$

for each k as $n \to \infty$.

Lemma 7.7. [HP, Lemma 4.2] *For any $\delta > 0$, there exist n_0 and k_0 so that at least $(1-\delta) d^n$ elements of $f^{-n}(z_0)$ (counted with multiplicity) lie in a connected component of $f^{-n}(D_k)$ having diameter $< \delta$, for all $k \geq k_0$ and for all $n \geq n_0$.*

Proof idea. The key input is the **Koebe Distortion Theorem**, which implies that the image of a small round disk under a univalent map is close to being a round disk. It states that, for any univalent $h : \mathbb{D} \to \mathbb{C}$ with $h(0) = 0$ and $h'(0) = 1$, we have

$$\frac{|z|}{(1+|z|)^2} \leq |h(z)| \leq \frac{|z|}{(1-|z|)^2} \tag{7.1}$$

We apply this to suitably chosen inverse branches $h = f^{-n}$.

Since there are at most $d-1$ critical points of f in $\mathbb{C}$, counted with multiplicity, there is a finite N so that any critical points c in the backward orbit of z_0 will satisfy $f^n(c) = z_0$ for some $n < N$. Taking further preimages, we can therefore find locally univalent branches of $f^{-(n-N)}$ from a neighborhood of $w \in f^{-N}(z_0)$ to a neighborhood of $z \in f^{-n}(z_0)$ for $n > N$. Then, by Distortion, we know that these locally-defined inverse branches will take a small-enough disk to a nearly round disk. □

Now fix any continuous test function φ with compact support in $\mathbb{C}$. Fix $\varepsilon > 0$. Because it has compact support, the function φ is uniformly continuous, so there is a $\delta > 0$ so that $|z - w| < \delta \implies |\varphi(z) - \varphi(w)| < \varepsilon$. For this δ, we find n_0 and k_0 from Lemma 7.7. Then for all $n \geq n_0$ and all $k \geq k_0$, we know that most of the components of $f^{-n}(D_k)$ have diameter $< \delta$. This gives

$$\frac{1}{d^n}\left|\int_{\hat{\mathbb{C}}} \varphi \ \left((f^n)^*\nu_k - (f^n)^*\delta_{z_0}\right)\right| \leq \varepsilon + \delta \sup|\varphi|$$

for all $n \geq n_0$ and for each $k \geq k_0$. The ε on the right-hand-side comes from the components of $f^{-n}(D_k)$ with diameter bounded by δ, and the second term ($\delta \sup|\varphi|$) comes from the proportion of components that might have large diameter. Choosing $\varepsilon \to 0$ shows that the measures $d^{-n}(f^n)^*\delta_{z_0}$ must have the same weak limit as the sequence $d^{-n}(f^n)^*\nu_k$.

8. p-ADIC DYNAMICS: AN INTRODUCTION

8.1. p-adic metric.

Recall that, for each prime number p, we can equip $\mathbb{Q}$ with a non-archimedean absolute value defined by

$$\left|p^r \frac{a}{b}\right|_p = p^{-r}$$

when $a, b \in \mathbb{Z}$ are nonzero integers coprime to p. It satisfies the strong triangle inequality

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}.$$

We denote by $\mathbb{Q}_p$ the metric completion of $\mathbb{Q}$ with respect to the distance function $|x-y|_p$; it is a field, and elements are identified with

$$\mathbb{Q}_p = \left\{\sum_{n \geq n_0} a_n p^n : a_n \in \{0, 1, \ldots, p-1\}, n_0 \in \mathbb{Z}\right\}.$$

The ring of integers is

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \left\{ \sum_{n \geq 0} a_n p^n : a_n \in \{0, 1, \ldots, p-1\} \right\}$$

and has maximal ideal

$$m_p = \{x \in \mathbb{Z}_p : |x|_p < 1\} \quad \text{with} \quad \mathbb{Z}_p/m_p \simeq \mathbb{F}_p.$$

It is convenient to work with complete and algebraically closed fields, so we let $\mathbb{C}_p$ denote the completion of an algebraic closure $\overline{\mathbb{Q}}_p$; it is also algebraically closed. We have

$$\mathcal{O}_p = \{x \in \mathbb{C}_p : |x|_p \leq 1\} \text{ and } \mathfrak{m}_p = \{x \in \mathcal{O}_p : |x|_p < 1\}$$

with

$$k_p := \mathcal{O}_p/\mathfrak{m}_p \simeq \overline{\mathbb{F}}_p.$$

For each $a \in \mathbb{C}_p$ and $r > 0$, we set

$$D(a, r) = \{x \in \mathbb{C}_p : |x - a|_p < r\}$$

and

$$\overline{D}(a, r) = \{x \in \mathbb{C}_p : |x - a|_p \leq r\}.$$

We sometimes say $D(a, r)$ is "open" and $\overline{D}(a, r)$ is "closed", though each is both open and closed in the topology on $\mathbb{C}_p$.

Since $\mathbb{C}_p$ is metrically complete, a nested intersection of disks with shrinking radii $r_n \to 0$ will always have non-empty intersection, but it can happen that

$$\bigcap_{n \geq 0} \overline{D}(a_n, r_n) = \emptyset$$

when $r_n \to r > 0$, even if $\overline{D}(a_n, r_n) \supset \overline{D}(a_{n+1}, r_{n+1})$ for all n. (See examples in Chapter 2 of [Be].) Indeed, the topology on $\mathbb{C}_p$ fails to be locally compact. For example, sequences such as $a_n = p^{1/n}$ are bounded (and converge to 1 in absolute value) but have no convergent subsequence. Note that $\mathbb{C}_p$ is not a discretely-valued field; in fact, $|\mathbb{C}_p^*| = \{p^s : s \in \mathbb{Q}\} \subset \mathbb{R}$. Observe that, for any $a \in \mathbb{C}_p$ and $r > 0$, we have $D(a, r) = \overline{D}(a, r)$ if and only if $r \notin |\mathbb{C}_p^*|$.

The **chordal distance** on $\mathbb{P}^1(\mathbb{C}_p)$ is defined by

$$d_\sigma((x_1 : y_1), (x_2 : y_2)) = \frac{|x_1 y_2 - x_2 y_1|}{\max\{|x_1|, |y_1|\} \max\{|x_2|, |y_2|\}}.$$

8.2. **Classical Fatou and Julia sets.** Now suppose that

$$f(z) = \frac{P(z)}{Q(z)}$$

is a rational function with coefficients in $\mathbb{C}_p$ of degree $d > 1$. It defines a morphism $\mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p)$. Its **classical** (or **Type I**) **Fatou set** is defined as

$$\Omega_I(f) = \left\{x \in \mathbb{P}^1(\mathbb{C}_p) : \{f^n\} \text{ is equicontinuous on a disk containing } x\right\}.$$

Here, we work with the chordal distance d_σ, and recall that a family $\mathcal{F}$ of maps is equicontinuous if, for every $\varepsilon > 0$ there is a δ so that

$$d_\sigma(x, y) < \delta \implies d_\sigma(f(x), f(y)) < \varepsilon$$

for all $f \in \mathcal{F}$. Over $\mathbb{C}$, this definition coincides with the definition given in §4.1. The **classical Julia set** is

$$J_I(f) = \mathbb{P}^1(\mathbb{C}_p) \setminus \Omega_I(f).$$

Over $\mathbb{C}$, the Julia set is never empty; see Theorem 4.1. By contrast, here we can have empty Julia set, but we will see that the *Fatou set* is never empty!

Example 8.1. Let $f(z) = z^2$. As over $\mathbb{C}$, the sequence of iterates $\{f^n\}$ will converge, locally uniformly, to the constant 0 on the open disk $D(0, 1)$. Similarly, for any prime p, the iterates will converge locally uniformly to ∞ on $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}(0, 1)$. This shows that

$$\Omega_I(f) \supset D(0,1) \cup \left(\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}(0,1)\right)$$

Note that $|f(z)|_p = |z|_p$ for all $|z|_p = 1$. In fact,

$$|f^n(z) - f^n(z_0)|_p = \left|z^{2^n} - z_0^{2^n}\right|_p = \left|(z - z_0)(z + z_0)(z^2 + z_0^2) \cdots (z^{2^{n-1}} + z_0^{2^{n-1}})\right|$$

for all $n > 0$, so that

$$|f^n(z) - f^n(z_0)|_p \leq |z - z_0|_p$$

whenever z and z_0 are both of absolute value 1. It follows that the iterates $\{f^n\}$ are equicontinuous on a disk around z_0 for every $|z_0|_p = 1$. This shows that

$$J_I(f) = \emptyset.$$

Compare Example 4.2.

Example 8.2. Let $f(z) = z^2 - \frac{1}{9}$ and $p = 3$. Then f maps each of the closed disks

$$D_\pm := \overline{D}\left(\pm\frac{1}{3}, 1\right)$$

onto the larger disk $D_0 = \overline{D}(0, 3)$ with degree 1. All points $z \notin D_0$ will tend to ∞ under iteration (and in fact, uniformly on disks), because

$$|z| > \left|\frac{1}{3}\right|_3 \implies |f(z)|_3 = \left|z^2 - \frac{1}{9}\right|_3 = |z^2|_3.$$

It follows that the Julia set $J_I(f)$ lies in the union of D_+ and D_-. And in fact, taking preimages of D_+ and D_-, we find that $f^{-n}(D_0)$ consists of 2^n disks of radius 3^{-n+1} nested in D_+ and D_-. The Julia set $J_I(f)$ is homeomorphic to a Cantor set in $\mathbb{C}_p$. See [Be, Example 5.29].

8.3. **Good reduction.** Given $f : \mathbb{P}^1 \to \mathbb{P}^1$ with coefficients in $\mathbb{C}_p$, we can always write f in homogeneous coordinates as

$$f(x : y) = (P(x, y) : Q(x, y))$$

where $P, Q \in \mathcal{O}_p[x, y]$ are homogeneous polynomials of degree $d = \deg f$, and at least one of the coefficients of P or Q has absolute value 1. If this is the case, we say that $(P : Q)$ is a **normalized** presentation of f.

If $f = (P : Q)$ is a normalized presentation, then we say that f has **good reduction** if – after reducing the coeffiicients mod $\mathfrak{m}_p$ – the polynomials $\bar{P}$ and $\bar{Q}$ have no common zeroes in $\mathbb{P}^1(k_p)$.

Equivalently, writing $f = P/Q$ in local coordinates for a normalized presentation $(P : Q)$, this f will have good reduction if and only if the polynomials $\bar{P}$ and $\bar{Q}$ continue to satisfy $d = \max\{\deg \bar{P}, \deg \bar{Q}\}$ and have no common roots in the residue field k_p.

We say that f has **potential good reduction** if we can change coordinates by an element $A \in \mathrm{PSL}_2(\mathbb{C}_p)$ so that AfA^{-1} has good reduction. We say f has **bad reduction** if it does not have potential good reduction.

Example 8.3. Let $f(z) = 3z^2 + 1$ and $p = 3$. In z coordinates, we can take $P(z) = 3z^2 + 1$ and $Q(z) = 1$, so that $\bar{P}(z) = 1$ and $\bar{Q}(z) = 1$. So f fails to have good reduction, because $\max\{\deg P, \deg Q\} = 0 < 2 = \deg f$. But it has potential good reduction. Indeed, we can take $A(z) = 3z$ and observe that

$$AfA^{-1}(z) = 3(3(z/3)^2 + 1) = z^2 + 3,$$

which has good reduction.

Example 8.4. Let $f(z) = z^2 - \frac{1}{9}$ and $p = 3$, as in Example 8.2. In homogeneous coordinates, we can take $P(x, y) = 9x^2 - y^2$ and $Q(x, y) = 9y^2$. This example fails to have good reduction because $Q(x, y) = 0$ and so vanishes at all of the roots of $\bar{P}(x, y) = -y^2$. In fact, this f has bad reduction, but we need to know a bit more to be certain of that. It will follow from Theorem 8.6.

8.4. **The homogeneous resultant.** A homogeneous polynomial $P(x, y)$ of degree d, defined over an algebraically closed field K, can be factored into linear terms as

$$P(x, y) = \prod_{i=1}^{d} (\beta_i x - \alpha_i y).$$

Note that, as a form on $P^1(K)$, P vanishes at the d points $(\alpha_i : \beta_i)$. The homogeneous resultant of two such polynomials P and $Q = \prod_{j=1}^{d}(\delta_j x - \gamma_j y)$ is defined to be

$$\operatorname{Res}(P, Q) = \prod_i \prod_j (\delta_j \alpha_i - \gamma_j \beta_i).$$

The resultant vanishes if and only if P and Q share a root in $\mathbb{P}^1$. It can be expressed as a polynomial over $\mathbb{Z}$ of degree $2d$ in the coefficients of P and Q.

When $f = (P : Q)$ is normalized, we define

$$|\operatorname{Res}(f)|_p := |\operatorname{Res}(P, Q)|_p.$$

Note that the pair (P, Q) is unique up to multiplication by an element $\alpha \in \mathcal{O}_p^*$, so the absolute value is well defined, even if a value of $\operatorname{Res}(f)$ is not. Note also that $|\operatorname{Res}(f)|_p \leq 1$ for all f. *Exercise: we have* $|\operatorname{Res}(f)|_p = 1$ *if and only if* f *has good reduction.*

Proposition 8.5. *Suppose* $f = (P : Q)$ *is normalized. Then*

$$d_\sigma(f(z), f(w)) \leq |\operatorname{Res}(f)|_p^{-2} d_\sigma(z, w).$$

Proof. The key observation is that there are homogeneous polynomials $A_1, A_2, B_1, B_2 \in \mathcal{O}_p[x, y]$ of degree $d - 1$ such that

$$A_1(x, y)P(x, y) + B_1(x, y)Q(x, y) = \operatorname{Res}(f)\, x^{2d-1}$$

and

$$A_2(x, y)P(x, y) + B_2(x, y)Q(x, y) = \operatorname{Res}(f)\, y^{2d-1}.$$

See, for example, [Si4, Proposition 2.13]. Evaluating these polynomials at any point (x_0, y_0) with $\max\{|x_0|, |y_0|\} = 1$, we can deduce that

$$|\operatorname{Res}(f)| \leq \max\{|P(x_0, y_0)|, |Q(x_0, y_0)|\}.$$

Therefore, writing $z = (x_1, y_1)$ and $w = (x_2, y_2)$, both with $\max\{|x_i|, |y_i|\} = 1$, we have

$$\begin{aligned} d_\sigma(f(z), f(w)) &= \frac{|P(x_1, y_1)Q(x_2, y_2) - P(x_2, y_2)Q(x_1, y_1)|}{\max\{|P(x_1, y_1)|, |Q(x_1, y_1)|\} \max\{|P(x_2, y_2)|, |Q(x_2, y_2)|\}} \\ &\leq \frac{|P(x_1, y_1)Q(x_2, y_2) - P(x_2, y_2)Q(x_1, y_1)|}{|\operatorname{Res}(f)|^2} \end{aligned}$$

Viewing x_i and y_i as variables, observe that the polynomial $x_1 y_2 - x_2 y_1$ would be a factor of $P(x_1, y_1)Q(x_2, y_2) - P(x_2, y_2)Q(x_1, y_1)$, with the other factor being in $\mathcal{O}_p[x_1, y_1, x_2, y_2]$, so we have

$$|P(x_1, y_1)Q(x_2, y_2) - P(x_2, y_2)Q(x_1, y_1)| \leq |x_1 y_2 - x_2 y_1|,$$

concluding the proof. Details can be found in [Si4, Theorem 2.14]. □

In particular, maps with good reduction have $|\operatorname{Res}(f)| = 1$, so they are distance non-increasing. In particular, we see that Example 8.1 is an example of a more general phenomenon:

Theorem 8.6. *Potential good reduction implies empty classical Julia set.*

Proof. For maps with good reduction, we have $|\operatorname{Res}(f)| = 1$. Therefore,

$$d_\sigma(f(z), f(w)) \leq d_\sigma(z, w),$$

and so, by induction, we have

$$d_\sigma(f^n(z), f^n(w)) \leq d_\sigma(z, w),$$

for all $n \geq 1$. This shows that the sequence $\{f^n\}$ is equicontinuous on all of $\mathbb{P}^1(\mathbb{C}_p)$. □

9. p-ADIC DYNAMICS: PERIODIC POINTS AND LATTÈS EXAMPLES

9.1. **Classification of periodic points.** As for rational functions defined over $\mathbb{C}$, we classify the periodic points as follows. Suppose that f is a rational function defined over $\mathbb{C}_p$, and suppose that $z_0 \in \mathbb{P}^1(\mathbb{C}_p)$ satisfies $f^n(z_0) = z_0$. Work in local coordinates where $z_0 \neq \infty$. We say that z_0 is

- **attracting** if $|(f^n)'(z_0)|_p < 1$;
- **repelling** if $|(f^n)'(z_0)|_p > 1$; and
- **neutral** if $|(f^n)'(z_0)|_p = 1$.

We have the following analog of Theorem 4.4. Here, neutral cycles are much easier to handle:

Theorem 9.1. *Attracting and neutral periodic points are in the classical Fatou set. Repelling periodic points are in the classical Julia set.*

Proof. If z_0 has period n with $\lambda = (f^n)'(z_0)$, we can expand f^n as a power series near z_0 as

$$f^n(z) = z_0 + \lambda(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

As over $\mathbb{C}$, there is a neighborhood around z_0 on which the leading term will dominate.

For attracting periodic points, this shows that there is a neighborhood of the cycle on which orbits converge (uniformly) to that cycle. So this means that a small disk around each attracting periodic point must be Fatou.

For neutral periodic points, we find that $|f^n(z) - z_0| = |z - z_0|$ for all z close enough to z_0. This implies – by a Schwarz-type lemma that we will discuss in Section 10 – that the iterates are uniformly Lipschitz (with Lipschitz constant 1) near the cycle of z_0, and so they must be equicontinuous, thus in the Fatou set.

