MATH 274: COMPLEX AND ARITHMETIC DYNAMICS SPRING 2023

The goal of this course is to study tools from complex analysis and pluripotential theory that have proved useful in the study of algebraic dynamical systems and more broadly in complex, arithmetic, and algebraic geometry. We aim to cover:

- the theory of distributions and currents in $\mathbb{R}^N$ (quickly)
- currents on complex manifolds, pluripotential theory (most of the semester)
- basics of holomorphic dynamics in dimensions > 1 (interspersed throughout, with more towards the end)
- non-archimedean potential theory and applications (if time permits)

I am specifically interested in the use of these tools in the theory of adelically-metrized line bundles and the associated intersection theory, for example in the recent manuscript of Xinyi Yuan and Shouwu Zhang [YZ]. At the end of the semester, we will discuss the proofs of Theorems A and B from the recent preprint [GV] of Thomas Gauthier and Gabriel Vigny, on the theory of stability and geometric canonical heights in algebraic families of maps on projective space $\mathbb{P}^N_{\mathbb{C}}$.

Below are three examples to provide some motivation and context for what we will study. Definitions and details will be provided throughout this semester.

Example 0.1. Currents are objects dual to smooth differential forms with compact support. Let S be a smooth, oriented, compact surface in $\mathbb{R}^3$. Then S defines a linear functional L on the vector space of smooth real 2-forms

$$\omega = f_1\, dx_2 \wedge dx_3 + f_2\, dx_1 \wedge dx_3 + f_3\, dx_1 \wedge dx_2$$

by integration:

$$L(\omega) := \int_S \omega.$$

It is continuous in the topology of uniform convergence of forms. We call L a current of integration.

In the complex setting, suppose C is a complex algebraic curve in $\mathbb{P}^2$. The current of integration along C is a positive (1,1)-current that acts on smooth differential (1,1)-forms ω by

$$\omega \mapsto \int_{C^{sm}} \omega,$$

where C^{sm} is the smooth part of C. Just as we study the intersection of curves (and of algebraic varieties in any dimension), we are interested in the more general

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intersection theory of currents, especially those that are closed and positive and of bidegree (1,1).

Example 0.2. Here is a basic example in $\mathbb{C}^2$ of a closed and positive (1,1)-current which is not a current of integration. Consider the function

$$U(z,w) = \max\{\log|z|, \log|w|, 0\}.$$

This function is continuous and plurisubharmonic (which is a higher-dimensional complex-analytic analog of subharmonic). It follows that

$$T := dd^c U$$

is a positive (1,1)-current, where dd^c is a complex-analytic version of the Laplace operator, acting in the sense of distributions. (We will look into the operations of $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2\pi i}(\partial - \bar{\partial})$.)

The support of T turns can be computed to be

$$\operatorname{supp} T = \left\{(z,w) \in \mathbb{C}^2 \ : \ \max\{|z|, |w|\} = 1\right\} \cup \left\{|z| = |w| \geq 1\right\}.$$

We will see that the intersection of T with itself,

$$\mu := T \wedge T,$$

is the complex Monge-Ampère measure of the function U; it is a well-defined probability measure satisfying

$$\operatorname{supp} \mu = \{|z| = |w| = 1\}.$$

Note that this set is a torus $S^1 \times S^1$, and it turns out that μ is the Haar measure on this torus.

The function U and measure μ are also dynamically significant. Indeed, consider the polynomial mapping

$$F : \mathbb{C}^2 \to \mathbb{C}^2$$

defined by

$$F(z,w) = (z^2, w^2).$$

Then T is the dynamical Green current for F and μ is an invariant measure of maximal entropy.

Example 0.3. Here is another example in $\mathbb{C}^2$, but one that we do not fully understand, related to a recent research project of mine with Myrto Mavraki [DM]. It is not important that you understand this example completely either; I am using it to illustrate the roles played by the pluripotential-theoretic objects we will study this semester, even if interested in one-dimensional complex – or arithmetic! – dynamics.

For each $c \in \mathbb{C}$, consider the one-dimensional dynamical system

$$f_c : \mathbb{C} \to \mathbb{C}$$

defined by the quadratic polynomial $f_c(z) = z^2 + c$. The escape-rate function for f_c is defined by

$$G_c(z) = \lim_{n\to\infty} \frac{1}{2^n} \log \max\{|f_c^n(z)|, 1\};$$

it is continuous and subharmonic on $\mathbb{C}$. The probability measure

$$\mu_c := \frac{1}{2\pi} \Delta\, G_c$$

is defined in the sense of distributions and has support equal to the Julia set of f_c. (References will be given later.)

Given any two such polynomials, f_{c_1} and f_{c_2}, we consider an energy pairing

$$E(c_1, c_2) = \iint \log|z - w|\, d\mu_{c_1}(z)\, d\mu_{c_2}(w).$$

It turns out that E defines a continuous and plurisubharmonic function

$$E : \mathbb{C}^2 \to \mathbb{R}$$

satisfying $E \geq 0$ with $E(c_1, c_1) = 0$ if and only if $c_1 = c_2$. In particular, we may define

$$S := dd^c\, E$$

as a positive (1,1)-current on $\mathbb{C}^2$. What is the support of S? What can we say about the Monge-Ampère measure

$$\mu = S \wedge S,$$

and do we even know if μ is nonzero? What does it tell us about the dynamics of the maps f_c?

Contents

1. Distributions in $\mathbb{R}^N$

Throughout this section, X is a connected and open subset of $\mathbb{R}^N$. We follow the outline in the Dinh-Sibony course notes. Proofs can be found in, for example, [Ru]; see also [Dem] and the references therein.

1.1. **Riesz theorem.** We will work with regular Borel measures, not necessarily positive. Let $C_c(X)$ denote the set of continuous $\mathbb{R}$-valued functions with compact support. We can endow it with a topology of local-uniform convergence. That is, we say that $\varphi_n \to \varphi$ in $C_c(X)$ if for every compact subset K of X, we have

$$\sup_K |\varphi_n - \varphi| \to 0.$$

Observe that a measure μ on X defines a continuous linear functional by

$$f \mapsto \int f \, d\mu$$

on the space $C_c(X)$. Conversely we have:

Theorem 1.1. *Every continuous linear functional L on $C_c(X)$ is represented by a measure μ, meaning that $L(f) = \int_X f\ d\mu$.*

Recall that a measure μ is *positive* if $\int f\,d\mu \geq 0$ for all functions $f \geq 0$.

1.2. **Test functions.** Now let $\mathcal{D}(X)$ denote the space of C^∞ functions on X with compact support. The elements of $\mathcal{D}(X)$ are called *test functions*. We can endow $\mathcal{D}(X)$ with many different topologies. For each integer $k \geq 0$ and compact $K \subset X$, we define

$$\|\varphi\|_{k,K} = \max_{|\alpha|\leq k} \sup_K |\partial^\alpha \varphi|$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index with each α_i non-negative integers and $|\alpha| = \sum \alpha_i$ and

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_1^{\alpha_N}}.$$

Then we say that $\varphi_n \to \varphi$ *in* C^k if for every compact set $K \subset X$, we have $\|\varphi_n - \varphi\|_{k,K} \to 0$. We say that $\varphi_n \to \varphi$ *in* C^∞ if the functions converge in C^k for every k.

Fact 1.2. *The space $\mathcal{D}(X)$ is dense in $C^k(X)$ in the C^k topology, for every $k \geq 0$.*

1.3. **Distributions.** A distribution T on X is a linear functional on $\mathcal{D}(X)$ which is continuous in the C^∞ topology. We write $\langle T, \varphi\rangle$ for the value of T at $\varphi \in \mathcal{D}(X)$. The set of all distributions is often written $\mathcal{D}'(X)$. We endow $\mathcal{D}'(X)$ with the weak topology, meaning that $T_n \to T$ if $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$ for every test function φ.

The *order* of a distribution is the smallest $k \geq 0$ for which, for all compact sets K, there exists a constant $c(K) > 0$ so that

$$|\langle T, \varphi\rangle| \leq c(K)\, \|\varphi\|_{k,K}$$

for all test functions φ supported in K. We say T has infinite order if no such k exists.

Theorem 1.3. *Order k distributions extend (uniquely) to continuous linear functionals on $C_c^k(X)$.*

In particular, order 0 distributions coincide with measures.

Convention 1.4. We identify functions with their associated order 0 distributions. A continuous function $f \in C(X)$, or just an integrable function $f \in L^1_{loc}(X)$, can be identified with the measure $\mu = f\,dx_1 \wedge \cdots \wedge dx_N$, defining a distribution of order 0.

Example 1.5. $X = \mathbb{R}$. $\langle T, \varphi\rangle := \varphi'(0)$ is a distribution of order 1.

Example 1.6. $X = \mathbb{R}$. Infinite order distributions exist, for example $\langle T, \varphi\rangle = \sum_{n=1}^\infty \varphi^{(n)}(n)$.

We can take derivatives of distributions, mimicking the rule of Integration by Parts. Recall, for example in $\mathbb{R}$, we learn that

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du.$$

So we define

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle := -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle$$

Example 1.7. $X = \mathbb{R}$. $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$. Let $T = f(x)\,dx$. Then

$$\left\langle \frac{\partial T}{\partial x}, \varphi \right\rangle \;=\; -\left\langle T, \frac{\partial \varphi}{\partial x} \right\rangle \;=\; -\int_0^\infty \frac{\partial \varphi}{\partial x}\,dx \;=\; \varphi(0)$$

for all test functions φ. So $\frac{\partial T}{\partial x} = \delta_0$.

Example 1.8. $X = \mathbb{R}$. Consider the function $g(x) = 0$ for $x < 0$ and $g(x) = x$ for $x \geq 0$, and let $S = g(x)\,dx$. Then

$$\left\langle \frac{\partial S}{\partial x}, \varphi \right\rangle \;=\; -\left\langle S, \frac{\partial \varphi}{\partial x} \right\rangle \;=\; -\int_0^\infty x\,\frac{\partial \varphi}{\partial x}\,dx \;=\; \int_0^\infty \varphi(x)\,dx$$

so $\frac{\partial S}{\partial x}$ is the distribution T of the previous example. It follows that $\Delta S = \frac{\partial^2 S}{\partial x^2} = \delta_0$.

1.4. **Smoothing.** Choose any smooth function $\psi \geq 0$ with support in the ball $B(0,1) \subset \mathbb{R}^N$ and $\int \psi = 1$. Fix $\varepsilon > 0$ and let

$$\psi_\varepsilon(x) = \varepsilon^{-N}\psi(x/\varepsilon).$$

If T is a distribution in $\mathbb{R}^N$, then

$$T * \psi_\varepsilon(x) = \langle T, \psi_\varepsilon(x - \cdot)\rangle$$

is a smooth function on $\mathbb{R}^N$. As measures/distributions, we have $\psi_\varepsilon \to \delta_0$ weakly as $\varepsilon \to 0$, and $T * \psi_\varepsilon \to T$ in the sense of distributions.

2. Subharmonic functions

Recall that the Laplacian in $\mathbb{R}^N$ is defined as

$$\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}.$$

As before, we let X be an open and connected subset of $\mathbb{R}^N$.

2.1. **Definitions.** A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is *upper-semi-continuous* (or *usc* for short) if $u^{-1}[-\infty, y)$ is open for all $y \in \mathbb{R}$. So

$$u(x_0) \geq \limsup_{x \to x_0} u(x)$$

at all points $x_0 \in X$. Given any function $v : X \to \mathbb{R} \cup \{-\infty\}$ which is locally bounded from above, its *usc regularization* is

$$v^*(x_0) := \limsup_{x \to x_0} v(x)$$

for all $x_0 \in X$; it is the smallest usc function satisfying $v^* \geq v$.

Following the Dinh-Sibony notes, we say a function $u : X \to \mathbb{R} \cup \{-\infty\}$ is *subharmonic* if it is usc, in $L^1_{loc}(X)$, and the Laplacian Δu (defined in the sense of distributions) is a positive measure. In other words, the integral

$$\int u \, \Delta\varphi \, dx_1 \wedge \cdots \wedge dx_N \;\geq\; 0$$

for any test function $\varphi \geq 0$.