For repelling periodic points of period n, we have $|(f^n)'(z_0)|_p = |\lambda|_p > 1$ so that $|(f^{nm})'(z_0)| \to \infty$ as $m \to \infty$. But why does this mean that the sequence of iterates cannot be equicontinuous on a neighborhood of z_0? Using the power series expansion of f^n, we can find a small disk $D(z_0, r)$ so that every $z \neq z_0$ in $D(z_0, r)$ will have an iterate $f^{nm}(z) \notin D(z_0, r)$. This is enough to show that the sequence $\{f^{nm}\}_m$ is not equicontinuous. □

9.2. **Detecting bad reduction.** Given $f : \mathbb{P}^1 \to \mathbb{P}^1$ with degree $d > 1$ and defined over $\mathbb{C}_p$, if it fails to have good reduction, how can we determine if f has bad reduction or if there is a change of coordinates that might have good reduction?

Combining Theorem 9.1 with Theorem 8.6, we have:

Corollary 9.2. *Suppose $f : \mathbb{P}^1 \to \mathbb{P}^1$ is defined over $\mathbb{C}_p$ and has degree $d > 1$. If f has a repelling periodic point, then f has bad reduction.*

Example 9.3. Fix prime $p \neq 2$ and consider $f(z) = z^2 + c$. Then f has bad reduction if and only if $|c|_p > 1$. For $|c|_p \leq 1$, we see from the definition that f has good reduction. For $|c|_p > 1$, we compute that f has a fixed points at

$$x_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-4c}\right)$$

each having absolute value $|x_\pm|_p = |c|_p^{1/2}$. Therefore $|f'(x_\pm)|_p = |2x_\pm|_p = |c|_p^{1/2} > 1$. So f has repelling fixed points and cannot have potential good reduction.

Not all maps with bad reduction will have repelling cycles, and it turns out there is another way (in fact, one that can be checked in practice) to determine if f has bad reduction:

Theorem 9.4. [Be, Theorem 4.20] *Suppose $f : \mathbb{P}^1 \to \mathbb{P}^1$ is defined over $\mathbb{C}_p$ and has degree $d > 1$. Suppose f has no repelling fixed points. If all of its fixed points are attracting, then they must be distinct; choose any 3 and label them as z_1, z_2, z_3. If f has an indifferent fixed point, then call it z_1, and choose distinct points z_2 and z_3 so that $f(z_3) = z_2$ and $f(z_2) = z_1$.*

Now choose linear fractional transformation $A \in \mathrm{PSL}_2(\mathbb{C}_p)$ sending $\{z_1, z_2, z_2\}$ to points lying in three distinct residue classes in $\mathbb{P}^1(k_p)$. (For example, we could send them to $0, 1, \infty$.) Then f has potential good reduction if and only if AfA^{-1} has good reduction.

Sketch of proof. We need a few more general facts about power series and transformations over $\mathbb{C}_p$ to make this rigorous, but here is the idea. If AfA^{-1} has good reduction, then f has potential good reduction by definition. So assume, for the converse implication, that f has potential good reduction. Choose $B \in \mathrm{PSL}_2(\mathbb{C}_p)$ so that BfB^{-1} has good reduction. By a further change of coordinates in $\mathrm{PSL}_2\mathcal{O}_p$, we can assume that none of $B(z_1), B(z_2), B(z_3)$ have absolute value > 1 while maintaining good reduction. So it suffices to show that the images $B(z_1), B(z_2), B(z_3)$ lie in distinct residue classes in $\mathbb{P}^1(k_p)$; for then the conjugation of f by A will have the same reduction type, as AB^{-1} will be in $\mathrm{PSL}_2(\mathcal{O}_p)$. (We will see more about this action when we discuss the Berkovich space.)

If z_1 is an indifferent fixed point for f, then $B(z_1)$ is an indifferent fixed point for $g := BfB^{-1}$. Since g has good reduction, this implies that the residue class $[B(z_1)]$ is fixed by $\bar{g}$, and g has degree 1 when restricted to the disk $D_1 := D(B(z_1), 1)$. See [Be, Theorem 4.18]. In particular, $B(z_2)$ cannot be in the same residue class, because $g(B(z_2)) = B(z_1)$. Thus g maps $D_2 = D(B(z_2), 1)$, which is disjoint from D_1, to D_1. It follows that $B(z_3)$ must lie in a third distinct residue class, because $g(D_1) = D_1$ with degree 1 and $g(D_2) = D_1$, while $g(B(z_3)) = B(z_2)$.

Note that a map f of degree $d \geq 2$ will have $d + 1$ fixed points counted with multiplicity. If all of the fixed points of f are attracting, then they cannot be multiple, because multiple fixed points must be neutral: they satisfy $f'(z_0) = 1$. Suppose that z_1, z_2, z_3 are attracting fixed points for f, so the same will be true for the three fixed points $B(z_i)$ of $g = BfB^{-1}$. Being fixed points, the residue class $[B(z_i)]$ of each must be fixed by $\bar{g}$. As detailed in [Be, Theorem 4.18], g must map the disk $D(B(z_i), 1)$ onto itself with degree $k_i > 1$. But then a Schwarz-type lemma (as we will discuss in Section 10) implies that there is a unique attracting fixed point in each of these disks. □

9.3. **Lattès examples.** Recall the description of the Lattès maps coming from elliptic curves, in §4.4. Working similarly with elliptic curves defined over $\mathbb{C}_p$, we obtain maps on $\mathbb{P}^1$ induced from their endomorphisms.

Theorem 9.5. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be any Lattès map defined over $\mathbb{C}_p$ with degree $d > 1$. The classical Julia set $J_I(f)$ is empty.*

Remark 9.6. Theorem 9.5 holds independent of the reduction type of f. The map f will have bad reduction if and only if the elliptic curve E has multiplicative reduction.

Example 9.7. For each $t \in \mathbb{C}_p \setminus \{0, 1\}$, recall that multiplication-by-2 on the elliptic curve $y^2 = x(x-1)(x-t)$ induces, via $(x, y) \mapsto x$, the rational function f_t with formula (4.1). For $p \neq 2$, this f_t has bad reduction if and only if $|t(t-1)|_p \neq 1$. Indeed, the point ∞ is an indifferent fixed point, and $f_t(0) = \infty$ and $f_t(\sqrt{t}) = 0$. We apply the criterion from Theorem 9.4. In homogeneous coordinates, we can write

$$f(z : w) = ((z^2 - tw^2)^2 : 4zw(z - w)(z - tw)).$$

This is normalized if and only if $|t|_p \leq 1$. In this case, the reduction is good if and only if $|t(t-1)|_p = 1$. For $|t|_p > 1$, a normalized presentation is given by

$$f(z : w) = \left(\left(\frac{1}{t} z^2 - w^2 \right)^2 : \frac{4}{t} zw(z - w) \left(\frac{1}{t} z - w \right) \right);$$

this has bad reduction.

Sketch proof of Theorem 9.5. A rigorous proof for all Lattès maps will be given after we discuss Berkovich spaces. But here I mention the key ideas for the examples

$$f_t(z) = \frac{(z^2 - t)^2}{4z(z-1)(z-t)},$$

coming from the multiplication-by-2 endomorphisms on the elliptic curve $E_t = \{y^2 = x(x-1)(x-t)\}$. As explained in Example 9.7, this f has good reduction for $|t(t-1)|_p = 1$. So the Julia set $J_I(f)$ is empty for these t.

Assume $|t|_p < 1$. The strong triangle inequality for $|\cdot|_p$ shows that $|f_t(z)|_p = |z|_p$ for all $|z|_p > 1$. In particular, the disk $U = \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}(0,1)$ satisfies $f_t(U) \subset U$, so it must be contained in the Fatou set. (This uses the Schwarz-type lemma that will be discussed in Section 10.) Similarly, repeated use of the strong triangle inequality implies that any small enough disk $D(a,r)$ will either map to U under a finite number of iterates (and so be Fatou) or we will have $f_t^n(D(a,r)) \subset \overline{D(0,1)}$ for all $n \geq 0$. This will imply that every disk is Fatou. Therefore $J_I(f) = \emptyset$.

The argument for $|t-1|_p < 1$ and $|t|_p > 1$ is similar. □

10. p-ADIC POWER SERIES AND DISKS

Here we follow some of Chapters 3 and 14 in [Be]. A basic fact about p-adic series is that, given any sequence $a_n \in \mathbb{C}_p$, we have

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \lim_{n\to\infty} a_n = 0.$$

So, given $z_0 \in \mathbb{C}_p$ and sequence $a_n \in \mathbb{C}_p$, a power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)$$

naturally converges on a disk (possibly of radius 0 or ∞). We have

$$\text{for all } r > 0, \quad \lim_{n\to\infty} |a_n| r^n = 0 \implies f \text{ converges on } \overline{D}(z_0, r) \tag{10.1}$$

and,

$$\text{if } r \in |\mathbb{C}_p^*|, \quad f \text{ converges on } \overline{D}(z_0, r) \implies \lim_{n\to\infty} |a_n| r^n = 0. \tag{10.2}$$

10.1. **Ring of convergent power series.** Suppose that D is a disk in $\mathbb{C}_p$ of radius $r \in |\mathbb{C}_p^*|$. It may be open or closed. Let $\mathcal{A}_D$ denote the ring of power series convergent on D. Fix $z_0 \in D$. Every element $f \in \mathcal{A}_D$ can be written as

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n.$$

Convergence is independent of the choice of center z_0; see [Be, Proposition 3.3] and its proof in his §14.1.

For any disk D' (open or closed) contained in D and of radius r', with $z_0 \in D'$, we define

$$\|f\|_{D'} = \sup_{n \geq 0} \{|a_n|(r')^n\}. \tag{10.3}$$

If D' is closed, or if $r' < r$, the norm will be finite for all f in $\mathcal{A}_D$, and the supremum is achieved. But it can be infinite if $r' = r$ and D is open. That is, there exist convergent power series on an open disk where the supremum in (10.3) is infinite. *Exercise: check that this is well defined: choosing a different center* $z_0 \in D'$ *leads to the same value of* $\|f\|_{D'}$.

Proposition 10.1. [Be, Proposition 3.7] *Suppose that* D *is a disk (open or closed) in* $\mathbb{C}_p$ *of radius* $r \in |\mathbb{C}_p^*|$. *For closed disks* $D' \subset D$, $\|f\|_{D'}$ *defines a multiplicative, non-archimedean norm on* $\mathcal{A}_D$. *That is, we have* $\|f\|_{D'} < \infty$ *for all* $f \in \mathcal{A}_D$, *and*

1. $\|f\|_{D'} = 0 \iff f = 0$;
2. $\|cf\|_{D'} = |c|_p \|f\|_{D'}$ *for all* $c \in \mathbb{C}_p$ *and* $f \in \mathcal{A}_D$;
3. $\|f + g\|_{D'} \leq \max\{\|f\|_{D'}, \|g\|_{D'}\}$ *for all* $f, g \in \mathcal{A}_D$; *and*
4. $\|fg\|_{D'} = \|f\|_{D'}\|g\|_{D'}$ *for all* $f, g \in \mathcal{A}_D$.

Proof. First note that, for $r' < r$, the convergence of $f \in \mathcal{A}_D$ implies that

$$\lim_{n \to \infty} |a_n|(r')^n = 0$$

so the supremum in the definition of $\|f\|_{D'}$ is achieved. For $r' = r$, since we required that D' be closed, this means that D was closed, so we again have that the supremum is achieved in the definition of $\|f\|_{D'}$.

To prove (1), it is clear that $\|f\|_{D'} = 0$ if and only if $a_n = 0$ for all n, so $f = 0$.

For (2), it is clear from the definitions.

For (3), assume $z_0 \in D'$, and write $f(z) = \sum a_n (z - z_0)^n$ and $g(z) = \sum b_n (z - z_0)^n$, so that

$$f + g = \sum_n (a_n + b_n)(z - z_0)^n$$

and

$$\max_n \{|a_n + b_n|(r')^n\} \leq \max_n \{\max\{|a_n|, |b_n|\}(r')^n\} \leq \max\{\|f\|_{D'}, \|g\|_{D'}\}.$$

The proof of (4) is exactly as for Gauss's Lemma, used in the proof of Proposition 1.2, though that was for the case where f and g are polynomials and $r' = 1$. The coefficient of $(z - z_0)^n$ in fg is $a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$, so

$$\begin{aligned}\|fg\|_{D'} &= \max_n\{|a_0b_n + \cdots a_nb_0|(r')^n\} \\ &\leq \max_n\{\max\{|a_0b_n|, \ldots, |a_nb_0|\}(r')^n\} \\ &= \max_n\left\{\max_{j\leq n}\{(|a_j|(r')^j)(|b_{n-j}|(r')^{n-j})\}\right\} \\ &\leq \|f\|_{D'}\|g\|_{D'}\end{aligned}$$

On the other hand, let n_0 be the smallest integer so that $\|f\|_{D'} = |a_{n_0}|(r')^{n_0}$, and let m_0 be the smallest integer so that $\|g\|_{D'} = |b_{m_0}|(r')^{m_0}$. Then the product $a_{n_0}b_{m_0}$ appears as a summand in the coefficient of $(z - z_0)^{n_0+m_0}$ and it is strictly larger in absolute value than the other terms. This implies that the coefficient will have absolute value $= |a_{n_0}b_{m_0}|$ and therefore

$$\|fg\|_{D'} \geq |a_{n_0}b_{m_0}|(r')^{n_0+m_0} = \|f\|_{D'}\|g\|_{D'}.$$

□

10.2. **Weierstrass degree and disk images.** Fix radius $r \in |\mathbb{C}_p^*|$, and point $z_0 \in \mathbb{C}_p$. Let $D = D(z_0, r)$ and $\overline{D} = \overline{D}(z_0, r)$. Assume that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is convergent on $\overline{D}$. Note that, by definition, $\|f\|_D = \|f\|_{\overline{D}}$, and convergence on $\overline{D}$ implies that the supremum in (10.3) is achieved and $\lim_{n\to\infty} |a_n|r^n = 0$. The **Weierstrass degrees** of f are

$$\bar{d}(f) = \max\{n : |a_n|r^n = \|f\|_D\}$$

and

$$\underline{d}(f) = \min\{n : |a_n|r^n = \|f\|_D\}$$

Theorem 10.2. [Be, Theorem 3.15 & Proposition 3.17] *Assume that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is convergent on $\overline{D} = \overline{D}(z_0, r)$ *with* $r \in |\mathbb{C}_p^*|$. *Then*

(1) $\|f\|_D = \sup\{|f(x)|_p : x \in D\} = \sup\{|f(x)|_p : x \in \overline{D}\}$
(2) *$f(D)$ is an open disk of radius $\|f - a_0\|_D$, and $f(\overline{D})$ is a closed disk of radius $\|f - a_0\|_D$.*
(3) *$f|D$ is $\underline{d}(f - a_0)$-to-one, and $f|\overline{D}$ is $\bar{d}(f - a_0)$-to-one.*

Proof idea. This theorem is a consequence of the non-archimedean **Weierstrass Preparation** theorem: any power series f can be expressed as a product $P\,h$ on $\overline{D}$, where P is a polynomial of degree $\bar{d}$ having all its zeroes in $\overline{D}$, and h is a convergent power series that does not have any zeroes [Be, Theorem 14.2].

We apply this to the given power series $f(z) - a_0 = \sum_{n\geq 1} a_n(z-z_0)^n$, to show that $f - a_0$ has exactly $\bar{d}(f - a_0)$ zeroes, counted with multiplicity.

Furthermore, for any $c_0 \in \mathbb{C}_p$ with $|c_0 - a_0|_p \leq \|f - a_0\|_D$, we can apply Weierstrass Preparation to $f - c_0$. Note that its 0th coefficient is chosen small enough so that

$$\bar{d}(f - c_0) = \bar{d}(f - a_0).$$

So $f - c_0$ has the same number of zeroes. On the other hand, for any c_0 with $|c_0 - a_0|_p > \|f - a_0\|_D$, the 0th coefficient will dominate the norm, and we will have $\bar{d}(f - c_0) = 0$, this implies that $f - c_0$ has no zeroes in $\overline{D}$. This proves (2) and (3) for $\overline{D}$.

To see the same results for the open disk D, we express D as an increasing union of closed disks. The important observation is that $\underline{d}(f-a_0)$ on D is equal to $\bar{d}(f-a_0)$ on disks of radius just smaller than r. So Weierstrass Preparation will tell us that there are $\underline{d}(f-a_0)$ zeroes of $f-a_0$ on D, and similarly for $f-c_0$ with $|c_0-a_0|_p \leq \|f-a_0\|_D$.

Finally, to see statement (1), first suppose that $\bar{d}(f) = 0$. Then $\|f\|_D = |a_0|_p$ and $\|f - a_0\|_D < \|f\|_D$, so $|f(x)|_p = |a_0|_p$ for all x. If $\bar{d}(f) > 0$, then $\bar{d}(f) = \bar{d}(f - a_0)$ and $|a_0|_p \leq \|f\|_D = \|f - a_0\|_D$, so that $f(\overline{D}) = \overline{D}(a_0, \|f\|_D)$. This shows that $\|f\|_D = \sup\{|f(x)|_p : x \in \overline{D}\}$. The same bound holds for the open disk. □

Remark 10.3. The conclusion of Theorem 10.2 also holds if we only assume f is convergent on an open disk, as long as $\underline{d}(f - a_0)$ is finite. It is possible for the Weierstrass degree on an open disk to be infinite, if the supremum in (10.3) is not achieved. If $\sup_n\{|a_n|r^n\}$ is infinite, then f maps D onto all of $\mathbb{C}_v$ with infinite degree. If $R = \sup_{n\geq 1}\{|a_n|r^n\}$ is finite but the supremum is not achieved, then f maps D to a disk of radius R but with infinite degree. See Exercise 3.13 in [Be].