Remark 2.1. The positivity of $T = \Delta u$ as a distribution on X is enough to know that it is a measure. Indeed, from the positivity, if $\varphi_1 \leq \varphi_2$, then $\langle T, \varphi_1 \rangle \leq \langle T, \varphi_2 \rangle$. Restricting to a compact set K, choose any non-negative function $\psi \in \mathcal{D}(X)$ which is $\equiv 1$ on K. Given any $\varphi \in \mathcal{D}(X)$ supported in K, positivity implies that

$$|\langle T, \varphi \rangle| \leq \left\langle T, \left(\sup_K |\varphi| \right) \psi \right\rangle = \langle T, \psi \rangle \|\varphi\|_{0,K}.$$

We say that u is *harmonic* if both u and $-u$ are subharmonic.

Remark 2.2. You might usually see the definition that a function h is harmonic if it is C^2 and $\Delta h \equiv 0$. And, at least in the case of $\mathbb{R}^2 = \mathbb{C}$, you will likely have learned that harmonic functions are actually C^∞, being (locally) the real part of a holomorphic function. Subharmonic is also defined differently in the literature: we usually say that a usc function is subharmonic if, for all balls $\overline{B} \subset X$ and harmonic functions h on B that are continuous to $\overline{B}$, we have $u \leq h$ on B whenever $u \leq h$ on ∂B. These turn out to be equivalent definitions, as can be seen from the next few theorems.

2.2. **Fundamental solutions.**

Theorem 2.3. *The fundamental solution for the Laplacian is*

$$U_2(z) = \frac{1}{2\pi} \log |z|$$

in $\mathbb{R}^2 = \mathbb{C}$, *and*

$$U_N(x) = \frac{C_N}{\|x\|^{N-2}}$$

for $\mathbb{R}^N$ *with* $N > 2$*, for some appropriate constant* $C_N < 0$*. That is, this function satisfies*

$$\Delta U_N = \delta_0$$

in the sense of distributions.

To prove this, one needs to show that for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we have

$$\int U \, \Delta\varphi \, dx_1 \wedge \cdots \wedge dx_N = \varphi(0).$$

You can check by direct computation that $\Delta U = 0$ at all nonzero points of $\mathbb{R}^N$. The proof of the proposition is an application of Green's theorem on the domain $B(0,R) \setminus \overline{B(0,\varepsilon)}$ for some large R and tiny $\varepsilon > 0$. See, for example, [Kr, Proposition 1.3.2].

The Poisson kernel in $\mathbb{R}^N$ is

$$P(\xi, y) = \frac{1 - \|y\|^2}{\|\xi - y\|^2}$$

for $y \in B(0,1)$ and $\xi \in S^{N-1}$.

Theorem 2.4. *A function* $h \in C(\overline{B(0,1)})$ *is harmonic if and only if*

$$h(y) = \int_{S^{N-1}} P(\xi, y) h(\xi) \, d\sigma(\xi)$$

for all $y \in B(0,1)$*, where* σ *is normalized Lebesgue measure on the sphere.*

Via this integral formula, we see that h is actually C^∞. You will find details in [Kr, §1.3].

2.3. Sub-mean-value property.

Theorem 2.5. *A usc function* u *on* X*, not* $\equiv -\infty$*, is subharmonic if and only if*

$$u(x_0) \le \int_{S^{N-1}} u(x_0 + r\xi) \, d\sigma(\xi) \tag{2.1}$$

for all $\overline{B(x_0, r)} \subset X$*, where* σ *is the unit area spherical measure on the unit sphere in* $\mathbb{R}^N$*; and if and only if*

$$u(x_0) \le \int_{\overline{B(0,1)}} u(x_0 + r\xi) \, dV(\xi)$$

for all $\overline{B(x_0, r)} \subset X$*, where* V *is the unit volume Lebesgue measure on the unit ball in* $\mathbb{R}^N$*.*

We will call inequality (2.1) the *sub-mean-value property* of subharmonic functions. Proof details can be found in §2.5 of [Kl], but keep in mind that Klimek uses a different definition of subharmonic. So the ideas here are a combination of Theorems 2.4.1, 2.5.1, 2.5.5, and 2.5.8 in [Kl].

Proof sketch. We want to replace u by smooth approximations. We will focus on the inequality (2.1).

Fix a subharmonic function u and consider smoothings $u * \psi_\varepsilon \in C^\infty(X^\varepsilon)$ for a radially symmetric choice of ψ, as in §1.4; these are well defined because $u \in L^1_{loc}$. As distributions, we have $\Delta(u * \psi_\varepsilon) \to \Delta u$. By Fubini, we have that $\Delta(u * \psi_\varepsilon) \geq 0$ in the sense of distributions and therefore also as smooth functions.

If u is a usc function with $u \not\equiv -\infty$, and if it satisfies the sub-mean-value property, then in fact $u \in L^1_{loc}(X)$ [Kl, Corollary 2.4.7]. It follows that its smoothings are well-defined, and they will also satisfy the sub-mean-value property. Indeed, the averages over spheres become "averages of averages", and the value of $u * \psi_\varepsilon$ at a point is – by definition – the average of u over small balls. So $u * \psi_\varepsilon$ will satisfy the inequality if $\varepsilon > 0$ is small enough, and it descends to u, converging pointwise. See [Kl, Theorem 2.5.5].

Now if u is C^∞ and if $\Delta u \geq 0$, then we can replace u by $u_\delta := u + \delta\|x - x_0\|^2$ on a small ball centered at x_0, for $\delta > 0$, so that $\Delta u_\delta > 0$. (See the proof of [Kl, Theorem 2.5.1].) Integrating $u_\delta|\partial B(x_0, r)$ against the Poisson kernel we obtain a harmonic function on the ball with the same boundary values. Then $\Delta(u_\delta - h) = \Delta u_\delta > 0$ and $u_\delta - h \equiv 0$ on the boundary of the ball. If $(u_\delta - h)(y) \geq 0$ at some y inside the ball, then at a local maximum b, we would have all second (directional, one-dimensional) derivatives being non-positive, implying that $\Delta u_\delta(b) \leq 0$, which is impossible. So $u_\delta - h < 0$ throughout this ball, so u_δ satisfies the sub-mean-value property on this ball, with a strict inequality. It follows that u itself satisfies the sub-mean-value property on all of X.