Theorem 10.4 (Schwarz-type Lemma)**.** *Suppose $f(z) = \sum_{n\geq 0} a_n(z-z_0)^n$ is convergent on a disk D_1 of radius r_1 (which can be open or closed) and such that f maps D_1 with degree $d \geq 1$ to a disk D_2 of radius r_2. Then for all $x, y \in D_1$, we have*

$$|f(x) - f(y)|_p \;\leq\; \frac{r_2}{r_1}\,|x - y|_p.$$

Moreover, if $d = 1$, then equality holds for all $x, y \in D_1$.

Proof. Since f has finite degree, we know that

$$r_2 = \|f - a_0\|_{D_1} = \max_{n\geq 1}\{|a_n|(r_1)^n\}$$

from Theorem 10.2 and Remark 10.3. Then

$$\begin{aligned}
|f(x)-f(y)|_p &= \left|\sum_{n\geq 1} a_n(x^n-y^n)\right|_p \\
&= \left|\sum_{n\geq 1} a_n\left(\sum_{j<n} x^j y^{n-1-j}\right)\right|_p |x-y|_p \\
&\leq \max_{n\geq 1}\left(|a_n|_p \max_{j<n}\{|x^j|_p|y^{n-1-j}|_p\}\right)|x-y|_p \\
&\leq \max_{n\geq 1}\left\{|a_n|_p\,(r_1)^{n-1}\right\}|x-y|_p \\
&= \frac{r_2}{r_1}\,|x-y|_p
\end{aligned}$$

Now suppose that $d=1$. If D_1 is a closed disk, then d is equal to $\bar{d}(f-a_0)$, and $r_2=|a_1|_p r_1 > |a_n|_p(r_1)^n$ for all $n>1$. This implies that $|a_1|_p > |a_n|_p r^{n-1}$ for all $n>1$, so that both of the inequalities above are actually equalities, and $r_2/r_1=|a_1|_p$. If D_1 is an open disk, then d is equal to $\underline{d}(f-a_0)$, and $r_2=|a_1|_p r_1 \geq |a_n|_p(r_1)^n$ for all $n>1$. But for all $r<r_1$, we have $|a_1|_p \geq |a_n|_p(r_1)^{n-1} > |a_n|_p r^{n-1}$ for $n>1$, so again we see that the two inequalities above are equalities. $\square$

11. The Berkovich space over $\mathbb{C}_p$

We are now ready to define the Berkovich affine line $\mathbb{A}_p^{1,an}$ over the field $\mathbb{C}_p$ and the associated Berkovich projective line $\mathbb{P}_p^{1,an}$. We follow Chapter 6 of [Be] and give references there.

11.1. **What's wrong with $\mathbb{P}^1(\mathbb{C}_p)$?** Working with the complete and algebraically closed field $\mathbb{C}_p$ has many advantages, but – topologically – it also has many drawbacks. The most obvious is that it fails to be locally compact. In closed disks of any size around any point, there are sequences of points that have no convergent subsequences. In fact, closed disks can always be covered by infinitely many open disks without a finite subcover. Indeed, recall that the residue field $k_p=\mathcal{O}_p/\mathfrak{m}_p \simeq \overline{\mathbb{F}}_p$ is infinite. Each residue class forms an open disk of radius 1 in $\mathbb{C}_p$, disjoint from all the others. So, for any $r\in|\mathbb{C}_p^*|$ and looking in $\overline{D}(z_0,r)$, the union of disks $rD+z_0$, as D ranges over all the residue classes, forms an open cover of $\overline{D}(z_0,r)$ having no finite subcover. We can choose sequences in $\overline{D}(z_0,r)$ having one element in each of these disks, and all points will be distance r from one another.

In Bilu's equidistribution theorem (Theorem 1.1), we looked at the geometry of points of small Weil height in $\overline{\mathbb{Q}}$. Working with the usual topology on $\mathbb{C}$, the Galois orbits of these points became uniformly distributed on the unit circle. Can we say anything about the geometry of these points in $\mathbb{C}_p$? Do the roots of unity (the points of height 0) have convergent subsequences in $\mathbb{C}_p$?

To study measures on a space, we usually assume the space is locally compact. Recall that the Riesz Representation Theorem provides an isomorphism between Radon measures and linear functionals on the space of continuous functions with compact support. What functions can have "compact support" on a space such as $\mathbb{C}_p$? How to define measures?

The Berkovich analytification of $\mathbb{P}^1(\mathbb{C}_p)$ is one way (of many ways) to resolve some of these issues.

11.2. **Definition.** Let R denote the polynomial ring $\mathbb{C}_p[x]$. The Berkovich affine line, as a set, is

$$\mathbb{A}^{1,an}_p = \{\text{multiplicative seminorms on } R \text{ extending } |\cdot|_p \text{ on } \mathbb{C}_p\}.$$

That is, the function $\|\cdot\| : R \to [0, \infty)$ must satisfy

(1) $\|c_0\| = |c_0|_p$ for all $c_0 \in \mathbb{C}_p$;
(2) $\|fg\| = \|f\|\|g\|$ for all $f, g \in R$; and
(3) $\|f + g\| \leq \|f\| + \|g\|$.

Note that $\|\cdot\|$ is only required to be a seminorm and not a norm; that is, we can have $\|f\| = 0$ for nonzero f.

Remark 11.1. Since $|\cdot|_p$ is non-archimedean, the seminorms will also be non-archimedean, so condition (3) can be replaced with the condition that $\|f + g\| \leq \max\{\|f\|, \|g\|\}$.

The **topology** on $\mathbb{A}^{1,an}_p$ is the weakest such that $\zeta \mapsto \|f\|_\zeta$ will be continuous for each fixed $f \in R$. We will see a basis for this topology shortly.

The **Berkovich projective line** $\mathbb{P}^{1,an}_p$ is defined as the quotient of two disjoint copies of $\mathbb{A}^{1,an}$ identified along $\mathbb{A}^{1,an}_p$ via $\zeta \mapsto 1/\zeta$. The action of $1/\zeta$ on seminorms is given by

$$\|f\|_{1/\zeta} := \frac{\|x^{\deg f} f(1/x)\|_\zeta}{\|x\|_\zeta^{\deg f}}$$

for each $f \in R$. Alternatively, it can be defined naively as the union $\mathbb{A}^{1,an}_v \cup \{\infty\}$ with an "obvious" topology.

11.3. **The elements of the Berkovich space, more concretely.** We have already seen examples of seminorms $\|\cdot\|$ on $R = \mathbb{C}_p[x]$ in Section 10. For each closed disk $\overline{D} = \overline{D}(z_0, r) \subset \mathbb{C}_p$, we have an element

$$\zeta(z_0, r) := \|\cdot\|_{\overline{D}} \in \mathbb{A}^{1,an}_p$$

defined by

$$\|f\|_{\overline{D}} = \sup\{|f(z)|_p : z \in \overline{D}\}.$$

Proposition 10.1 and Theorem 10.2 show that it is multiplicative. It is also a genuine norm because $\|f\|_{\overline{D}} = 0 \iff f = 0$. The norm associated $\zeta(0,1)$ associated to the unit disk $\overline{D}(0,1)$ is called the **Gauss point**.

Other examples come from the elements of $\mathbb{C}_p$. Evaluation

$$\|f\|_{z_0} := |f(z_0)|_p$$

defines a multiplicative seminorm for each $z_0 \in \mathbb{C}_p$. In this case, it is only a seminorm and not a norm. This shows that $\mathbb{C}_p \hookrightarrow \mathbb{A}_p^{1,an}$, because each element gives a distinct seminorm on R. In fact, these are essentially all of the possibilities, as the following theorem shows.

Theorem 11.2. [Be, Theorem 6.9] *Suppose $\zeta \in \mathbb{A}_p^{1,an}$. Then ζ is one of the following types of seminorms:*

- **Type I***: it comes from an element $z_0 \in \mathbb{C}_p$ and is given by $f \mapsto |f(z_0)|_p$;*
- **Type II***: it is the supremum norm on a disk $\overline{D}(z_0, r)$ for $z_0 \in \mathbb{C}_p$ and $r \in |\mathbb{C}_p^*|$;*
- **Type III***: it is the supremum norm on a disk $\overline{D}(z_0, r)$ for $z_0 \in \mathbb{C}_p$ and positive $r \notin |\mathbb{C}_p^*|$;*
- **Type IV***: it is the limit of norms on closed disks $D_1 \supset D_2 \supset D_3 \supset \cdots$ with empty intersection in $\mathbb{C}_p$.*

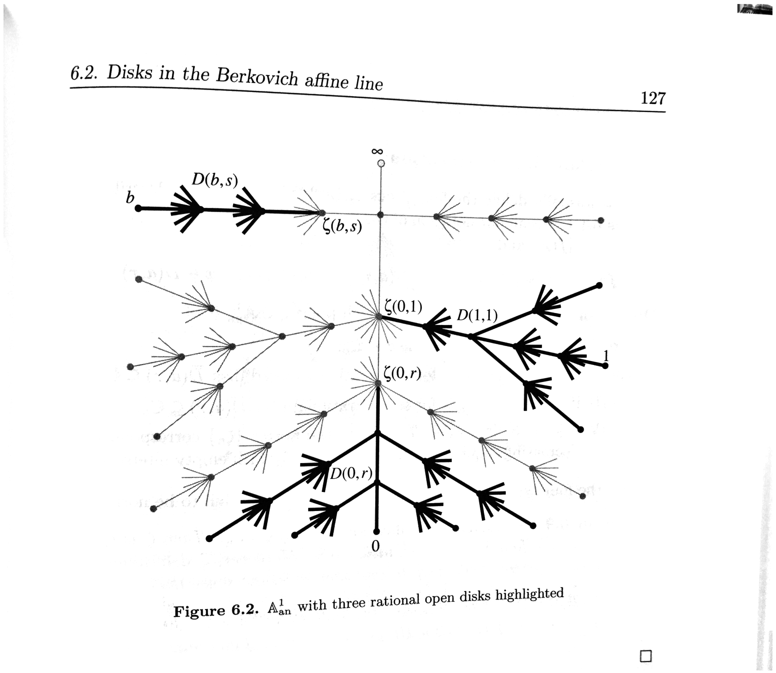
Type IV points are the most mysterious. An example is given by the closed disks D_n centered at

$$z_n = \sum_{i=1}^{n} p^{-1/n}$$

of radius $r_n = |p^{-1/n}| = p^{1/n} \to 1$. Fortunately we can ignore these points in most applications, but they must be included to guarantee the compactness of $\mathbb{P}_p^{1,an}$.

Theorem 11.3. [Be, Theorems 6.22, 6.25, 6.32] *The Berkovich projective line $\mathbb{P}_p^{1,an}$ is compact, Hausdorff, and (uniquely) path connected. It contains $\mathbb{P}^1(\mathbb{C}_p)$ as a dense subset.*

11.4. **The $\mathbb{R}$-tree structure.** We can picture $\mathbb{P}_p^{1,an}$ as an infinitely branched tree. First picture $\mathbb{P}^1(\mathbb{C}_p)$ as a circle. Then fill in the disk by drawing a ray connecting each point $z_0 \in \mathbb{C}_p$ to ∞; these represent the points $\zeta(z_0, r)$ as r grows from 0 to ∞. Recall that $\zeta(z_0, r) = \zeta(w_0, s)$ if and only if $r = s$ and $w_0 \in \overline{D}(z_0, r)$. So pairs of rays come together when the disks containing the two distinct points coincide. We see immediately that there should be infinitely many rays coming together at every Type II point (disk of radius $r \in |\mathbb{C}_p^*|$). These Type II points are dense along each ray, because $|\mathbb{C}_p^*| = p^{\mathbb{Q}}$.



6.2. Disks in the Berkovich affine line 127

Figure 6.2. $\mathbb{A}^1_{\mathrm{an}}$ with three rational open disks highlighted

□

FIGURE 11.1. Stolen from [Be].

11.5. **The topology of $\mathbb{P}^{1,an}_p$.** The definition of the topology given in §11.2 is not the most helpful. Somewhat more concretely, for every open $V \subset \mathbb{R}$ and every $f \in R = \mathbb{C}_p[x]$, we want all sets of the form

$$V_f = \{\zeta \in \mathbb{A}^{1,an}_p : \|f\|_\zeta \in V\}$$

to be open. Then we allow finite intersections and arbitrary unions of such sets. Most importantly, consider the **Berkovich open disk**, defined by

$$D^{an}(z_0, r) = \{\zeta : \|x - z_0\| < r\}.$$

It is a V_f for $f(x) = x - z_0$ and $V = (-\infty, r)$. Note that it contains all elements $z \in D(z_0, r) \subset \mathbb{C}_p$, because

$$\|x - z_0\|_z = |z - z_0|_p$$

for all $z \in \mathbb{C}_p$. Similarly, the **Berkovich closed disk** is defined by

$$\overline{D}^{an}(z_0, r) = \{\zeta : \|x - z_0\|_\zeta \leq r\}$$

and is closed, as the complement of the V_f with $f(x) = x - z_0$ and $V = (r, \infty)$. It satisfies $\overline{D}^{an}(z_0, r) \cap \mathbb{C}_p = \overline{D}(z_0, r)$. Unlike the disks in $\mathbb{C}_p$, we shall see that the disk $D^{an}(z_0, r)$ is open and *not* closed, while $\overline{D}^{an}(z_0, r)$ is closed and *not* open.

A **Berkovich open disk around infinity** is a set of the form $\mathbb{P}^{1,an}_p \setminus \overline{D}$ where $\overline{D}$ is a Berkovich closed disk in $\mathbb{A}^{1,an}_p$.

An **open affinoid** is a set of the form

$$A = D_1 \cap D_2 \cap \cdots \cap D_n$$

for a finite collection of Berkovich open disks. Equivalently, an open affinoid can be expressed as

$$A = \mathbb{P}^{1,an}_p \setminus \left(\overline{D}_1 \cup \cdots \cup \overline{D}_m\right)$$

for a finite collection of closed disks $\overline{D}_i$.

Theorem 11.4. [Be, Theorem 6.17] *The set of open affinoids forms a basis for the topology on* $\mathbb{P}^{1,an}_p$.

Proof idea. A proof is given in §15.3 of [Be]. The key ingredient is an understanding of the structure of open sets such as

$$V_f = \{\zeta \in \mathbb{A}^{1,an} : \|f\|_\zeta < r\}$$

for $r > 0$ and $f \in R$. If f is constant and equal to $c \in \mathbb{C}_p$ with $|c|_p < r$, then $V_f = \mathbb{A}^{1,an}_p$. For $|c|_p \geq r$, we have $V_f = \emptyset$. For nonconstant f, we can write

$$f(x) = \alpha \prod_{j=1}^{s} (x - z_j)^{m_j}.$$

For each z_k, we consider functions of $t \in [0, \infty)$ defined by

$$R_k(t) = |\alpha|_p \prod_{j=1}^{s} \max\left\{t, |z_k - z_j|_p\right\}^{m_j}.$$

Note that $R_k(0) = 0$, because of the $j = k$ term in the product, and the function R_k is *strictly* increasing with t. So there exists a unique $t_k > 0$ at which $R_k(t_k) = r$. The claim is that

$$V_f = \bigcup_{k=1}^{s} D^{an}(z_k, t_k). \tag{11.1}$$

Indeed, for $\zeta \in D^{an}(z_k, t_k)$, we know that $\|x - z_k\|_\zeta < t_k$, so that

$$\begin{aligned} \|x - z_j\|_\zeta &= \|x - z_j + z_k - z_k\|_\zeta \\ &\leq \max\{\|x - z_k\|_\zeta, \|z_k - z_j\|_\zeta\} \\ &= \max\{\|x - z_k\|_\zeta, |z_k - z_j|_p\} \end{aligned}$$

for each j. Therefore

$$\begin{aligned}\|f\|_\zeta &= |\alpha|_p \prod_{j=1}^{s} \|x - z_j\|_\zeta^{m_j} \\ &\le |\alpha|_p \prod_{j=1}^{s} \max\{\|x - z_k\|_\zeta, |z_k - z_j|_p\}^{m_j} \\ &< |\alpha|_p \prod_{j=1}^{s} \max\{t_k, |z_k - z_j|_p\}^{m_j} \;=\; r,\end{aligned}$$

with the strict $<$ because of the $j = k$ term in the product.

On the other hand, suppose that $\zeta \in \mathbb{A}_p^{1,an}$ satisfies $\|x - z_k\|_\zeta \ge t_k$ for each k. Now fix k so that $\|x - z_k\|_\zeta$ is minimal. Note that

$$|z_k - z_j|_p = \|z_k - z_j\|_\zeta = \|z_k - x + x - z_j\|_\zeta \le \max\{\|x - z_k\|_\zeta, \|x - z_j\|_\zeta\}$$

for every j. So

$$\begin{aligned}\|f\|_\zeta &= |\alpha|_p \prod_{j=1}^{s} \|x - z_j\|_\zeta^{m_j} \\ &= |\alpha|_p \prod_{j=1}^{s} \max\{\|x - z_k\|_\zeta, \|x - z_j\|_\zeta\}^{m_j} \\ &\ge |\alpha|_p \prod_{j=1}^{s} \max\{t_k, |z_k - z_j|_p\}^{m_j} \;=\; r.\end{aligned}$$

This completes the proof of (11.1).

I leave the rest of the proof as an exercise. □

Example 11.5. The closure of the open Berkovich disk $D^{an}(0,1)$ is $D^{an}(0,1) \cup \{\zeta(0,1)\}$ and not $\overline{D}^{an}(0,1)$. Indeed, open set containing $\zeta(0,1)$ must contain an open affinoid, and so it must intersect $D^{an}(0,1)$. The same reasoning shows that the closed Berkovich disk $\overline{D}^{an}(0,1)$ cannot be open, because affinoid neighborhoods of $\zeta(0,1)$ must intersect the open $\mathbb{P}^1 \setminus \overline{D}^{an}(0,1)$.

Example 11.6. Suppose $\{z_n\}$ is any sequence on $\mathbb{C}_p$ satisfying $|z_n|_p = 1$ for all n and $|z_n - z_m|_p = 1$ for all $n \ne m$. Then the sequence converges to $\zeta(0,1)$. Indeed, every open affinoid neighborhood of $\zeta(0,1)$ must contain infinitely many of the open disks $D^{an}(z_n, 1)$.

11.6. **Path-connectedness.** For any z_0, there is a continuous embedding of $(0,\infty)$ into $\mathbb{A}_p^{1,an}$ defined by

$$\varphi_{z_0}(t) = \zeta(z_0, t).$$

To show continuity, it suffices to check that preimages of affinoids are open. But, since affinoids are finite intersections of open disks, the preimage is always an open interval. It is an embedding because all open intervals in $(0,\infty)$ can be obtained as $\varphi^{-1}(U)$ for U an open Berkovich disk or an open Berkovich annulus $\{\zeta \in \mathbb{A}^{1,an}_p : r_1 < \|x - z_0\|_\zeta < r_2\}$. Taking a limit as $t \to 0$, we see that the image points $\zeta(z_0, t)$ must converge to z_0, since every open affinoid around z_0 must contain these points for all t small. Taking a limit as $t \to \infty$, the points $\zeta(z_0, t)$ converge to ∞ in $\mathbb{P}^{1,an}_p$.

This shows that how we might join any pair of points in $\mathbb{P}^{1,an}_p$ by a path. The above joins any $z_0 \in \mathbb{C}_p$ to all of the Type II or III points corresponding to disks that contain z_0. Any pair of points or disks in $\mathbb{A}^{1,an}_p$ lie in a common larger disk, so we can connect them through that larger disk. To reach a Type IV corresponding to the intersection of closed disks $D_1 \supset D_2 \supset \cdots$, we need to take a concatenation of paths between the corresponding sup-norms, from ζ_1 to ζ_2 to ζ_3, etc.

As claimed in §11.4, the space $\mathbb{P}^{1,an}$ has the structure of an $\mathbb{R}$-tree. In particular, each pair of points $\zeta, \xi \in \mathbb{P}^{1,an}_p$ can be connected by a *unique* path. The uniqueness hinges on the following basic fact: any path from a point in a disk $D^{an}(z_0, r)$ to its complement must pass through the boundary point $\zeta(z_0, r)$. Indeed, suppose we connect $\zeta_0 \in D^{an}(z_0, r)$ to $\zeta_1 \notin D^{an}(z_0, r)$ by a path $\gamma : [0,1] \to \mathbb{P}^{1,an}_p$. Let $t_0 = \sup\{t : \gamma(t) \in D^{an}(z_0, r)\}$. If $\zeta_1 \neq \zeta(z_0, r)$, then it lies in the open set $\mathbb{P}^{1,an}_p \setminus (D^{an}(z_0, r) \cup \zeta(z_0, r))$, we see that $0 < t_0 < 1$. If $\gamma(t_0) \neq \zeta(z_0, r)$, then we can find an open set around $\gamma(t_0)$ so that it is fully contained in $D^{an}(z_0, r)$ or fully contained in $\mathbb{P}^{1,an}_p \setminus (D^{an}(z_0, r) \cup \zeta(z_0, r))$. But this violates the definition of t_0 and continuity of γ.

12. Julia sets in the Berkovich space

12.1. **Rational functions as maps.** Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a map of degree $d \geq 1$ defined over the field $\mathbb{C}_p$. As we know, this f can be expressed as a rational function

$$f(z) = \frac{P(z)}{Q(z)}$$

for polynomials $P, Q \in \mathbb{C}_p[z]$ having no common roots, with $d = \max\{\deg P, \deg Q\}$.

The map f extends in a natural way to define a map

$$f : \mathbb{P}^{1,an}_p \to \mathbb{P}^{1,an}_p.$$

Formally, for $\zeta \in \mathbb{A}^{1,an}_p$ and $g(x) = a_0 x^n + \cdots + a_n \in R = \mathbb{C}_p[x]$, we put

$$\|g\|_{f(\zeta)} := \|g \circ f\|_\zeta = \frac{\|a_0 P^n + \cdots + a_n Q^n\|_\zeta}{\|Q\|^n_\zeta}. \tag{12.1}$$

Note that $\|Q\|_\zeta = 0$ only at the Type I points that are the zeroes of Q (i.e., the poles of f). Informally, we have already seen that f will take each (small enough) round disk in $\mathbb{P}^1(\mathbb{C}_p)$ to another round disk, and so it will act on the corresponding sup-norms.

Here are three important facts that I won't have time to prove in class:

Theorem 12.1. [Be, Theorem 7.4, ...]

1. *Equation* (12.1) *describes the unique continuous extension of f from $\mathbb{P}^1(\mathbb{C}_p)$ to all of $\mathbb{P}^{1,an}_p$.*
2. *If $\zeta \in P^{1,an}_p$ is an element of Type $X \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, then $f(\zeta)$ is also Type X.*
3. *If D is any open Berkovich disk in $\mathbb{P}^{1,an}_p$ with boundary point ζ, then either $f(D)$ is another open Berkovich disk with boundary point $f(\zeta)$, or $f(D) = \mathbb{P}^{1,an}_p$.*

And here are some fundamental facts about the action of $\mathrm{PSL}_2(\mathbb{C}_p)$ on $\mathbb{P}^{1,an}_p$ by linear fractional transformations (the maps of degree $d = 1$):

Theorem 12.2. [Be, ?]

1. *The group* $\mathrm{PSL}_2(\mathbb{C}_p)$ *acts triply-transitively on Type* I *points (meaning that any triple of distinct points of $\mathbb{P}^1(\mathbb{C}_p)$ can be sent – by a unique linear fractional transformation – to $\{0, 1, \infty\}$).*
2. *It acts transitively on the Type* II *points of $\mathbb{P}^{1,an}_p$, so any Type* II *point can be sent to the Gauss point $\zeta_G = \zeta(0, 1)$.*
3. *The stabilizer of the Gauss point ζ_G is* $\mathrm{PSL}_2(\mathcal{O}_p)$*.*

And finally, via the action of $\mathrm{PSL}_2(\mathbb{C}_p)$, we can give another description of the image $f(\zeta)$ for Type II points:

Theorem 12.3. *For any $\zeta_1 \in \mathbb{A}^{1,an}_p$ of Type II, the image $\zeta_2 = f(\zeta_1)$ is the unique Type II point for which the reduction of*

$$A_2 \circ f \circ A_1^{-1}$$

is nonconstant, where $A_i \in \mathrm{PSL}_2\mathbb{C}_p$ is chosen so that $A_i(\zeta_i) = \zeta_G$.

12.2. **The Berkovich Julia set.** Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ has degree $d \geq 2$. Its **Berkovich Fatou set** $\Omega^{an}(f)$ is the set of all $\zeta \in \mathbb{P}^{1,an}_p$ for which there exists an open neighborhood U of ζ so that

$$\overline{\bigcup_{n \geq 0} f^n(U)} \neq \mathbb{P}^{1,an}_p.$$

The **Berkovich Julia set** $J^{an}(f)$ is the complement, $\mathbb{P}^{1,an}_p \setminus \Omega^{an}(f)$.

Example 12.4. Let $f(z) = z^2$. Then $J^{an}(f) = \{\zeta_G\}$, the Gauss point. As we have already seen in Example 8.1, $f^n(D(0, 1)) \subset D(0, 1)$ for all n, and the same holds for

$D^{an}(0,1)$. For example, taking $\zeta = \zeta(0,r)$ to be the supremum norm on the disk $\overline{D}(0,r)$ for $r < 1$, we have

$$\|g(x)\|_{f(\zeta)} := \|g(x^2)\|_\zeta = \sup\{|g(z)|_p : |z|_p \leq r^2\}$$

for all $g \in \mathbb{C}_p[x]$, so that $f(\zeta) = \zeta(0,r^2)$. This shows that $f^n(\zeta) \to 0$; similarly for all $\zeta \in D^{an}(0,1)$. So $D^{an}(0,1) \subset \Omega^{an}(f)$. We can argue similarly for the disk $\mathbb{P}^{1,an}_p \setminus \overline{D}^{an}(0,1)$ around ∞, so that Berkovich disk is also Fatou.

For a point z with $|z|_p = 1$, we have $f^n(D^{an}(z,1)) \subset D^{an}(f^n(z),1)$ where $|f^n(z)|_p = 1$ for all n, so that the union of the images is disjoint from $D^{an}(0,1)$. This shows that $J^{an}(f) \subset \{\zeta_G\}$. On the other hand, suppose that U is an affinoid neighborhood of ζ_G. Then it is the complement of finitely many closed Berkovich disks. Consider the action of $\bar{f}$ on $\mathbb{P}^1(k_p)$, which can be identified with the set of all connected components of $\mathbb{P}^{1,an}_p \setminus \{\zeta_G\}$. Every disk in $\overline{D}^{an}(0,1) \setminus D^{an}(0,1)$ has infinitely many preimage disks under the iterates of f, so the union of domains $f^n(U)$ must contain all of those disks. The set U will also contain Berkovich annuli of the form $\{1 < \|x\|_\zeta < R\}$ and $\{r < \|x\|_\zeta < 1\}$, and the union of their forward iterates will cover all of $D^{an}(0,1) \cup \big(\mathbb{P}^{1,an}_p \setminus \overline{D}^{an}(0,1)\big)$. This shows that $J^{an}(f) = \{\zeta_G\}$.

More generally, we have:

Theorem 12.5. [Be, Proposition 8.12] *A map f has good reduction if and only if $J^{an}(f) = \{\zeta_G\}$.*

Proof idea. If f has good reduction, then an argument similar to that of Example 12.4 will show that $J^{an}(f) = \{\zeta_G\}$. This is because the reduction map $\bar{f}$ gives the action of f on the connected components of $\mathbb{P}^{1,an}_p \setminus \{\zeta_G\}$. Suppose, on the other hand, that the reduction of f fails to have degree d. If the degree of $\bar{f}$ is > 0, then $f(\zeta_G) = \zeta_G$ by Theorem 12.3. The reduction map $\bar{f}$ determines the action on (most) components of $\mathbb{P}^{1,an}_p \setminus \{\zeta_G\}$, but there will be a component of $\mathbb{P}^{1,an}_p \setminus \{\zeta_G\}$ that contains both a zero and a pole of f. This disk will be forced to map over all of $\mathbb{P}^{1,an}_p$ by f, from Theorem 12.1 (3). This shows that there is another preimage of ζ_G in that disk, so the set $\{\zeta_G\}$ is not totally invariant for f and cannot be the Julia set. If $\deg \bar{f} = 0$, then ζ_G is again not invariant for f, so it cannot be the Julia set. □

And:

Theorem 12.6. [Be, Theorem 8.3] *For the Type I points, we have*

$$J^{an}(f) \cap \mathbb{P}^1(\mathbb{C}_p) = J_I(f).$$

Proof. If z_0 lies in $\Omega^{an}(f) \cap \mathbb{P}^1(\mathbb{C}_p)$, then there is an open Berkovich disk D^{an} around z_0 and another open Berkovich disk D' so that the closure of $\bigcup_{n\geq 0} f^n(D)$ is disjoint from D'. Changing coordinates so that D' is centered at ∞ and taking the intersection of $D = D^{an} \cap \mathbb{C}_p$, this means that $f^n(D) \subset \overline{D''}$ for a disk $D'' \subset \mathbb{C}_p$ and for all $n \geq 0$.

Using Theorem 10.4, we deduce that the iterates f^n are uniformly Lipschitz on D and so equicontinuous in the chordal metric on $\mathbb{P}^1(\mathbb{C}_p)$. Therefore $z_0 \in \Omega_I(f)$.

Conversely, given $z_0 \in \Omega_I(f)$, change coordinates so that $z_0 \in D(0,1)$. There exists a disk $D = D(z_0, r)$ with $r < 1$ on which the iterates f^n are equicontinuous. So taking any positive $\varepsilon < 1$, there exists $\delta > 0$ so that $d_\sigma(f^n(z), f^n(w)) < \varepsilon$ for all n and for all $d_\sigma(z,w) < \delta$ in D, for the chordal distance d_σ. (Note that the chordal distance and the usual distance on $\mathbb{C}_p$ agree in $D(0,1)$.) In particular, we have $f^n(D(z_0,\delta)) \subset D_\sigma(f^n(z_0), \varepsilon)$ for all n, where D_σ denotes a disk in the chordal distance. We aim to show there is a disk $D' \subset \mathbb{C}_p$ which is disjoint from all iterates of $D(z_0,\delta)$. For then the image of the Berkovich disks $f^n(D^{an}(z_0,\delta))$ will be disjoint from the Berkovich $(D')^{an}$.

Write D_n for the disk $f^n(D(z_0),\delta)$. If there exist $n_0 > m_0$ so that D_{n_0} is contained in D_{m_0}, then the union $\bigcup_n D_n$ will lie in the finite union $D_0 \cup \cdots \cup D_{n_0-1}$, and we are done. If there exist $n_0 > m_0$ so that D_{n_0} contains but is not equal to D_{m_0}, then we can show $z_0 \notin \Omega_I(f)$. Indeed, iterates of the form $f^{k(n_0-m_0)}$ for $k \geq 1$, restricted to D_{m_0}, will stretch distances of points near $f^{m_0}(z_0)$ and will violate the condition that $f^n(D(z_0,\delta)) \subset D_\sigma(f^n(z_0),\varepsilon)$ for all n. So we may assume that the disks D_n are disjoint for all $n > 0$. But then, choosing any $\delta' < \delta$, the forward images of $D(z_0,\delta')$ will be disjoint from the annulus $\delta' \leq |z - z_0|_p < \delta$. So we can find a Berkovich disk $(D')^{an}$ containing a point z with $|z - z_0|_p = \delta'$ that will be disjoint from $f^n(D^{an}(z_0,\delta'))$ for all n. □

12.3. **Polynomials.** Let $f \in \mathbb{C}_p[z]$ be a polynomial map of degree $d \geq 2$. As for polynomials over $\mathbb{C}$, the point at infinity is an attracting fixed point for f, also in the Berkovich space. The **Berkovich filled Julia set** is

$$K^{an}(f) = \{\zeta \in \mathbb{A}^{1,an}_p : f^n(\zeta) \not\to \infty\}.$$

Theorem 12.7. *The Julia set of a polynomial is*

$$J^{an}(f) = \partial K^{an}(f)$$

in $\mathbb{A}^{1,an}_p$.

We will soon see that there is a natural equilibrium measure on $K^{an}(f)$, and the escape rate – defined exactly as for complex polynomials – will be a potential for this measure.

Proof. The proof of Theorem 12.7 is left as Exercise 8.16 in [Be], but I don't believe it is an obvious argument, so I include it here. It follows the ideas presented in the proof of [Be, Theorem 9.5].

First let's understand the Type I part of the Julia set. The set of all points in $\mathbb{C}_p$ that go to ∞ under iteration is an open set of the form

$$B = \bigcup_{n\geq 0} f^{-n}(\{z \in \mathbb{C}_p : |z|_p > R\}$$

for some suitably large R. So any element of B must be in $\Omega_I(f)$, since we can find a disk around it for which all large enough iterates are contained in a disk around ∞. Similarly, if $z \in K_I(f)$, and if there is a disk around z which is fully contained in $K_I(f)$, then all iterates of this disk will lie in $D(0,R)$, and so the iterates will be equicontinuous. On the other hand, if there exists a point $z_0 \in \partial K_I(f)$, then there are points z in any small disk around z_0 that lie in B. Consequently, equicontinuity must fail, because the chordal distance $d_\sigma(f^n(z), f^n(z_0))$ will eventually be larger than some fixed constant $r > 0$, while $d_\sigma(z, z_0)$ can be arbitrarily small. This shows that $\partial K_I(f) = J_I(f)$.

In general, for points in $\mathbb{A}_p^{1,an}$, we have that the basin of infinity is the open set

$$B^{an} = \bigcup_{n\geq 0} f^{-n}(V_0)$$

for a Berkovich open disk V_0 around ∞, and the same reasoning as above shows that $J^{an}(f) \subset \partial K^{an}(f)$. Note that B^{an} is connected, just as it was over $\mathbb{C}$, because there is a maximum principle for polynomials.

Now suppose that $\zeta_0 \in \partial K^{an}(f)$. Conjugating f by an affine transformation, we can assume that $\zeta_0 = \zeta_G$, the Gauss point, so that $0 \in K^{an}(f)$. Let U be any affinoid neighborhood of ζ_0. We need to show that the union of the iterates $f^n(U)$ is dense in $\mathbb{P}_p^{1,an}$. Let D_∞ be the disk component of $\mathbb{P}_p^{1,an} \setminus \{\zeta_0\}$ containing ∞, so that $B^{an} \subset D_\infty$. The structure of an open affinoid implies that U must contain an annulus A inside D_∞, with one of its boundary points equal to ζ_0. Consider the open sets

$$V_n = f^{-n}(V_0).$$

Note that $V_n \subset V_{n+1}$ for all n; in fact $\overline{V_n} \subset V_{n+1}$. If V_n is a Berkovich disk around ∞ for all n, then B^{an} is also a disk and must coincide with D_∞ because $\zeta_0 \in \partial K^{an}(f) = \partial B^{an}$. Then, by continuity, we must have $f(\zeta_0) = \zeta_0$. As the restriction of f to B^{an} has degree d, it follows that the reduction of f at ζ_0 must have degree d (for otherwise there would be a disk component of $\mathbb{P}_p^{1,an} \setminus \{\zeta_0\}$ containing a zero and a pole of f, and it would map by f over all of $\mathbb{P}_p^{1,an}$, contradicting the invariance of B^{an}). In other words, f has good reduction and $J^{an}(f) = \{\zeta_0\}$, and we are done.