On the other hand, if u is C^∞ and satisfies the sub-mean-value property, and if there were a point x_0 at which $\Delta u(x_0) < 0$, then $-\Delta u(x_0) > 0$, and we apply the previous argument to $-u$ on a small ball around x_0. We would find that $-u$ satisfies a strict version of the sub-mean-value property throughout this small ball, which is a contradiction. □

Two corollaries of the sub-mean-value property of subharmonic functions:

Theorem 2.6 (Maximum principle)**.** *If u is subharmonic on X and if there is a point $x_0 \in X$ at which $u(x_0) = \sup_X u$, then u is constant.*

Proposition 2.7. *If $\{u_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of subharmonic functions on X, uniformly bounded from above, then the usc regularization*

$$U = \left(\sup_{\alpha \in \mathcal{A}} u_\alpha\right)^*$$

is subharmonic on X.

Example 2.8. We need to take the usc regularization in Proposition 2.7. Consider, for example, the subharmonic function $v(z) = \sup_{\varepsilon > 0} \varepsilon \log|z|$ on the unit disk $\{|z| <$

1} in $\mathbb{C}$. This supremum is 0 for all $z \neq 0$ and $-\infty$ at $z = 0$ and so cannot be subharmonic.

Example 2.9. There is a reason we work with usc functions and not only continuous functions to $\mathbb{R} \cup \{-\infty\}$. Consider, for example

$$u(z) = \sum_{n=1}^{\infty} 2^{-n} \log |z - 1/2^n|.$$

Then $\Delta u = \sum_n 2^{-n} \delta_{2^{-n}}$, and $u(0)$ is finite.

2.4. **Polar sets.** A set $E \subset X$ is *polar* if there is a subharmonic function u so that $E \subset \{u = -\infty\}$. An important extension theorem is:

Theorem 2.10 (Removable singularities)**.** *If $E = \{v = -\infty\}$ is a closed polar set in X, and if u is subharmonic on $X \setminus E$ and bounded from above, then the function defined by*

$$\tilde{u}(x_0) = \limsup_{x \to x_0,\, x \notin E} u(x)$$

is subharmonic on X. If h is harmonic on $X \setminus E$ and bounded near E, then the extension is harmonic.

Proof. By subtracting a constant from v and possibly shrinking X, we may assume that $v < 0$ throughout X. Consider the function $u_\varepsilon := u + \varepsilon v$ on X, for $\varepsilon > 0$. The function is subharmonic on $X \setminus E$, and it satisfies the sub-mean-value inequality on all of X. Since u is bounded from above, the function u_ε must also be upper-semicontinuous at the points of E, so u_ε is subharmonic. Note that

$$u = \sup_{\varepsilon > 0} u_\varepsilon$$

on $X \setminus E$, and

$$\tilde{u} = \left(\sup_{\varepsilon > 0} u_\varepsilon \right)^*,$$

so that $\tilde{u}$ is also subharmonic.

If h is harmonic on $X \setminus E$ and bounded near E, then h and $-h$ are subharmonic and bounded from above, so we can extend both to all of X as subharmonic functions $\tilde{h}$ and $\widetilde{-h}$. But their sum is also subharmonic, and $\tilde{h} + \widetilde{-h} = 0$ on $X \setminus E$, so we must have $\tilde{h} + \widetilde{-h} = 0$ on all of X. In other words, $\tilde{h}$ and $-\tilde{h}$ are both subharmonic on X, so $\tilde{h}$ is harmonic. □

EXERCISE: Can subharmonic functions in $\mathbb{R}$ have polar sets?

3. Currents in $\mathbb{R}^N$

In this section, X is a domain in $\mathbb{R}^N$ (or you may take it to be an oriented C^∞ manifold and work with local coordinates).

3.1. **Basic definitions.** For integers $0 \leq p \leq N$, we let $\mathcal{D}^p(X)$ denote the smooth p-forms with compact support in X. So $\mathcal{D}^0(X) = \mathcal{D}(X)$ are the test functions from before. The elements of $\mathcal{D}^p(X)$ are called *test forms.* Each can be expressed as

$$\sum_{|I|=p} f_I \, dx^I$$

with $f_I \in \mathcal{D}(X)$ and $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, N\}$ ordered so that $0 \leq i_1 < \ldots < i_p \leq N$ and

$$dx^I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

As for the case $p = 0$, we can endow $\mathcal{D}^p(X)$ with the C^k topologies for all $k \geq 0$ and the C^∞ topology, measuring the norms of the coefficients on compact subsets.

A *current* of *degree* p (or of *dimension* $N-p$) is a continuous – in the C^∞ topology – linear functional on $\mathcal{D}^{N-p}(X)$. Note that the degree N currents are the distributions. Caution: many mathematicians also identify the degree 0 currents with distributions, by identifying a function $\varphi \in \mathcal{D}^0(X)$ with the form $\varphi(x)\, dx_1 \wedge \cdots \wedge dx_N$ in $\mathcal{D}^N(X)$.

A p-current T on X is *closed* if $dT = 0$; that is, if

$$\langle dT, \alpha \rangle := (-1)^{p+1} \langle T, d\alpha \rangle = 0$$

for all $\alpha \in \mathcal{D}^{N-p-1}(X)$.

Example 3.1. A form $\omega \in \mathcal{D}^p(X)$ or even with only L^1_{loc} coefficients defines a current of degree p by integration:

$$\langle \omega, \alpha \rangle = \int_X \omega \wedge \alpha.$$

Example 3.2. If $Y \subset X$ is a closed, oriented, submanifold of codimension p, then it defines a current of integration of degree p (and dimension $N - p$):

$$\langle [Y], \alpha \rangle = \int_Y \alpha.$$

In general, the action of T on α is denoted both as $\langle T, \alpha \rangle$ and as $\int_X T \wedge \alpha$. This is reasonable because:

Theorem 3.3. *Any current T of degree p on X can be expressed (uniquely) as*

$$T = \sum_{|I|=p} T_I \, dx^I$$

where each T_I is a degree 0 current, identified with a distribution.