Now assume that there is an $n_0 \geq 0$ so that V_{n_0} is a Berkovich disk but V_{n_0+1} is not. Then for all $n > n_0$, the open set V_n has at least two boundary points (both of Type II) lying in $\mathbb{P}_p^{1,an} \setminus \overline{V_{n-1}}$. Moreover, as explained in Step 1 of Case 2 in the proof of [Be, Theorem 9.5], every connected component of $\mathbb{P}_p^{1,an} \setminus \overline{V_{n-1}}$ will contain at least two boundary points of V_n. Since $\zeta_0 \in \partial K^{an}(f) = \partial B^{an}$, there must be a choice of boundary points $\zeta_n \in \partial V_n$ so that $\zeta_n \to \zeta_0$ as $n \to \infty$. Choose N so that ζ_N lies in the annulus A. Then there are (at least) two boundary points ξ_1 and ξ_2 of V_{N+1} that are also contained in A. Each ξ_i defines a closed Berkovich disk in $\mathbb{A}^{1,an}_p$ which is disjoint from V_{N+1}; as the two disks are disjoint, at most one can contain ζ_0. Suppose the disk $\overline{D}_1$ (with boundary point ξ_1) does not contain ζ_0. Then $\overline{D}_1$ is fully contained in A. Note that $f^{N+1}(\xi_1) = \partial V_0$. From Theorem 12.1 (3), the component of $\mathbb{P}^{1,an}_p \setminus \{\xi_1\}$ containing ∞ must map over V_0, and $\overline{D}_1$ must map surjectively over all of $\mathbb{P}^{1,an} \setminus V_0$. In other words, we see that $\bigcup_n f^n(U) \supset \bigcup_n f^n(A)$ is dense in $\mathbb{P}^{1,an}_p$. This shows that $\zeta_0 \in J^{an}(f)$. □

12.4. **Lattès maps.** ... coming ...

13. The Laplacian on a finite graph

We now follow Chapter 3 of Baker and Rumely's book [BR2]. Our goal is to construct the Laplacian Δ on a suitable class of functions defined on $\mathbb{A}^{1,an}_p$.

13.1. **Piecewise affine functions on graphs.** Let Γ be a finite metrized tree. It consists of finitely many vertices and edges, and each edge is identified (isometrically) with a closed interval in $\mathbb{R}$.

We first define a Laplacian operator Δ on the space

$$\mathrm{CPA}(\Gamma) = \{\text{continuous } f : \Gamma \to \mathbb{R} : f \text{ is affine on each edge}\}$$

of continuous, piecewise affine functions.

The **tangent space** to Γ at a point $x \in \Gamma$ is

$$T_x\Gamma = \{\text{the set of unit tangent vectors to } \Gamma \text{ at } x\}$$

so there is a one-to-one correspondence between elements of $T_x\Gamma$ and connected component of $\Gamma \setminus \{x\}$. The **directional derivative** of a function $f \in \mathrm{CPA}(\Gamma)$ at x is

$$d_{\vec{v}} f(x) = \lim_{t \to 0^+} \frac{f(x + t\vec{v}) - f(x)}{t}$$

For x interior to an edge, note that

$$\sum_{\vec{v} \in T_x\Gamma} d_{\vec{v}} f(x) = 0$$

because the two directional derivatives will be negative of each other.

For $f \in \mathrm{CPA}(\Gamma)$, we define a signed measure on Γ by

$$\Delta f = \sum_{x \in \Gamma} \left(\sum_{\vec{v} \in T_x\Gamma} d_{\vec{v}} f(x) \right) \delta_x. \tag{13.1}$$

Note that Δf is an atomic measure, supported on the vertices of Γ. *Remark: this is the negative of the operator Δ introduced in* [BR2, §3.2].

Example 13.1. Suppose that Γ is a graph consisting of 3 vertices and 2 edges, identified with the interval $[-1,1]$ in $\mathbb{R}$, with vertices at -1, 0, and 1. Consider the function $f(x) = 2x$ on $[-1,0]$ and $f(x) = 3x$ on $[0,1]$. Then $f \in \mathrm{CPA}(\Gamma)$. We compute that

$$\Delta f = 2\delta_{-1} - 2\delta_0 + 3\delta_0 - 3\delta_1 = 2\delta_{-1} + \delta_0 - 3\delta_1.$$

Proposition 13.2. [BR2, Proposition 3.2] *For functions in* $\mathrm{CPA}(\Gamma)$ *and the* Δ *of* (13.1)*, we have*

1. $\Delta f = 0$ *if and only if* f *is constant*
2. $\int_\Gamma f \, \Delta g = \int_\Gamma g \, \Delta f = -\int_\Gamma f'(x) g'(x) \, dx$
3. $\Delta f(\Gamma) = 0$
4. $\int_\Gamma f \, \Delta f \le 0$*, with equality if and only if* f *is constant.*

Proof. If the function f is constant, then cleary $\Delta f = 0$. So assume $\Delta f = 0$ and let $M = \max\{f(x) : x \in \Gamma\}$. Let $\Gamma_M = \{x \in \Gamma : f(x) = M\}$, so Γ_M is a closed subset of Γ. By definition, we have $d_{\vec{v}} f(x) \le 0$ at all points of Γ_M. If $\Gamma_M \neq \Gamma$, the derivative will be nonzero at some point of $\partial \Gamma_M$, so the sum over all directions at this point will be nonzero, violating our assumption on Δf. We conclude that $\Gamma_M = \Gamma$, so f is constant. This proves (1).

Property (2) is an Integration By Parts. Fix an edge E of Γ, identified with interval $[a,b] \subset \mathbb{R}$. Let $u = f'$ which is a constant function on E, and let $v = g$, so that $dv = g'(x)dx$ on E. IBP on $[a,b]$ tells us that

$$\int_E f'(x) g'(x) \, dx = \int_E u \, dv = g \, f'|_{\partial E} - \int_E v \, du = g(b) f'(b) - g(a) f'(a) = -\int_E g \, \Delta_E f$$

where Δ_E is the Laplacian associated to the graph E. Interchanging f and g gives the same equality with their roles reversed. Summing over all edges, and since $\Delta_\Gamma = \sum_E \Delta_E$, gives the result (2).

Properties (3) and (4) follow from (2). □

13.2. **Piecewise C^2 functions on graphs.** Let Γ be a finite metrized tree. We now extend our Laplacian operator Δ to the space

$$\mathrm{CPC2}(\Gamma) = \{\text{continuous } f : \Gamma \to \mathbb{R} : f \text{ is } C^2 \text{ on each (open) edge and } f'' \in L^1(\Gamma, dx)\}$$

of continuous, piecewise C^2 functions. Baker and Rumely call this the Zhang class $\mathrm{Zh}(\Gamma)$ in [BR2], as this Δ was introduced by Shouwu Zhang in a 1993 publication:

$$\Delta f = f''(x) \, dx + \sum_{x \in \Gamma} \left(\sum_{\vec{v} \in T_x \Gamma} d_{\vec{v}} f(x) \right) \delta_x. \tag{13.2}$$

The directional derivatives exist at all points, because of the hypothesis that $f'' \in L^1(\Gamma, dx)$. This Δf can have absolutely continuous components on the edges of Γ, but it coincides with the Δf of (13.1) for f in CPA(Γ). *Remark: this is the negative of the operator* Δ *introduced in* [BR2, §3.4].

Proposition 13.3. *Properties (1)–(4) of Proposition 13.2 hold for the* Δ *of* (13.2) *for the Zhang class* CPC2(Γ).

Proof. The proof is identical to that of Proposition 13.2. For (2), note that the $\int v\,du$ term does not vanish now, but we have

$$\int_E f'(x)g'(x)\,dx = \int_E u\,dv = g\,f'|_{\partial E} - \int_E v\,du = \\ g(b)f'(b) - g(a)f'(a) - \int_E g(x)f''(x)\,dx = -\int_E g\,\Delta_E f$$

□

Given a closed and connected subset S of Γ, and given any point $x \in \partial S$, we define

$$T_x^{out}(\Gamma, S)$$

to be the set of vectors that are **outward pointing** from S. That is, $\vec{v} \in T_x\Gamma$ is outward pointing if $x + t\vec{v} \notin S$ for all $t > 0$ sufficiently small.

Proposition 13.4. *Let* S *be a closed and connected subset of* Γ. *We have*

$$\Delta f(S) = \sum_{x\in\partial S}\ \sum_{\vec{v}\in T_x^{out}(\Gamma,S)} d_{\vec{v}}f(x)$$

for all $f \in$ CPC2(Γ).

Proof. This follows from the Fundamental Theorem of Calculus on edges. If an edge $E = [a, b]$ is fully contained in S, then

$$\int_a^b f''(x)dx = f'(b) - f'(a) = -d_{\vec{v}_b}f - d_{\vec{v}_a}f$$

for the directional derivatives on E at a and b. Summing over all edges and partial edges within S, we conclude that all of the inward-pointing derivative contributions in S cancel with the contributions from integrating f''. □

13.3. **BDV and the Laplacian on Γ.** Proposition 13.4 suggests the definition of a more general class of functions and associated Laplacian operator Δ. If $A \subset \Gamma$ is a finite, disjoint union of intervals (closed, open, or half-open), then we can use Proposition 13.4 to compute that

$$\Delta f(A) = \sum_{\{x\in\partial A\cap A\}}\ \sum_{\vec{v}\in T_x^{out}(\Gamma,\overline{A})} d_{\vec{v}}f(x) - \sum_{\{x\in\partial A\cap(\Gamma\setminus A)\}}\ \sum_{\vec{v}\in T_x^{out}(\Gamma,\overline{\Gamma\setminus A})} d_{\vec{v}}f(x) \tag{13.3}$$

for all $f \in$ CPC2(Γ), recalling that $\Delta f(\Gamma) = 0$. Note that f'' does not enter into the formula for the Δf measure of such a set A. And so, simply assuming that the directional derivatives of f exist at all points, we may take (13.3) as a definition. We set

$$\mathrm{BDV}(\Gamma) = \Big\{ f : \Gamma \to \mathbb{R} : \text{ directional derivatives exist at all points and}$$

$$\sum |\Delta f(A_i)| < \infty \text{ for all countable, pairwise disjoint sets of intervals } \{A_i\} \Big\}.$$

This is the class of functions with **bounded differential variation**. Note that the existence of the directional derivatives forces all $f \in \mathrm{BDV}(\Gamma)$ to be continuous.

Theorem 13.5. [BR2, Theorem 3.6, Propostions 3.14, 3.17] *For all $f \in \mathrm{BDV}(\Gamma)$, equation* (13.3) *extends to define a finite, signed Borel measure Δf on Γ. Moreover, for all $f, g \in \mathrm{BDV}(\Gamma)$, we have:*

(1) $\Delta f = 0$ *if and only if f is constant*
(2) $\int_\Gamma f\, \Delta g = \int_\Gamma g\, \Delta f = -\int_\Gamma f'(x) g'(x)\, dx$
(3) $\Delta f(\Gamma) = 0$
(4) $\int_\Gamma f\, \Delta f \leq 0$*, with equality if and only if f is constant.*

For condition (3), the derivative f' is well-defined outside of the (countable) set of atoms of Δf.

It is easy to see that Δf coincides with the definition (13.1) for $f \in \mathrm{CPA}(\Gamma)$ and with (13.2) for $f \in \mathrm{CPC2}(\Gamma)$.

13.4. **Inverting the Laplacian on Γ.** Let Γ be a finite metrized tree, and suppose y, z are any two points in Γ. There is a unique function

$$j_z(\cdot, y) \in \mathrm{BDV}(\Gamma)$$

defined by the conditions that

$$\Delta j_z(\cdot, y) = \delta_y - \delta_z \quad \text{and} \quad j_z(z, y) = 0.$$

For example, if $\Gamma = [-1, 1]$, then

$$j_0(x, -1) = \begin{cases} x & \text{for } x \leq 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

and

$$j_1(x, 0) = \begin{cases} -1 & \text{for } x \leq 0 \\ x - 1 & \text{for } x \geq 0 \end{cases}$$

Lemma 13.6. *We have*

$$j_z(x, y) = j_z(y, x)$$

for all $x, y, z \in \Gamma$.

Proof. Fix $x_0, y_0, z \in \Gamma$. Consider the functions

$$h(x) = j_z(x, y_0) \quad \text{and} \quad k(x) = j_z(x, x_0).$$

Then

$$\Delta h = \delta_{y_0} - \delta_z \quad \text{and} \quad \Delta k = \delta_{x_0} - \delta_z.$$

Since $h(z) = k(z) = 0$, we have

$$j_z(x_0, y_0) = h(x_0) = \int_\Gamma h\,\Delta k = \int_\Gamma k\,\Delta h = k(y_0) = j_z(y_0, x_0).$$

□

Theorem 13.7. [BR2, Proposition 3.11] *Suppose that μ is any finite (signed) Borel measure on Γ. Fix $z \in \Gamma$, and set*

$$U_\mu(x) = \int_\Gamma j_z(x, y)\, d\mu(y).$$

Then $U_\mu \in \mathrm{BDV}(\Gamma)$ and

$$\Delta U_\mu = \mu - \mu(\Gamma)\delta_z.$$

Sketch of proof. Suppose first that $\mu = \sum_j c_j \delta_{y_j}$ is an atomic measure. Then

$$U_\mu = \sum_j c_j j_z(x, y_j)$$

is a sum of piecewise affine functions. And

$$\Delta U_\mu = \sum_j c_j(\delta_{y_j} - \delta_z) = \mu - \mu(\Gamma)\delta_z.$$

For the general measure μ, we first observe that U_μ has directional derivatives that exist everywhere. By definition, for $\vec{v} \in T_x\Gamma$, we have

$$d_{\vec{v}}U(x) = \lim_{t\to 0^+} \int_\Gamma \frac{j_z(x + t\vec{v}, y) - j_z(x, y)}{t}\, d\mu(y).$$

For fixed x, and for y bounded away from x, the integrand is constant in t, for all t sufficiently small. So we can break the integral into two parts, over an interval $e_{x,\vec{v}}$ and $\Gamma \setminus e_{x,\vec{v}}$, for $e_{x,\vec{v}}$ a small interval starting at x and extending in the direction of $\vec{v}$. The integral over $\Gamma \setminus e_{x,\vec{v}}$ becomes

$$\int_{\Gamma\setminus e_{x,\vec{v}}} d_{\vec{v}} j_z(\cdot, y)|_x\, d\mu(y).$$

for all sufficiently small t. The integral over $e_{x,\vec{v}}$ will shrink to 0 as the interval shrinks, and we find that $d_{\vec{v}}U_\mu$ exists.

To complete the proof, showing $U_\mu \in \mathrm{BDV}(\Gamma)$ and to compute ΔU_μ, we can approximate μ be a sequence of discrete measures μ_n that converge “moderately well” to μ. See the proof of [BR2, Proposition 3.11] for details. □

14. Potential functions and the Laplacian on $\mathbb{A}_p^{1,an}$

The following material is from Chapters 4 and 5 of [BR2].

14.1. **The Hsia kernel.** Given two points $\zeta_1, \zeta_1 \in \mathbb{A}_p^{1,an}$, we define their **join** to be

$$\zeta_1 \vee \zeta_2 \in \mathbb{A}_p^{1,an},$$

the point corresponding to the smallest disk containing ζ_1 and ζ_2. In other words, letting γ_i be the path connecting ζ_i to ∞, the point $\zeta_1 \vee \zeta_2$ is the point where γ_1 and γ_2 first intersect. The **diameter**

$$\operatorname{diam} \zeta \geq 0$$

of a point $\zeta \in \mathbb{A}_p^{1,an}$ is the radius (or diameter, equivalently) of the disk corresponding to ζ. Note that $\operatorname{diam} \zeta = 0$ if and only if ζ has Type I. For Type IV points ζ, the diameter is the (positive) limit of the diameters of the nested disks limiting on ζ.

The **Hsia kernel** on $\mathbb{A}_p^{1,an}$ is

$$\delta_H(\zeta_1, \zeta_2) = \operatorname{diam}(\zeta_1 \vee \zeta_2).$$

If $z, w \in \mathbb{C}_p$ are Type I points, then

$$\delta_H(z, w) = |z - w|_p.$$

The Hsia kernel is clearly non-negative and symmetric. Note that

$$\delta_H(\zeta, \zeta) = \operatorname{diam} \zeta$$

for all $\zeta \in \mathbb{A}_p^{1,an}$, so it is not a distance function on $\mathbb{A}_p^{1,an}$, though it extends the p-adic distance function $|z - w|_p$.

Proposition 14.1. [BR2, Proposition 4.1] *The Hsia kernel is the unique extension of $|z - w|_p$ from $\mathbb{C}_p$ to $\mathbb{A}_p^{1,an}$ satisfying*

$$\delta_H(\zeta_1, \zeta_2) = \limsup_{(z,w) \to (\zeta_1, \zeta_2)} |z - w|_p$$

for Type I points $z, w \in \mathbb{C}_p$.

The Hsia kernel δ_H is continuous as a function in each variable separately, but it is only upper-semi-continuous as a function of $(\zeta_1, \zeta_2) \in (\mathbb{A}_p^{1,an})^2$. It is continuous away from the diagonal, and it is continuous at points (z, z) for z of Type I, but it fails to be continuous along the diagonal at points (ζ, ζ) where ζ has Type II, III, or IV.