The proof is straightforward. We can simply define T_I as a distribution by

$$\langle T_I, \varphi \rangle \;:=\; \sigma_I \, \langle T, \varphi(x)\, dx^{I^c} \rangle$$

for all $\varphi \in \mathcal{D}(X)$, where I^c is the complement of the index set, and $\sigma_I = \pm 1$ so that

$$dx^I \wedge dx^{I^c} = \sigma_I \, dx_1 \wedge \cdots \wedge dx_N$$

As for distributions, we can define the *order* of a current; it is the smallest $k \geq 0$ for which the current extends as a C^k-continuous linear functional on the C^k forms with compact support. That is, for every compact $K \subset X$, there is a constant $c(K)$ so that

$$|\langle T, \alpha \rangle| \leq c(K) \|\alpha\|_{k,K}$$

for all $\alpha \in \mathcal{D}^{N-p}(X)$ with $\operatorname{supp} \alpha \subset K$. The two examples above have order 0.

The *support* of a current is the smallest closed subset C in X so that T vanishes on all forms supported in $X \setminus C$. For a current of integration $[Y]$ on a submanifold, we have $\operatorname{supp}[Y] = Y$.

3.2. **Pushforward.** Suppose $F : X \to Y$ is a C^∞ map, and set $N_X = \dim X$ and $N_Y = \dim Y$. Recall that we can pull back a form $\alpha \in \mathcal{D}^p(Y)$ as

$$F^*\alpha = F^* \left(\sum_I f_I \, dx^I \right) = \sum_I f_I \circ F \, dF^I$$

where $dF^I = dF_{i_1} \wedge \cdots \wedge dF_{i_p}$ when writing $F = (F_1, \ldots, F_{N_Y})$. But note that $F^*\alpha$ is not necessarily in $\mathcal{D}^p(X)$, as it may not have compact support.

The map $F : X \to Y$ is *proper* if $F^{-1}(K)$ is compact for every compact K in Y. If F is proper, then we can push forward currents by F, as

$$\langle F_*T, \alpha \rangle := \langle T, F^*\alpha \rangle.$$

Noe that if T has degree p on X, then $F^*\alpha$ must be an $(N_X - p)$-form, so α was also degree $N_X - p$, so

$$\deg F_*T = N_Y - (N_X - p) = p - (N_X - N_Y).$$

More generally, the current F_*T is well defined if F is *proper on* $\operatorname{supp} T$, meaning that for all compact sets K in Y, the set $F^{-1}(K) \cap \operatorname{supp} T$ is compact.

Compare this to the usual notion of pushforward of a measure:

$$F_*\mu(A) := \mu(F^{-1}(A)).$$

The measure μ is a currents of degree N_X on X, and so – as expected – of degree N_Y on Y.

3.3. **Pullback.** Now suppose T is a current on Y; when can we pull it back by a C^∞ map $F : X \to Y$. When sensible, we can define F^*T by duality with integration of forms over the fibers of F.

Recall that $F : X \to Y$ is a *submersion* if F is surjective and the derivative map $DF_x : T_xX \to T_{F(x)}Y$ is surjective at all points $x \in X$. Locally, X will look like a product of Y with a fiber, and F will be the projection to Y.

Suppose F is a submersion, and suppose $\alpha \in \mathcal{D}^{N_X - N_Y}(X)$. Then we can define $F_*\alpha$ as a smooth function on Y by

$$F_*\alpha(y) = \int_{F^{-1}(Y)} \alpha.$$

More generally, for $p \geq N_X - N_Y$ we can define

$$F_* : \mathcal{D}^p(X) \to \mathcal{D}^{p-(N_X - N_Y)}(Y)$$

in the same way, integrating over fibers. Roughly, we do the following: we choose local coordinates on X of the form $(s_1, \ldots, s_e, y_1, \ldots, y_{N_Y})$ where the y's are local coordinates on Y and the s's are coordinates on the fiber E of dimension $e = N_X - N_Y$. Then

$$F_*\alpha = \sum_{|J|=p-e} \left(\int_{F^{-1}(y)} \sum_{|I|=e} f_{I,J}\, ds^I \right) dy^J(y)$$

where

$$\alpha = \sum_{|I|+|J|=p} f_{I,J}\, ds^I \wedge dy^J.$$

So if $F : X \to Y$ is a submersion, and if T is a p-current on Y, then we may set

$$\langle F^*T, \alpha \rangle := \langle T, F_*\alpha \rangle.$$

Note that F^*T is also a p-current!

EXERCISE: If T is a smooth p-form defining a p-current, is this the same as pullback of the form?

Example 3.4. Suppose μ is a positive measure on $\mathbb{R}$, for example the delta mass δ_0 at $x = 0$. Consider the projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ sending (x, y) to x. Then $\pi^*\mu$ is a 1-current on $\mathbb{R}^2$ defined by

$$\langle \pi^*\mu, \alpha \rangle = \langle \mu, \pi_*\alpha \rangle = \int \left(\int \alpha_2(x, y)\, dy \right) d\mu(x)$$

for all $\alpha = \alpha_1\, dx + \alpha_2\, dy \in \mathcal{D}^1(\mathbb{R}^2)$. Taking $\mu = \delta_0$, this would just be

$$\langle \pi^*\delta_0, \alpha \rangle = \int \alpha_2(0, y)\, dy,$$

so $\pi^*\delta_0 = [\{x = 0\}]$ is the current of integration over the y axis.

4. Intro to SCV

Here we give a quick introduction to the most basic concepts in the function theory of Several Complex Variables.

4.1. **Forms and currents.** With $z = x + iy \in \mathbb{C}$, we set

$$dz = dx + i\,dy \quad \text{and} \quad d\bar{z} = dx - i\,dy.$$

Note that $2\,dx \wedge dy = i\,dz \wedge d\bar{z}$. We define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In one-dimensional complex analysis, we learn that a C^1 function f is holomorphic if and only if it satisfies the Cauchy-Riemann equations; these can be written as $\frac{\partial f}{\partial \bar{z}} = 0$.

In coordinates $(z_1, \ldots, z_N) \in \mathbb{C}^N$, the exterior derivative d operator can be decomposed as

$$d = \partial + \bar{\partial} = \sum_j \frac{\partial}{\partial z_j} dz_j + \sum_k \frac{\partial}{\partial \bar{z}_j} d\bar{z}_k.$$

We define

$$d^c = \frac{1}{2\pi i}(\partial - \bar{\partial}).$$

In complex manifolds or complex algebraic varieties, recall that a smooth form α has bidegree (p, q) if it can be expressed in local coordinates as

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{I,J}\, dz^I \wedge d\bar{z}^J$$

where $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$.

A current on a domain $\Omega \subset \mathbb{C}^N$ can also have a bidgree: A current of bidegree (p, q) is a linear functional on the space $\mathcal{D}^{(N-p,N-q)}(\Omega)$ of smooth $(N-p, N-q)$-forms with compact support.

4.2. **Holomorphic functions.** For a domain $\Omega \subset \mathbb{C}^N$, we will say that a function $f : \Omega \to \mathbb{C}$ is *holomorphic* if it is C^1 and if it is holomorphic in each variable separately. That is, if $\bar{\partial} f = 0$. In fact, the C^1 condition is superfluous, though this requires some work to show. But assuming C^1, it is straightforward to deduce from the 1-dimensional theory the Cauchy Integral Formula

$$f(z) = \frac{1}{(2\pi i)^N} \int_{\partial D_1} \cdots \int_{\partial D_N} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_N - z_N)}\, d\zeta_1 \cdots d\zeta_N$$

for holomorphic functions, for z in a polydisk $D^N(a, r) = \prod_j D_j(a_i, r_i)$. It follows that f is C^∞.

In learning the theory of these holomorphic functions, we very quickly encounter striking differences between the one-dimensional theory and the theory in dimensions $N > 1$. For example:

Theorem 4.1 (Hartogs Extension Lemma)**.** *Suppose $\Omega \subset \mathbb{C}^N$ is a domain with $N > 1$; suppose that $K \subset \Omega$ is compact and $\Omega \setminus K$ is connected. If a function f on $\Omega \setminus K$ is holomorphic, then there exists a (unique) holomorphic extension*

$$\tilde{f} : \Omega \to \mathbb{C}$$

of f.

Remark 4.2. For a domain $U \subset \mathbb{C}$ and compact subset $K \subset U$, we can easily find examples where holomorphic $f : U \setminus K \to \mathbb{C}$ does not extend to U; take $f(z) = 1/(z-a)$ for some point $a \in K$. But examples can be much more extreme. For any K, there exists a holomorphic $f_K : U \setminus K \to \mathbb{C}$ that does not extend to any larger domain. For example, you could take a discrete sequence of points $\{a_n\}_{n=1}^{\infty} \subset U \setminus K$ that accumulate everywhere on the boundary of $U \setminus K$. Then, by a theorem of Weierstrass (?), there exists a non-constant holomorphic function on $U \setminus K$ that vanishes at each a_n. It cannot extend.

The proof of Theorem 4.1 is illustrative, so let's go through the details. It relies on the more general form of the Cauchy Integral Formula in dimension 1: for domain $U \subset \mathbb{C}$ and C^1 function $g : U \to \mathbb{C}$, we have

$$g(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{g(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_D \frac{\partial g/\partial \bar{\zeta}}{\zeta - z}\, d\bar{\zeta} \wedge d\zeta. \tag{4.1}$$

on disks $\overline{D} \subset U$.

Now suppose that $\Omega \subset \mathbb{C}^N$ is a domain in any dimension. Let $\psi = \sum_{j=1}^N \psi_j \, d\bar{z}_j$ be a smooth form of type (0,1) with compact support in Ω, and assume that $\bar{\partial}\psi = 0$. (Note that, in dimension $N = 1$, the condition that $\bar{\partial}\psi = 0$ is always satisfied for any (0,1)-form ψ. In higher dimensions, it implies that $\partial\psi_k/\partial\bar{z}_j = \partial\psi_j/\partial\bar{z}_k$ for all j, k.) Then we can always solve the equation

$$\bar{\partial} g = \psi \tag{4.2}$$

for a function g on Ω by integrating: we fix any variable z_j and set

$$g(z) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\psi_j(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_N)}{\zeta - z_j}\, d\bar{\zeta} \wedge d\zeta. \tag{4.3}$$

See, for example, [Kr, Theorem 1.1.10]; the proof relies on (4.1). Note that

$$\begin{aligned}\frac{\partial g}{\partial \bar z_k} &= -\frac{\partial}{\partial \bar z_k}\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi_j(z_1,\ldots,z_{j-1},\zeta+z_j,z_{j+1},\ldots,z_N)}{\zeta}\, d\bar\zeta\wedge d\zeta \\ &= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial\psi_j}{\partial \bar z_k}(z_1,\ldots,z_{j-1},\zeta+z_j,z_{j+1},\ldots,z_N)}{\zeta}\, d\bar\zeta\wedge d\zeta \\ &= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial\psi_k}{\partial \bar z_j}(z_1,\ldots,z_{j-1},\zeta+z_j,z_{j+1},\ldots,z_N)}{\zeta}\, d\bar\zeta\wedge d\zeta \\ &= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\frac{\partial\psi_k}{\partial \bar\zeta}(z_1,\ldots,z_{j-1},\zeta+z_j,z_{j+1},\ldots,z_N)}{\zeta}\, d\bar\zeta\wedge d\zeta\end{aligned}$$

Moreover, this solution g will have *compact support in* Ω *for all choices of* ψ *if and only if* $N = 1$ [Kr, §0.3.9]. See [Dem, Chapter I §3E] for another solution.

Remark 4.3. The proof that (4.3) is a solution to (4.2) relies crucially on the fact that ψ is compactly supported. [Do you see that? The boundary term in (4.1) must vanish.] Solving (4.2) for noncompactly supported data ψ is a much more complicated story! The existence of a solutions on a domain Ω for noncompact data is tied up in the theory of domains of holomorphy and the geometry of the boundary of Ω. I'll say a few words about this in §4.3. The existence of locally-defined solutions will be addressed in §5.5.