14.2. **The potential function.** Suppose that μ is a signed, finite measure with compact support on $\mathbb{A}_p^{1,an}$. We define its **potential function** by the integral

$$V_\mu(\zeta) = \int_{\mathbb{A}_p^{1,an}} \log_p \delta_H(\zeta, \xi)\, d\mu(\xi) \tag{14.1}$$

where δ_H is the Hsia kernel.

We shall see that $\Delta V_\mu = \mu$ on $\mathbb{A}_p^{1,an}$, for a Laplacian operator we will next.

Example 14.2. Suppose that $\mu = \delta_{z_0}$ for a Type I point $z_0 \in \mathbb{C}_p$. Then its potential is

$$V_{z_0}(\zeta) = \log_p \delta_H(\zeta, z_0) = \log_p \|x - z_0\|_\zeta$$

for all $\zeta \in \mathbb{A}^{1,an}_p$. Note that V_{z_0} is continuous as a function from $\mathbb{A}^{1,an}_p$ to $\mathbb{R} \cup \{-\infty\}$, and $V_{z_0}(\zeta) \to -\infty$ as $\zeta \to z_0$. This is similar to the behavior of the function $V(z) = \log |z - z_0|$ on $\mathbb{C}$, the potential function for the delta mass δ_{z_0} in $\mathbb{C}$.

Suppose that $\mu = \delta_{\zeta_0}$ for a Type II or III point $\zeta_0 = \zeta(z_0, r_0) \in \mathbb{A}^{1,an}_p$. Then its potential is

$$V_{\zeta_0}(\zeta) = \log_p \delta_H(\zeta, \zeta_0) = \log_p \max\{r_0, \|x - z_0\|_\zeta\}$$

for all $\zeta \in \mathbb{A}^{1,an}_p$. Note that V_{ζ_0} is continuous as a function from $\mathbb{A}^{1,an}_p$ to $\mathbb{R}$. This is similar to the behavior of the function $V(z) = \log \max\{r_0, |z-z_0|\}$ on $\mathbb{C}$, the potential function for the uniform measure on the circle $\{|z - z_0| = r_0\}$ in $\mathbb{C}$.

The case of a delta mass at a Type IV point is similar. Its diameter is positive, and again the potential is bounded from below and continuous on $\mathbb{A}^{1,an}_p$.

14.3. **Hyperbolic distance.** There is a natural metric on

$$\mathbb{H}_{\mathbb{C}_p} := \mathbb{P}^{1,an}_p \setminus \mathbb{P}^1(\mathbb{C}_p),$$

though inducing a *finer* topology than the one we defined, that will be convenient to use for defining the Laplacian. If points $\zeta_1 = \zeta(z_1, r_1)$ and $\zeta_2 = \zeta(z_2, r_2)$ are points of Type II or III for which $D(z_1, r_1) \subset D(z_2, r_2)$ in $\mathbb{C}_p$, we set

$$\rho_{\mathbb{H}}(\zeta_1, \zeta_2) = \log_p(r_2/r_1).$$

Remark: Over $\mathbb{C}$, this quantity is (2π times) the modulus of the concentric round annulus $D(z_1, r_2) \setminus \overline{D}(z_1, r_1)$. The hyperbolic metric is sometimes called the modulus metric. This extends to all points of $\mathbb{A}^{1,an}_p \setminus \mathbb{C}_p$ by letting

$$\rho_{\mathbb{H}}(\zeta_1, \zeta_2) = 2 \log_p \operatorname{diam}(\zeta_1 \vee \zeta_2) - \log_p \operatorname{diam} \zeta_1 - \log_p \operatorname{diam} \zeta_2.$$

Proposition 14.3. [BR2, Proposition 2.29] *The metric space $(\mathbb{H}_{\mathbb{C}_p}, \rho_{\mathbb{H}})$ is complete.*

Now let S be a finite set of points in $\mathbb{H}_{\mathbb{C}_p}$, and let Γ_S be the connected (convex) hull of S. Then Γ_S is a closed subset of $\mathbb{A}^{1,an}_p$ (in the usual topology). Note that Γ inherits the structure of a finite metrized tree, from the $\rho_{\mathbb{H}}$ metric. We call such trees **finite graphs** in $\mathbb{H}_{\mathbb{C}_p}$.

14.4. **Coherent system of measures.** Suppose that $S_1 \subset S_2$ are closed connected subsets of $\mathbb{P}^{1,an}_p$. There is a **retraction map**

$$r_{S_2,S_1} : S_2 \to S_1$$

defined as follows. Choose any point $s \in S_1$. Then for each $\zeta \in S_2$, the point $r_{S_2,S_1}(\zeta)$ is the first point in S_1 on the path from ζ to s. Given any open and connected set

$U \subset \mathbb{P}^{1,an}_p$, a collection of (finite, signed) measures $\{\mu_\Gamma\}$ on finite graphs $\Gamma \subset U$ is called **coherent** if

1. for each pair of finite graphs $\Gamma_1 \subset \Gamma_2 \subset U$, we have $(r_{\Gamma_2,\Gamma_1})_*\mu_{\Gamma_2} = \mu_{\Gamma_1}$; and
2. there is a (uniform) constant B so that $|\mu_\Gamma|(\Gamma) \leq B$.

The finite graphs inside U form a directed system by inclusion. A coherent system of measures induces then induces a measure μ supported on *the closure* $\overline{U}$ so that

$$\int_{\overline{U}} f(x)\, d\mu(x) = \lim_{\Gamma} \int_\Gamma f(x)\, d\mu_\Gamma(x)$$

for all functions f that are continuous on $\overline{U}$. See [BR2, Proposition 5.10]. The measure μ satisfies $(r_{\overline{U},\Gamma})_*\mu = \mu_\Gamma$ for every finite graph $\Gamma \subset U$.

14.5. **The Laplacian.** We are now ready to define the Laplacian operator on functions of bounded differential variation. Let $U \subset \mathbb{P}^{1,an}_p$ be an open and connected subset. We say that $f \in \mathrm{BDV}(U)$ if

1. $f|_\Gamma \in \mathrm{BDV}(\Gamma)$ for every finite graph $\Gamma \subset U$; and
2. there is a constant $B = B(f)$ so that $|\Delta_\Gamma f|(\Gamma) \leq B$ on every finite graph $\Gamma \subset U$.

Theorem 14.4. [BR2, Proposition 5.14, Definition 5.15] *Let $U \subset \mathbb{P}^{1,an}_p$ be an open and connected set. For each $f \in \mathrm{BDV}(U)$, the Laplacians $\Delta_\Gamma f$ on the finite graphs $\Gamma \subset U$ form a coherent collection of measures and induce a finite, signed Borel measure, the* **complete Laplacian** *$\Delta_{\overline{U}} f$, on $\overline{U}$.*

Proof. The boundedness condition for coherence follows from the boundedness condition in the definition of $\mathrm{BDV}(U)$. So it suffices to show that $(r_{\Gamma_2,\Gamma_1})_*\Delta_{\Gamma_2} f = \Delta_{\Gamma_1} f$ for all $\Gamma_1 \subset \Gamma_2 \subset U$. And for this, it suffices to assume that $\Gamma_2 = \Gamma_1 \cup E$ for a single edge E attached at a point $p \in \Gamma_1$. But we see from the definition (13.3) that

$$\begin{aligned} \Delta_{\Gamma_2} f((r_{\Gamma_2,\Gamma_1})^{-1}(p)) \;=\; \Delta_{\Gamma_2} f(E) \;&=\; \sum_{\vec{v} \in T_p^{out}(\Gamma_2,E)} d_{\vec{v}} f(p) \\ &=\; \sum_{\vec{v} \in T_p\Gamma_1} d_{\vec{v}} f(p) \;=\; \Delta_{\Gamma_1} f(\{p\}) \end{aligned}$$

For all other points $\zeta \in \Gamma_1 \setminus \{p\}$, we clearly have

$$\Delta_{\Gamma_2} f((r_{\Gamma_2,\Gamma_1})^{-1}(\zeta)) = \Delta_{\Gamma_1} f(\{\zeta\}),$$

and this completes the proof. □

The **Laplacian of f on U** is the finite, signed measure

$$\Delta_U f = \Delta_{\overline{U}} f - (\Delta_{\overline{U}} f)|_{\partial U}.$$

Example 14.5. Let $U = D^{an}(0,1)$, the open Berkovich unit disk. For each $y \in U$, let

$$f_y(x) = \log_p \delta_H(x,y) = \log_p \operatorname{diam}(x \vee y)$$

for the Hsia kernel $\delta_H(x,y)$. Then, on each finite graph $\Gamma \subset U$, we have

$$\Delta_\Gamma f_y = \delta_{r(y)} - \delta_{r(\zeta_G)},$$

where $r = r_\Gamma$ is the retraction to Γ and $\zeta_G = \partial U$ is the Gauss point. To see this, it is most important to recall that the coordinates on Γ in the $\rho_{\mathbb{H}}$-metric are logarithmic in the diameter. Let γ_y be the path in Γ from y to ζ_G. The function f_y is constant on connected components of $\Gamma \setminus (\gamma_y \cap \Gamma)$, and it is the log of the diameter of x along $\gamma_y \cap \Gamma$. But $\log_p \operatorname{diam} x$ defines a linear coordinate on the interval $[r(y), r(\zeta_G)]$, so the function f_y is piecewise affine on Γ, with slope $+1$ along $\gamma_y \cap \Gamma$ and slope 0 elsewhere. We conclude that

$$\Delta_{\overline{U}} f = \delta_y - \delta_{\zeta_G}$$

so that

$$\Delta_U f = \delta_y.$$

14.6. **The potential function is a potential function.** The Hsia kernel δ_H can be related to the kernel function $j_z(x,y)$ on finite graphs $\Gamma \subset \mathbb{H}_{\mathbb{C}_p}$ defined in §13.4. Fixing any point $z \in \mathbb{H}_{\mathbb{C}_p}$ and any finite graph Γ that contains z, let ∞_Γ be the retraction of ∞ to Γ. Then

$$\log_p \delta_H(x,y) = j_z(x,y) - j_z(x,\infty_\Gamma) - j_z(\infty_\Gamma, y) + \log_p \operatorname{diam}(z) \tag{14.2}$$

for all $x \in \Gamma$. To see this, we observe that, as in Example 14.5, for each fixed $y \in \Gamma$, the function

$$f_y(x) = \log_p \delta_H(x,y) = \log_p \operatorname{diam}(x \vee y)$$

is constant on connected components of $\Gamma \setminus \gamma_y$, and it is the log of the diameter of x along γ_y. Its Laplacian is computed as

$$\Delta_\Gamma f_y = \delta_y - \delta_{\infty_\Gamma}.$$

This implies that, up to adding a constant $C = C(y,z)$, we have

$$\log_p \delta_H(x,y) = j_z(x,y) - j_z(x,\infty_\Gamma) + C(y,z)$$

on Γ. We repeat the argument with variable y, and normalize by the fact that $\delta_H(z,z) = \operatorname{diam} z$. See [BR2, Proposition 4.3].

We are now ready to show that the potential function V_μ defined in (14.1) is truly a potential for the measure μ on the affine line.

Theorem 14.6. [BR2, Example 5.21] *Suppose that μ is a (finite, signed) measure on $\mathbb{A}_p^{1,an}$ with compact support. Then*

$$V_\mu(\zeta) = \int_{\mathbb{A}_p^{1,an}} \log_p \delta_H(\zeta, \xi)\, d\mu(\xi)$$

is a function in $\mathrm{BDV}(\mathbb{P}_p^{1,an})$ *and satisfies*

$$\Delta_{\mathbb{P}_p^{1,an}} V_\mu = \mu - \mu(\mathbb{P}_p^{1,an})\, \delta_\infty$$

and

$$\Delta V_\mu = \mu \qquad \text{on } \mathbb{A}_p^{1,an}.$$

Proof. Since μ has compact support, we can choose a radius R so that the support of μ is contained in the Berkovich disk $\overline{D}^{an}(0, R)$. Let $z = \zeta(0, R) \in \mathbb{A}_p^{1,an}$.

Now let $\Gamma \subset \mathbb{H}_{\mathbb{C}_p}$ be any finite graph containing z. Let

$$r_\Gamma : \mathbb{P}_p^{1,an} \to \Gamma$$

be the retraction map. From (14.2), we know that

$$\log_p \delta_H(x, y) = j_z(x, y) - j_z(x, \infty_\Gamma) - j_z(\infty_\Gamma, y) + \log_p R$$

for all $x, y \in \Gamma$, where $\infty_\Gamma = r_\Gamma(\infty)$. Since $j_z(z, y) = 0$ for all y and $\Delta_\Gamma j_z(\cdot, y) = \delta_y - \delta_z$, we have $j_z(x, y) = 0$ whenever $\operatorname{diam} x > R$ and $\operatorname{diam} y \le R$.

Let μ_Γ be the pushforward of μ by the r_Γ, so μ_Γ is a finite, signed measure on Γ. Note that $\{\mu_\Gamma\}$ is a coherent system of measures and that $\mu_\Gamma(\Gamma) = \mu(\mathbb{P}_p^{1,an})$ for all Γ.

From Proposition 13.7, we know that

$$U_{\mu_\Gamma}(x) = \int_\Gamma j_z(x, y)\, d\mu_\Gamma(y)$$

is a function in $\mathrm{BDV}(\Gamma)$ and

$$\Delta_\Gamma U_{\mu_\Gamma} = \mu_\Gamma - \mu_\Gamma(\Gamma)\, \delta_z = \mu_\Gamma - \mu(\mathbb{P}_p^{1,an})\, \delta_z.$$

Moreover, observe that $U_{\mu_\Gamma}(\infty_\Gamma) = 0$ for all Γ, because of the choice of z. And so

$$\begin{aligned}
V_\Gamma(x) &:= \int_\Gamma \log_p \delta_H(x, y)\, d\mu_\Gamma(y) \\
&= \int_\Gamma \big(j_z(x, y) - j_z(x, \infty_\Gamma) - j_z(\infty_\Gamma, y) + \log_p R\big) d\mu_\Gamma(y) \\
&= U_{\mu_\Gamma}(x) - U_{\mu_\Gamma}(\infty_\Gamma) - \mu(\mathbb{P}_p^{1,an})\big(j_z(x, \infty_\Gamma) + \log_p R\big) \\
&= U_{\mu_\Gamma}(x) - \mu(\mathbb{P}_p^{1,an})\big(j_z(x, \infty_\Gamma) + \log_p R\big).
\end{aligned}$$

Computing the Laplacian on Γ, we have

$$\Delta_\Gamma V_\Gamma = \mu_\Gamma - \mu(\mathbb{P}_p^{1,an})\, \delta_{\infty_\Gamma}.$$

It remains to show that $V_\Gamma = V_\mu|_\Gamma$. For then, we would have

$$\Delta_\Gamma(V_\mu|_\Gamma) = (r_\Gamma)_*(\mu - \mu(\mathbb{P}^{1,an}_p)\,\delta_\infty)$$

and $V_\mu|_\Gamma \in \mathrm{BDV}(\Gamma)$ for all graphs Γ containing z, and the proof will be done. Note that, by our choice of z and Γ, the path from any $y \in \operatorname{supp}\mu$ to ∞ must pass through z. For $x \in \Gamma$ with $\operatorname{diam} x > R$, we have $x \vee y = x$ for all $y \in \operatorname{supp}\mu$. For $x \in \Gamma$ with $\operatorname{diam} x \leq R$, then its path to ∞ must include $[x, z] \subset \Gamma$, so again $x \vee y \in \Gamma$ for all $y \in \operatorname{supp}\mu$. This shows that

$$\delta_H(x, y) = \delta_H(x, r_\Gamma(y))$$

for all $x \in \Gamma$ and all $y \in \operatorname{supp}\mu$. Consequently, $V_\Gamma = V_\mu|_\Gamma$. □

15. Capacity and equilibrium measures in $\mathbb{A}^{1,an}_p$

Suppose that S is a bounded set in $\mathbb{A}^{1,an}_p$. For each probability measure μ supported in S, we define its energy by

$$E(\mu) = -\iint \log_p \delta_H(x, y)\, d\mu(x)\, d\mu(y).$$

The **capacity** of S is the quantity

$$\operatorname{cap}_p S = p^{-\inf E(\mu)}$$

where the infimum is taken over all probability measures supported in S.

Example 15.1. Let $S = \{\zeta_0\}$ consist of a single point in $\mathbb{A}^{1,an}_p$. The only measure supported in S is μ_{ζ_0}. From the formulas for the potential functions computed in Example 14.2, the energy of this measure must be

$$E(\mu_{\zeta_0}) = -\log_p(\operatorname{diam}\zeta_0).$$

So the capacity of S is

$$\operatorname{cap}_p(\{\zeta_0\}) = p^{\log_p(\operatorname{diam}\zeta_0)} = \operatorname{diam}\zeta_0.$$

In particular, we have $\operatorname{cap}_p(\{\zeta_0\}) = 0$ if and only if ζ_0 is Type I.

Example 15.2. Let $S = \mathbb{Z}_p$, the p-adic integers in $\mathbb{Q}_p$, and let μ_p be the (additive) Haar measure on $\mathbb{Z}_p$, normalized to have total mass 1. Note that

$$V_{\mu_p}(x) = \int_{\mathbb{Z}_p} \log_p \delta_H(x, y)\, d\mu_p(y) = \int_{\mathbb{Z}_p} \log_p |x - y|_p\, d\mu_p(y)$$

for all $x \in \mathbb{Z}_p$, so that, by translation invariance, for any other point $x' = x + \alpha \in \mathbb{Z}_p$, we have

$$\begin{aligned} V_{\mu_p}(x') &= V_{\mu_p}(x+\alpha) \\ &= \int_{\mathbb{Z}_p} \log_p |x+\alpha-y|_p \, d\mu_p(y) \\ &= \int_{\mathbb{Z}_p} \log_p |x+\alpha-y|_p \, d\mu_p(y-\alpha) \\ &= V_{\mu_p}(x). \end{aligned}$$

In other words, V_{μ_p} is constant on $\mathbb{Z}_p$. We shall see, as a consequence of the non-archimedean analog of Frostman's theorem, that this implies μ_p must be the (unique) equilibrium measure for $S = \mathbb{Z}_p$.

Note that $\mu_p(p^n\mathbb{Z}_p) = 1/p^n$ for all $n \geq 0$ by translation invariance of μ_p, so that $\mu_p(p^n\mathbb{Z}_p^*) = \frac{1}{p^n} - \frac{1}{p^{n+1}}$. We can now compute:

$$\begin{aligned} E(\mu_p) &= -\int_{\mathbb{Z}_p} V_{\mu_p}(x)\, d\mu_p(x) \\ &= -V_{\mu_p}(0) \\ &= -\int_{\mathbb{Z}_p} \log_p |y|_p \, d\mu_p(y) \\ &= -\sum_{n=0}^{\infty} \int_{p^n\mathbb{Z}_p^*} \log_p |y|_p \, d\mu_p(y) \\ &= -\sum_{n=0}^{\infty} (\log_p p^{-n}) \left(\frac{1}{p^n} - \frac{1}{p^{n+1}} \right) \\ &= -(p-1) \sum_{n=0}^{\infty} (-n)\, p^{-n-1} \\ &= -(p-1) \frac{d}{dp} \left(\sum_n p^{-n} \right) \;=\; -(p-1)\frac{d}{dp}\left(\frac{p}{p-1}\right) \\ &= \frac{1}{p-1} \end{aligned}$$

Finally, this gives

$$\operatorname{cap}_p \mathbb{Z}_p = p^{-1/(p-1)}.$$

15.1. **Equilibrium measure.** Let S be a compact set in $\mathbb{A}_p^{1,an}$. An **equilibrium measure** for S is a probability measure μ_S for which

$$E(\mu_S) = \inf_{\mu} E(\mu)$$

over all probability measures μ supported in S.

Theorem 2.2 has a p-adic incarnation, with a nearly identical proof:

Theorem 15.3. [BR2, Theorem 6.18, Proposition 6.8, Proposition 7.21] *Suppose that S is a compact set in $\mathbb{A}^{1,an}_p$ with $\operatorname{cap}_p S > 0$. There exists a unique equilibrium measure μ_S for S, and it satisfies*

1. $\operatorname{supp}\mu_S \subset \partial^o S$, *the outer boundary of S;*
2. $V_{\mu_S}(\zeta) \geq -E(\mu_S)$ *at all points* $\zeta \in \mathbb{A}^{1,an}_p$*; and*
3. $V_{\mu_S}(\zeta) = -E(\mu_S)$ *for all* $\zeta \in S \setminus S_0$, *where S_0 is a subset of capacity 0.*

Example 15.4. Let $S = \overline{D}^{an}(0,1)$, the Berkovich closed unit disk in $\mathbb{A}^{1,an}_p$. Then $\mu_S = \delta_{\zeta_G}$, the delta-mass supported at the Gauss point. Indeed, the outer boundary of S consists only of the single point ζ_G, and there is a unique probability measure supported on the point. As computed in Example 14.2, we see that this V_{μ_S} is constant 0 at all points of S.

Reading carefully through the (sketch of the) proof of Theorem 2.2 in these notes, you will see just a few places that will need further justification for this non-archimedean version. The first you might notice is the use of V_ν being upper semicontinuous. See [BR2, Proposition 6.12]. The proof also uses a (strong) maximum principle for harmonic functions. This, of course, requires having a notion of harmonic functions, and that can be a bit delicate, so I include a definition below. In the final step, the proof over $\mathbb{C}$ appeals to an identity principle for subharmonic functions; in this case, it is possible to again use a maximum principle for harmonic functions. See the proof of uniqueness of the equilibrium measure in [BR2, Proposition 7.21].

15.2. **Harmonic functions.** Suppose that $U \subset \mathbb{P}^{1,an}_p$ is a domain. We say that a function

$$h : U \to \mathbb{R}$$

is **strongly harmonic** if it is continuous on U, belongs to $\mathrm{BDV}(U)$, and satisfies

$$\Delta_U h \equiv 0.$$

We say that a function h is **harmonic** in U if, for each element $\zeta \in U$, there is a domain $U_\zeta \subset U$ containing ζ on which h is strongly harmonic.

Example 15.5. For any probability measure μ with compact support in $\mathbb{A}^{1,an}_p$, the potential function V_μ is strongly harmonic on each connected component of $\mathbb{A}^{1,an}_p \setminus (\operatorname{supp}\mu)$.

Example 15.6. Here is a construction of a harmonic function that is *not* strongly harmonic. Let $U = \mathbb{P}^{1,an}_p \setminus \mathbb{Z}_p$. Let $D \subset U$ be the infinite tree which is the convex hull of $\mathbb{Z}_p$ within U. The branch points of D are the Type II points $\zeta = \zeta(m, p^{-n})$ for integers $n \geq 0$ and $0 \leq m \leq p^n - 1$. The Gauss point ζ_G is a vertex of valence p, while all the other vertices have valence $p+1$, with p edges pointing away from ζ_G.

Each edge has length 1 in the hyperbolic metric. To define a function h, we set $h = 0$ at ζ_G and at all points above ζ_G in U. As $p \geq 2$, we can choose 2 edges adjacent to ζ_G in D, and let h have slope 1 on one of them and -1 on the other. Extend h by 0 on any remaining edges of D adjacent to ζ_G. Inductively, from any vertex v of D, we extend h on edges as follows: let s be the slope of h on the edge above v, choose 2 edges adjacent v that lead away from ζ_G, let h have slope $s + 1$ on one and $s - 1$ on the other, and extend h with slope 0 on any remaining edges. We then extend h to all of $U \setminus D$ to be constant on fibers of the retraction $r : U \to D$. In this way, we obtain a function h with $\Delta_U = 0$. It is not strongly harmonic, because the slopes on edges of D are unbounded, so $h \notin \mathrm{BDV}(U)$. But it is strongly harmonic on open neighborhoods of points, because we can always exclude the regions where the slopes grow. So h is harmonic on U.

The Strong Maximum Principle for harmonic functions is:

Theorem 15.7. [BR2, Propositions 7.16 and 7.17] *Suppose that $h : U \to \mathbb{R}$ is a nonconstant harmonic function on a domain U. Then h does not achieve a maximum or a mimium in U.*

Moreover, if h is bounded above on U, and if there exists M so that

$$\limsup_{\zeta \to z} h(\zeta) \leq M$$

for $\zeta \in U$ and $z \in \partial U \setminus S_0$ where $\mathrm{cap}_p S_0 = 0$, then

$$h(\zeta) \leq M$$

all points $\zeta \in U$.

I won't give the proof, but it uses the existence of potential functions for measures of infinite energy that show the sets of capacity 0 are "polar" in the traditional sense:

Lemma 15.8. [BR2, Lemma 7.18] *Suppose that S_0 is a compact set of capacity 0 in $\mathbb{A}^{1,an}_p$. Then there exists a probability measure μ supported in S_0 so that its potential function V_μ satisfies*

$$\lim_{\zeta \to x} V_\mu(\zeta) = -\infty$$

for all $x \in S_0$.

Compare the statement of Lemma 15.8 to Evans' Theorem [Ra, Theorem 5.5.6]. Such potentials V_μ are sometimes called Evans functions.

16. The Baker-Rumely (affine) adelic equidistribution theorem

We follow §7.9 of [BR2]. Let K be a number field, and let M_K be its set of places. Recall the product formula from (1.2):

$$\prod_{v\in M_K} |\alpha|_v^{N_v} = 1$$

for all $\alpha \neq 0$ in K, where $N_v = [K_v : \mathbb{Q}_v]$. Recall that K_v denotes the completion of K with respect to the given absolute value. Observe that we can reformulate the product formula as

$$\sum_{v\in M_K} N_v(\log_p |\alpha|_v)(\log p_v) = \sum_{v\in M_K} N_v \log|\alpha|_v = 0 \tag{16.1}$$

where $p_v = p$ if $v|p$ and $p_v = e$ if $v|\infty$, and log is the usual natural logarithm.

Let $\overline{K}$ denote an algebraic closure of K. For each place $v|p$, we fix an embedding of $\overline{K}$ into $\mathbb{C}_p$ that extends the canonical embedding of K into the completion K_v. For archimedean $v \in M_K^\infty$, we do the same: fix an embedding of $\overline{K} \subset \mathbb{C}$ that extends $K \subset K_v$.

16.1. **Global capacity.** Consider sets of the form

$$\mathbb{S} = \prod_{v\in M_K} S_v$$

where each S_v is a compact subset of $\mathbb{A}_p^{1,an}$ at places $v|p$ and of $\mathbb{C}$ for $v|\infty$. Assume, further, that

$$S_v = \overline{D}^{an}(0,1)$$

at all but finitely many (non-archimedean) places v. Such a set is called a **compact Berkovich adelic set**. Its capacity is defined to be

$$\operatorname{cap}\mathbb{S} = \prod_{p\in M_{\mathbb{Q}}} \prod_{v|p} (\operatorname{cap}_p S_v)^{N_v}$$

where, for $p=\infty$, the capacity $\operatorname{cap}_\infty$ denotes the usual capacity in $\mathbb{C}$.

16.2. **Heights associated to compact adelic sets.** Fix a compact Berkovich adelic set $\mathbb{S}$ for which $\operatorname{cap}_p S_v > 0$ at every place v of K. Let

$$G_{S_v}(\zeta) = \big(V_{\mu_{S_v}}(\zeta) + E(\mu_{S_v})\big) \log p_v \tag{16.2}$$

be the **Green's function** for S_v, where $p_v = p$ if $v|p$ and $p_v = e$ if $v|\infty$. The Green's function is a potential function for the measure $(\log p_v)\,\mu_{S_v}$ and is $\equiv 0$ on points of S_v (except possibly on a subset of capacity 0).

For any finite set $F \subset \overline{K}$ which is invariant under $\operatorname{Gal}(\overline{K}/K)$, we set

$$h_{\mathbb{S}}(F) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K} N_v \left(\frac{1}{|F|} \sum_{x\in F} G_{S_v}(x) \right).$$

This definition extends to a function $h_{\mathbb{S}} : \overline{K} \to \mathbb{R}$ by setting $h(x)$ to be constant on the $\mathrm{Gal}(\overline{K}/K)$-orbit of x.

Example 16.1. The standard (logarithmic) Weil height, defined by (1.1) or Proposition 1.2, is the height $h_{\mathbb{S}}$ for the compact adelic set

$$\mathbb{S} = \overline{\mathbb{D}} \times \prod_p \overline{D}^{an}(0,1)$$

over $\mathbb{Q}$, where $\overline{\mathbb{D}} \subset \mathbb{C}$ is the closed unit disk. This is because

$$\log^+ |z|_p = \max\{(\log p) \log_p |z|_p, 0\}$$

at each (finite) prime p of $\mathbb{Q}$.

Note that, by construction, these functions $h_{\mathbb{S}}$ are always non-negative on $\overline{K}$.

16.3. **Equidistribution.**

Theorem 16.2. [BR2, Theorem 7.52] *Suppose that* $\mathbb{S}$ *is a compact Berkovich adelic set with* $\operatorname{cap}\mathbb{S} = 1$*. Suppose that* F_n *is a sequence of finite sets in* $\overline{K}$ *that are* $\mathrm{Gal}(\overline{K}/K)$*-invariant, satisfying* $|F_n| \to \infty$ *and* $h_{\mathbb{S}}(F_n) \to 0$ *as* $n \to \infty$*. Then, at each place* $v|p$ *of* K*, we have weak convergence*

$$\frac{1}{|F_n|} \sum_{x \in F_n} \delta_x \longrightarrow \mu_{S_v}$$

in $\mathbb{A}^{1,an}_p$ *(if* v *is non-archimedean) or in* $\mathbb{C}$ *(if* $p = \infty$*).*

Without the capacity hypothesis, we run into trouble. Consider the following two examples:

Example 16.3. Let $K = \mathbb{Q}$ and put $S_p = \overline{D}^{an}(0,1)$ at all (finite) primes p. Let $S_\infty = \overline{D}(0,R)$ for some $R > 1$ at the archimedean place. Then $\operatorname{cap}\mathbb{S} = R > 1$. Note that the $h_{\mathbb{S}}$-height of all roots of unity is 0. Taking Galois-invariant sets of roots of unity for F_n, we obtain a sequence of discrete measures that converge to the Lebesgue measure on the unit circle in $\mathbb{C}$ and do not converge to the equilibrium measure at the archimedean place. Moreover, from the Fekete-Szegö Theorem 3.1, that there exist infinitely many Galois-invariant sets of algebraic integers that lie in a small neighborhood of the circle $\{|z| = r\}$ for any $1 \le r \le R$. This shows that for each such r, we can find a sequence of finite sets F_n satisfying the conditions of Theorem 16.2 that are converge in $\mathbb{C}$ to a measure supported in $\{|z| = r\}$.

Example 16.4. Let $K = \mathbb{Q}$ and put $S_p = \overline{D}^{an}(0,1)$ at all (finite) primes p. Let $S_\infty = \overline{D}(0,R)$ for some $0 < R < 1$ at the archimedean place. Then $\operatorname{cap}\mathbb{S} = R < 1$. Now we run into trouble finding sequences that are small for $h_{\mathbb{S}}$. Indeed, if there were an infinite sequence of points $x_n \in \overline{\mathbb{Q}}$ with $h_{\mathbb{S}}(x_n) \to 0$, we would find that x_n violates the product formula for all n sufficiently large. So the conclusion of Theorem 16.2 would only be satisfied vacuously.

But note that, in either setting of Example 16.3 or 16.4, if we express the radius R as

$$R = \prod_{i=1}^{n} p_i^{r_i}$$

for a finite set of primes p_i and real numbers r_i, then we can modify our set $\mathbb{S}$ by choosing S_{p_i} to be the closed Berkovich disk of radius $p_i^{-r_i}$ centered at 0. Then we will have $\operatorname{cap}\mathbb{S} = 1$. Small-height points will exist by an adelic version of the Fekete-Szegö Theorem 3.1, and their Galois orbits will converge to the equilibrium measures at all places, by Theorem 16.2.

Proof of Theorem 16.2. The proof is nearly identical to Rumely's proofs of Theorems 1.1 and 3.4. Define the discrete probability measures

$$\mu_n = \frac{1}{|F_n|} \sum_{x \in F_n} \delta_x.$$

There are 3 steps to the proof. We begin by working locally, at a fixed place $v|p$ of K with its chosen embedding $\overline{K} \to \mathbb{C}_p$. (The archimedean places will be handled similarly.)

Step 1 is to pass to a subsequence so that $\mu_n \to \nu$ weakly in $\mathbb{P}^{1,an}_p$. The hypothesis that $h_{\mathbb{S}}(F_n) \to 0$ implies that $\operatorname{supp}\nu \subset S_v$. Indeed, if there were a point $\zeta \in \operatorname{supp}\nu$ that lies outside S_v, then there must be a positive proportion of the points of F_n in a neighborhood of ζ, for all n large. But this would imply that

$$\liminf_{n\to\infty} \frac{1}{|F_n|} \sum_{x \in F_n} G_v(x)$$

is strictly positive (using the Maximum Principle, Theorem 15.7, applied to G_v on the component of $\mathbb{P}^{1,an}_p \setminus S_v$ containing ∞).

Step 2 is to consider the modified energy

$$E_v^*(\mu_n) = -\iint_{\mathbb{A}^{1,an}_p \times \mathbb{A}^{1,an}_p \setminus \mathrm{Diag}} \log_p \delta_H(x,y)\, d\mu_n(x)\, d\mu_n(y),$$

as in (1.4). We then mimic the proof of Lemma 1.9. For estimating integrals II and III, we observe that there is a uniform constant $C > 0$ so that

$$\log_p \delta_H(x,y) \leq (\log p_v)^{-1} \left(G_v(x) + G_v(y)\right) + C$$

for all $x, y \in \mathbb{A}^{1,an}_p$. This allows us to relate the integrals to the height $h_{\mathbb{S}}(F_n)$, exactly as in the proof of Lemma 1.9. We conclude that

$$\liminf_{n\to\infty} E_v^*(\mu_n) \geq E(\nu) \geq E(\mu_{S_v}) \tag{16.3}$$

with the final equality because μ_{S_v} is the equilibrium measure.

Step 3 proceeds as in the last part of the proof of Theorem 1.1. We consider a discriminant

$$D(F_n) = \prod_{x \neq y \in F_n} (x - y) \in K^*.$$

From the product formula, we know that

$$\prod_{v \in M_K} |D(F_n)|_v^{N_v} = 1$$

for every n. Note that

$$(\log p_v)\, E_v^*(\mu_n) = -(\log p_v) \frac{1}{|F_n|^2} \log_p |D(F_n)|_v = -\frac{1}{|F_n|^2} \log |D(F_n)|_v$$

at each place v, so that

$$\sum_{v \in M_K} (\log p_v)\, N_v\, E_v^*(\mu_n) = 0$$

for every n.