Remark 4.4. The above solution to (4.2) is really a 1-dimensional argument. It can be reformulated in the sense of distributions, as follows. The function

$$g_0(z) = \frac{1}{\pi z}$$

on $\mathbb{C}$ is the fundamental solution of the $\bar\partial$ operator. That is, $\frac{\partial g_0}{\partial \bar z} = \delta_0$. See, for example, [Dem, Chapter I, Cor 3.4]. Explicitly, for each test function $\varphi \in \mathcal{D}(\mathbb{C})$, we have

$$\begin{aligned}\left\langle \frac{\partial g_0}{\partial \bar z}, \varphi \right\rangle &:= -\int g_0\, \frac{\partial \varphi}{\partial \bar z}\, dx\wedge dy \\ &= -\frac{1}{2}\int g_0\,\frac{\partial\varphi}{\partial \bar z}\,(i\, dz\wedge d\bar z) \\ &= -\frac{1}{2i}\int g_0\,\frac{\partial\varphi}{\partial\bar z}\, d\bar z\wedge dz \\ &= -\frac{1}{2i}\int \frac{\partial\varphi/\partial\bar z}{\pi z}\, d\bar z\wedge dz \\ &= -\frac{1}{2\pi i}\int \frac{\partial\varphi/\partial\bar z}{z-0}\, d\bar z\wedge dz \\ &= \varphi(0) - \frac{1}{2\pi i}\int_{\partial D}\frac{\varphi(\zeta)}{\zeta - 0}\, d\zeta \;=\; \varphi(0),\end{aligned}$$

for all disks $D \subset \mathbb{C}$ containing the support of φ. It follows that, for any smooth function ψ with compact support (or indeed any distribution ψ with compact support), the convolution $g = g_0 * \psi$ will satisfy $\partial g/\partial \bar{z} = \psi$ in the sense of distributions.

We are now ready to give the proof of the Hartog Extension Lemma:

Proof of Theorem 4.1. Let $\varphi \geq 0$ be any smooth function with compact support in Ω satisfying $\varphi \equiv 1$ on K. Define a function $s(z) = (1-\varphi(z))f(z)$ on $\Omega \setminus K$ and $s(z) \equiv 0$ on K. Set

$$\psi = \bar{\partial} s$$

on Ω. Then ψ is a smooth $(0,1)$-form on Ω with compact support, and $\bar{\partial}\psi = \bar{\partial}\bar{\partial}s = 0$. It follows that there is a smooth function g – with compact support because $N > 1$ – so that

$$\bar{\partial} g = \psi$$

on Ω. We let $\tilde{f} = s - g$ on Ω. Note that $\tilde{f} \equiv f$ outside of the support of φ and of g, and so by unique analytic continuation, equality holds throughout $\Omega \setminus K$. See [Kr, Theorem 1.2.6]. □

4.3. **Domains of holomorphy.** Domains of holomorphy are open subsets Ω in $\mathbb{C}^N$ that are maximal domains of definition for some holomorphic function. In dimension $N = 1$, every domain is a domain of holomorphy. (The construction in Remark 4.2 with the sequence $\{a_n\}$ shows this.) But this is far from being the case once $N > 1$. It turns out the condition for being un-extendable is related to the geometry of the boundary of the domain Ω. It must satisfy a certain holomorphically-invariant convexity condition: roughly, where each piece of the boundary is biholomorphically equivalent to the boundary of a convex domain. It turns out that Ω is a domain of holomorphy if and only if it is pseudoconvex, which can be defined by requiring that the function

$$u(z) = -\log \operatorname{dist}(z, \partial\Omega)$$

is *plurisubharmonic*. Here

$$\operatorname{dist}(z, E) = \inf_{w \in E} \|z - w\|.$$

See, for example, [Hö] and [Kr] for background. And/or read the survey article that Professor Siu published in 1978 [S].

5. Plurisubharmonic functions and positivity

5.1. **Definition and examples.** Let $\Omega \subset \mathbb{C}^N$ be a domain. An upper-semi-continuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic* if $u|L$ is a subharmonic function on every complex line $L \subset \mathbb{C}^N$. That is, as a function of one complex variable,

$$\zeta \mapsto u(a + \zeta v)$$

is subharmonic (where defined) for all $a \in \Omega$ and $v \in \mathbb{C}^N$. A set $E \subset \mathbb{C}^N$ is *pluripolar* if, locally, it is contained in the $-\infty$ set for some subharmonic function.

Just as $\log|z|$ is subharmonic on $\mathbb{C}$, we can see that $\log|f(z)|$ is plurisubharmonic on $\Omega \subset \mathbb{C}^N$ for any holomorphic function $f : \Omega \to \mathbb{C}$ with $f \not\equiv 0$. It follows that analytic varieties of the form $\{f = 0\}$ are pluripolar. Higher-codimension analytic sets can also be the $-\infty$ sets of a subharmonic function, for example taking

$$u(z) = \log\left(|f_1|^{c_1} + \cdots + |f_m|^{c_m}\right).$$

This latter example is plurisubharmonic as the composition of plurisubharmonic functions $c_i \log|f_i|$ with the convex function $R(x_1, \ldots, x_m) = \log\left(e^{x_1} + \cdots + e^{x_m}\right)$.

As a special case, we have $\log\|z\|$ as a plurisubharmonic function on $\mathbb{C}^N$, for

$$\|z\| = \sqrt{|z_1|^2 + \cdots + |z_N|^2},$$

and therefore so is

$$\log^+\|z\| = \max\{\log\|z\|, 0\}.$$

And so, as we shall see when discussing dynamics, if $F : \mathbb{C}^N \to \mathbb{C}^N$ is a "regular" polynomial mapping (extending as a morphism to all of $\mathbb{P}^N(\mathbb{C})$) of degree $d > 1$, then the function

$$G_F(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+\|F^n(z)\|$$

is everywhere defined and plurisubharmonic on $\mathbb{C}^N$.