For each place v, define a sequence $b_{v,n} \in \mathbb{R}$ by

$$b_{v,n} = N_v \left(2 \frac{1}{|F_n|} \sum_{x \in F_n} G_v(x) + (\log p_v)\, E_v^*(\mu_n) - (\log p_v)\, E(\mu_{S_v}) \right). \tag{16.4}$$

Observe that

$$\liminf_{n \to \infty} b_{v,n} \geq 0$$

for every place v, because of (16.3) and since $h_{\mathbb{S}}(F_n) \geq 0$ for all n. For v outside a finite set M^- of places, we know that $S_v = \overline{D}^{an}(0,1)$, and therefore $E(\mu_{S_v}) = 0$ and $G_v(x) = \log^+ |x|_v$ at these places. At each of these places $v \notin M^-$, the estimates that appear in the final lines of the proof of Theorem 1.1 show that

$$(\log p_v)\, E_v^*(\mu_n) = -\frac{1}{|F_n|^2} \log |D(F_n)|_v \geq -2\, G_v(F_n).$$

In other words, we have

$$b_{v,n} \geq 0$$

for *all* n and for all $v \notin M^-$. We also know that

$$\sum_{v \in M_K} b_{v,n} = 2\, h_{\mathbb{S}}(F_n) \tag{16.5}$$

by the product formula and the hypothesis on the capacity. It follows that

$$\liminf_{n \to \infty} b_{v,n} = 0 \tag{16.6}$$

for all places v. Indeed, suppose there were a place v_0 for which $\liminf b_{v_0,n} = B_{v_0} > 0$. There exists N so that $h_{\mathbb{S}}(F_n) < \frac{1}{4} B_{v_0}$ and $b_{v_0,n} \geq \frac{1}{2} B_{v_0}$ for all $n \geq N$. So for each of these $n \geq N$, there is a finite set of places $V_n \subset M^-$ for which $b_{v,n} < 0$ for all

$v \in V_n$ and $\sum_{v\in V_n} b_{v,n} < -\frac{1}{4}B_{v_0}$. But, as M^- is finite, there must be some $v \in M^-$ so that $b_{v,n} \leq -B_{v_0}/(4|M^-|)$ for infinitely many n. This contradicts the fact that $\liminf b_{v,n} \geq 0$.

Combining (16.3) with (16.6), we find that $E(\nu) = E(\mu_{S_v})$ at every place v of K. Thus, from Theorem 15.3, we conclude that $\nu = \mu_{S_v}$. In other words, every subsequential limit if the discrete measures μ_n in $\mathbb{A}^{1,an}_p$ (or in $\mathbb{C}$ for the archimedean places) must be the equilibrium measure μ_{S_v}. The proof is complete. $\square$

17. Global dynamics and canonical heights

Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a map of degree $d \geq 2$ defined over a number field K. Recall that, in homogeneous coordinates, we can write

$$f(x:y) = (P(x,y), Q(x,y))$$

for homogeneous polynomials $P, Q \in K[x,y]$. Here we introduce the canonical height $\hat{h}_f$ of f on $\mathbb{P}^1(\overline{\mathbb{Q}})$, as defined by Call and Silverman in [CS].

When f is a *polynomial*, the equidistribution theorem of the previous section, Theorem 16.2 applies to show that small points for $\hat{h}_f$ are uniformly distributed – at all places of the field K – with respect to the equilibrium measure for the filled Julia sets of f. A similar result holds for all rational functions, but it requires another version of the equidistribution theorem.

17.1. **Functoriality.** Let $h : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the (logarithmic) Weil height, defined by (1.1) or Proposition 1.2, setting $h(\infty) = 0$. On $\mathbb{P}^1$, the height can also be expressed as

$$h(x:y) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K} N_v \max\{|x|_v, |y|_v\}$$

for any choice of number field K that contains x and y.

Theorem 17.1. [Si4, Theorem 3.11] *For each $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d defined over a number field, there exists a constant $C = C(f)$ so that*

$$d\,h(\alpha) - C \;\leq\; h(f(\alpha)) \;\leq\; d\,h(\alpha) + C$$

for all $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$.

Remark 17.2. The lower bound on $h(f(\alpha))$ holds for all morphisms $f : \mathbb{P}^n \to \mathbb{P}^m$, while the upper bound holds more generally for all rational maps $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$.

Proof. The upper bound on $h(f(\alpha))$ follow from the triangle inequality. Write $P(x,y) = \sum_i a_i x^{d-i} y^i$ and $\|P\|_v = \max_i |a_i|_v$. Set

$$h(f) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in K} N_v \log\max\{\|P\|_v, \|Q\|_v\},$$

Note that $h(f) \geq 0$ is simply the height of the point in $\mathbb{P}^{2d+1}(K)$ determined by the coefficients of f.

Fix $\alpha = (x : y) \in \mathbb{P}^1(\overline{\mathbb{Q}})$ and choose a number field K' extending K that contains x and y. For each place v of K', we have

$$|P(x,y)|_v \leq (d+1)^* \max_i |a_i x^{d-i} y^i|_v \leq (d+1)^* \|P\|_v \max\{|x|_v, |y|_v\}^d,$$

where the superscript $*$ indicates that we only include the term if the place v is archimedean. Similarly, we have

$$|Q(x,y)|_v \leq (d+1)^* \|Q\|_v \max\{|x|_v, |y|_v\}^d.$$

Summing over all places,

$$\begin{aligned} h(f(x:y)) &= \frac{1}{[K':\mathbb{Q}]} \sum_{v \in K'} N_v \log \max\{|P(x,y)|_v, |Q(x,y)|_v\} \\ &\leq d + 1 + h(f) + d\,h(x:y). \end{aligned}$$

For the lower bound on $h(f(\alpha))$, we use the fact that P and Q have no common zeroes in $\mathbb{P}^1(\overline{\mathbb{Q}})$. There exists an integer $e \geq 0$ and homogeneous polynomials $g_1, g_2, g_3, g_4 \in \overline{\mathbb{Q}}[x,y]$ of degree $e - d$ so that

$$x^e = g_1 P + g_2 Q \quad \text{and} \quad y^e = g_3 P + g_4 Q$$

Therefore, for our given $\alpha = (x : y)$ with coordinates in K', and for each place v of K', we have

$$\begin{aligned} \max\{|x|_v, |y|_v\}^e &= \max\{|g_1 P + g_2 Q|_v, |g_3 P + g_4 Q|_v\} \\ &\leq 2^* \max\{|g_1 P|_v, |g_2 Q|_v, |g_3 P|_v, |g_4 Q|_v\} \\ &\leq 2^*(e-d+1)^* \max_i \|g_i\|_v \max\{|x|_v, |y|_v\}^{e-d} \max\{|P(x,y)|_v, |Q(x,y)|_v\} \end{aligned}$$

where, again, the superscript $*$ indicates that we only include the term if the place v is archimedean. So, in other words, we have

$$\max\{|x|_v, |y|_v\}^d \leq 2^*(e-d+1)^* \max_i \|g_i\|_v \max\{|P(x,y)|_v, |Q(x,y)|_v\}.$$

Summing over all places, we conclude that

$$d\,h(x:y) \leq 2(e-d+1) + h(g) + h(f(x:y)),$$

where $h(g)$ is the height of the tuple of coefficients (g_1, g_2, g_3, g_4) in $\mathbb{P}^{4(e-d+1)-1}(\overline{\mathbb{Q}})$. □

17.2. **Preperiodic points.** As a consequence of Theorem 17.1, we have:

Proposition 17.3. *For each $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$, defined over a number field, the set of preperiodic points in $\mathbb{P}^1(\overline{\mathbb{Q}})$ has bounded height.*

Proof. For any point α and for any $n \geq 1$, we have $f^n(\alpha) = f(f^{n-1}(\alpha))$, so that Theorem 17.1 implies

$$h(f^n(\alpha)) \geq d\, h(f^{n-1}(\alpha)) - C \geq d^2\, h(f^{n-2}(\alpha)) - d\,C - C \geq \cdots,$$

giving

$$h(f^n(\alpha)) \geq d^n(h(\alpha) - C).$$

If α is preperiodic, then there exist $n > m \geq 0$ so that $f^n(\alpha) = f^m(\alpha)$. It follows that

$$h(f^m(\alpha)) = h(f^{n-m} f^m(\alpha)) \geq d^{n-m}(h(f^m(\alpha)) - C)$$

so that

$$h(f^m(\alpha)) \leq \frac{d^{n-m}}{d^{n-m} - 1} C \leq 2\,C.$$

And so

$$h(\alpha) \leq \frac{1}{d^m} h(f^m(\alpha)) + C \leq 3\,C.$$

□

From the Northcott property Proposition 1.5, we have

Corollary 17.4. *For each $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ defined over a number field K, and for any $D > 0$, there are at most finitely many preperiodic points of f with algebraic degree D over K.*

17.3. **Canonical height.** Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be any map of degree $d \geq 2$, defined over a number field. The **canonical height function**

$$\hat{h}_f : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}$$

is defined by

$$\hat{h}_f(\alpha) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(\alpha)).$$

The limit exists because of Theorem 17.1. Indeed, for any $n > m$, we have

$$\begin{aligned}\left|\frac{1}{d^n} h(f^n(\alpha)) - \frac{1}{d^m} h(f^m(\alpha))\right| &= \left|\sum_{j=m+1}^{n} \frac{1}{d^j}\big(h(f^j(\alpha)) - d\,h(f^{j-1}(\alpha))\big)\right| \\ &\leq \sum_{j=m+1}^{n} \frac{C}{d^j}\end{aligned}$$

which is small if m is sufficiently large. Furthermore, taking $m = 0$ and letting $n \to \infty$, we see that

$$|h - \hat{h}_f|$$

is uniformly bounded on $\mathbb{P}^1(\overline{\mathbb{Q}})$.

Proposition 17.5. *The height $\hat{h}_f$ is the unique function on $\mathbb{P}^1(\overline{\mathbb{Q}})$ for which $|h - \hat{h}_f|$ is uniformly bounded and $\hat{h}_f(f(\alpha)) = d\,\hat{h}_f(\alpha)$ for all $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$.*

Proof. We see easily that $\hat{h}_f$ satisfies the two properties. If h' is another such function, then there exists a constant C so that $g = \hat{h}_f - h'$ satisfies $|g| \leq C$ and $g(f^n(\alpha)) = d^n g(\alpha)$ for all α and all $n \geq 1$. Letting $n \to \infty$ shows that $g \equiv 0$. □

Proposition 17.6. *A point* $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$ *satisfies* $\hat{h}_f(\alpha) = 0$ *if and only if* α *is preperiodic for* f*.*

Proof. From the definition of $\hat{h}_f$ we see immediately that, if α has a finite orbit, then $\hat{h}_f(\alpha) = 0$. For the converse we use the Northcott Proposition 1.5. Indeed, if α and f are defined over a number field K, then all iterates $f^n(\alpha)$ will also lie in K, and will all be of $\hat{h}_f$-height 0, so of bounded h-height. □

17.4. **Polynomials and equidistribution.** Now suppose that f is a polynomial defined over a number field K. At each place v of K, we fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_{p_v}$ as in Section 16, where $\mathbb{C}_{p_v} = \mathbb{C}$ if v is archimedean. Let

$$K(f)_v = \{\zeta \in \mathbb{A}^{1,an}_{p_v} : f^n(\zeta) \not\to \infty\}$$

be the v-adic filled Julia set of f. The **escape rate** function,

$$G_{f,v}(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|_v$$

on Type I points extends to a continuous function on $\mathbb{A}^{1,an}_{p_v}$ which is 0 if and only if $\zeta \in K(f)_v$; see Theorem 6.1 for a proof in the archimedean case. At every place, the function $G_{f,v}$ is the Green's function for the filled Julia set $K(f)_v$, so its (normalized) Laplacian is the equilibrium measure, which we denote by $\mu_{f,v}$. At all but finitely many places, the polynomial f will have good reduction, so that $K(f)_v = \overline{D}^{an}(0,1)$ and $\mu_{f,v} = \delta_{\zeta_G}$, the delta-mass at the Gauss point. At all places, this measure will satisfy both $f_*\mu_{f,v} = \mu_{f,v}$ and the stronger pullback relation $\frac{1}{d} f^*\mu_{f,v} = \mu_{f,v}$.

The capacity of $K(f)_v$ can be computed as

$$\mathrm{cap}_{p_v} K(f)_v = |a_0|_v^{-1/(d-1)}$$

where a_0 is the leading coefficient of f. This is because

$$f^n(z) = a_0^{1+d+\cdots+d^{n-1}} z^{d^n} + \cdots$$

for all $n \geq 1$, so that

$$G_{f,v}(z) = \log |z|_v + \frac{1}{d-1} \log |a_0|_v + o(1)$$

as $|z|_v \to \infty$. But recall that

$$G_{f,v}(z) = (\log p_v)(V_{f,v} + E(\mu_{f,v}))$$

from (16.2), which shows that

$$E(\mu_{f,v}) = \frac{1}{d-1} \log_{p_v} |a_0|_v$$

and provides the desired formula for the capacity. Consequently, from the product formula, we have

$$\operatorname{cap} \mathbb{K}(f) = 1$$

for the compact, adelic, Berkovich filled Julia set of f.

It turns out that

$$\hat{h}_f = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} N_v \, G_{f,v} = h_{\mathbb{K}(f)}.$$

In other words, given any $\alpha \in \overline{\mathbb{Q}}$, we have

$$\hat{h}_f(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} N_v \, \frac{1}{|\operatorname{Gal}(\overline{K}/K)\cdot\alpha|} \sum_{x \in \operatorname{Gal}(\overline{K}/K)\cdot\alpha} G_{f,v}(x).$$

See [Si4, Theorem 3.27] and the proof in Chapter 5.

As a consequence of Theorem 16.2:

Theorem 17.7. *Suppose that f is a polynomial of degree $d \geq 2$ defined over a number field K. Then, for any sequence of finite,* $\operatorname{Gal}(\overline{K}/K)$*-invariant sets $F_n \subset \overline{\mathbb{Q}}$ with $\hat{h}_f(F_n) \to 0$ and $|F_n| \to \infty$, we have*

$$\frac{1}{|F_n|} \sum_{x \in F_n} \delta_x \longrightarrow \mu_{f,v}$$

weakly as $n \to \infty$, in $\mathbb{A}^{1,an}_{p_v}$ (if v is non-archimedean) or in $\mathbb{C}$ (if v is archimedean), at every place v of K.

In applications, the set F_n might be a collection of periodic points with period n, or a collection of points for which $f^n(x) = \alpha$ for some given (non-exceptional) point $\alpha \in \overline{\mathbb{Q}}$. Compare Theorems 7.1 and 7.2.

18. More equidistribution theorems

To prove a version of Theorem 17.7 for rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over a number field K, we must work with a broader classes of height functions than the affine heights introduced in Section 16. And there is no reason to restrict our attention to dynamically-defined heights ...

In this final section of the lecture notes, I provide a summary of some results in the literature where relevant arithmetic equidistribution theorems are proved.

[SUZ] As I already explained at the beginning of the course, the authors prove an equidistribution theorem for points of small canonical height on an abelian variety defined over a number field, at each archimedean place. This is the first result of its kind and had the important consequence of leading to a proof of the Bogomolov Conjecture.

[Bi] Bilu proves an quidistribution for points of small Weil height on the multiplicative $(\overline{\mathbb{Q}}^*)^N$.

[Ru1] Rumely puts Bilu's theorem, for $N = 1$, into the context of potential theory.

[BR1] [FRL] [CL] all published in 2006. The motivation for these more general theorems was the dynamics. Baker and Rumely only stated their result for canonical heights for maps on $\mathbb{P}^1$, though their proof generalizes to a larger class of heights. Favre and Rivera-Letelier wrote their equidistribution theorem for more general heights on $\mathbb{P}^1$ (any adelic measure on $\mathbb{P}^1$ with continuous potentials at all places) and provide a quantitative version; they also give a potential-theoretic proof. Chambert-Loir follows the original approach of Szpiro-Ullmo-Zhang, and he proves a more general result. The Chambert-Loir version is formulated in terms of continuous, semipositive adelic metrics on ample line bundles, in the sense of Zhang.

[Yu] Yuan extends the Szpiro-Ullmo-Zhang approach to treat arithmetically big line bundles. He proves a powerful version of equidistribution that holds on arbitrary projective varieties, for continuous, semipositive adelic metrics.

And though his hypotheses are very natural, it often requires work to show the conditions in Yuan's theorem are met in practice. For example, see the effort required in [DM1], building on three articles of Silverman [Si1, Si2, Si3], just to show that the metrics are adelic and continuous at all places. Or for a dynamical application, see [FG] where Favre and Gauthier prove an equidistribution theorem for the postcritically finite polynomials in the moduli space of all polynomials of a given degree on $\mathbb{P}^1$. In this case, the challenge is to prove that the "genericity" condition is satisfied for the sequence of points.

And the hypotheses needed for [Yu] do not always hold in dynamical applications, as we have discovered over time. See, for example, the equidistribution theorem of Mavraki and Ye in [MY] for "quasiadelic" heights on $\mathbb{P}^1$, because families of dynamical systems can fail to define adelic metrics, as the local contributions to the height functions can be nontrivial at infinitely many places of the number field. And in [DM2], Mavraki and I proved equidistribution for heights on curves (defined over number field) associated to metrized $\mathbb{R}$-divisors, with applications in the setting of elliptic surfaces.

I have skipped many results and applications, but I want to finish with some very recent work (2021):

[Kü] This preprint by Kühne contains a powerful application of an equidistribution result for families of abelian varieties. He proves the Uniform Manin-Mumford Conjecture, that is, the existence of a constant C, depending only on genus g, so that all compact Riemann surfaces of genus $g \geq 2$ embedded (in any way) into their Jacobians intersect at most C torsion points in Jac($\mathbb{C}$). He also provides uniform bounds on the number of rational points on a curve defined over a number field. Kühne's key insight was in finding a way to carry out the needed equidistribution for a (noncompact) family of abelian varieties over a quasiprojective base, with a Néron-Tate canonical height on each fiber.

[Ga] Gauthier has adapted Kühne's proof to obtain equidistribution for families of dynamical systems over a noncompact, quasiprojective base, equipped with the dynamical canonical height on each fiber.

[YZ] Yuan and Zhang will soon release a preprint proving an equidistribution theorem for heights on quasiprojective varieties, generalizing the results of Kühne and Gauthier.

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