A simple example of a function on $\mathbb{C}^2$ which is subharmonic but *not* plurisubharmonic would be

$$v(z_1, z_2) = 2\,|z_1|^2 - |z_2|^2.$$

Note that

$$\Delta v = \Delta(2x_1^2 + 2y_1^2 - x_2^2 - y_2^2) = 4 > 0$$

at all points, but $v(0, z_2) = -|z_2|^2$ is not subharmonic as a function of z_2.

By the sub-mean-value property for subharmonic functions, we see that plurisubharmonic is a stronger condition: the value at a point is less than or equal to the average value over all *circles* centered at the point in complex lines through that point, while the value for a subharmonic function only needs to be less than or equal to the average value over *spheres* centered at that point.

5.2. **Complex Hessian.** If $u : \mathbb{C} \to \mathbb{R}$ is C^2, we can compute that

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\Delta u.$$

We see that a C^2 function $u : \Omega \to \mathbb{R}$ is then plurisubharmonic if and only if the complex Hessian

$$\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$$

defines a positive semi-definite quadratic form on $\mathbb{C}^N$ at all points $a \in \Omega$. That is, if

$$\sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a)\, v_j \bar{v}_k \geq 0$$

for all $v \in \mathbb{C}^N$ and all $a \in \Omega$. In general, then, we see that a usc function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ which is locally L^1 is plurisubharmonic if and only if

$$\sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\, v_j \bar{v}_k,$$

defined in the sense of distributions, is a positive measure on Ω for all $v \in C^N$. See [Dem, Theorem 5.8]. Equivalently, and recalling the definitions of d and d^c from §4.1, this means that

$$dd^c u$$

exists as a *positive* $(1,1)$-current.

5.3. **Positivity.** A (p,p)-current T on $\Omega \subset \mathbb{C}^N$ is *positive* if

$$\left\langle T, (i\alpha_1 \wedge \bar{\alpha}_1) \wedge \cdots \wedge (i\alpha_{N-p} \wedge \bar{\alpha}_{N-p}) \right\rangle \geq 0$$

for all $\alpha_i \in \mathcal{D}^{(1,0)}(\Omega)$. Every positive (p,p) current

$$T = i^{p^2} \sum_{|I|=|J|=p} T_{I,J}\, dz^I \wedge d\bar{z}^J$$

is real and of order 0 [Dem, Ch III, Prop 1.14]. Indeed, positive forms are real, so positive currents have to be real by duality. More precisely, the coefficients are (complex) measures satisfying $\overline{T_{I,J}} = T_{J,I}$ and $T_{I,I} \geq 0$. To explain the i^{p^2} factor, Demailly points out in [Dem, Ch III, Ex 1.2] that $i^p(-1)^{p(p-1)/2} = i^{p^2}$ and

$$(i\alpha_1 \wedge \bar{\alpha}_1) \wedge \cdots \wedge (i\alpha_p \wedge \bar{\alpha}_p) = i^{p^2}(\alpha_1 \wedge \cdots \alpha_p) \wedge (\bar{\alpha}_1 \wedge \cdots \wedge \bar{\alpha}_p).$$

5.4. **Pluriharmonic.** A function $h \in C^2(\Omega)$ is *pluriharmonic* if $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0$ for all j, k at all points of Ω. This is equivalent to saying that $u|L$ is a harmonic function when restricted to each complex line L in $\mathbb{C}^N$. Without assuing $h \in C^2$, the function $h \in L^1_{loc}$ is pluriharmonic if both h and $-h$ are plurisubharmonic.

Note that pluriharmonic implies harmonic, but not conversely. Consider

$$h(z) = \frac{1}{\|z\|^2} = \frac{1}{x_1^2 + y_1^2 + x_2^2 + y_2^2}$$

on $\mathbb{C}^2 \setminus \{0\}$. It is harmonic but not pluriharmonic, as it is not harmonic on the line $z_2 = 0$, where $h(z, 0) = 1/|z|^2$ and $\frac{\partial^2 h}{\partial z \partial \bar{z}} = 1/|z|^4$.

Theorem 5.1. *A function h is pluriharmonic on a ball if and only if it is the real part of a holomorphic function.*

Proof. One implication is clear, by the Cauchy-Riemann equations. The converse is an immediate consequence of the Poincaré Lemma, that tells us that any d-closed form is exact on a contractible space, such as a ball. See, for example, [Dem, Chapter I (1.22)]. In detail, let

$$\omega = 2\pi d^c h$$

so that $d\omega = 0$. Then there exists a function v so that $dv = \omega$. In other words, we have

$$\frac{1}{i}(\partial - \bar{\partial})h = (\partial + \bar{\partial})v.$$

The $(1, 0)$ and $(0, 1)$ components must coincide, so $\bar{\partial} h = -i\bar{\partial} v$. Therefore

$$\bar{\partial}(h + iv) = -i\bar{\partial} v + i\bar{\partial} v = 0.$$

□

Remark 5.2. The analog of the Dirichlet problem for pluriharmonic functions on a ball in $\mathbb{C}^N$ is not solvable for all continuous boundary data. Recall that it is solvable with the Laplacian; that is, given any continuous function on the sphere, we can find a harmonic function with these boundary values via the Poisson kernel. But consider, for example, a function φ on the boundary of the unit ball in $\mathbb{C}^2$ which is $\equiv 1$ in a neighborhood of $(0, 1)$ and 0 at the point $(0, -1)$. Any pluriharmonic function with these boundary values would have to be $\equiv 1$ on slices of the form $(z_1, 1 - \varepsilon)$ for small positive ε. But then the function would have to be everywhere constant on the ball.

5.5. **Poincaré-Dolbeault Lemma.** First observe that the $\bar{\partial}$-equation (4.2) can always be solved locally, in the following sense:

Proposition 5.3 (Dolbeault-Grothendieck Lemma)**.** *Suppose that $\psi = \sum_{j=1}^N \psi_j \, d\bar{z}_j$ is a smooth $(0, 1)$-form on domain $\Omega \subset \mathbb{C}^N$ satisfying $\bar{\partial}\psi = 0$. Then, for each point $a \in \Omega$, there exists a smooth function g so that $\bar{\partial} g = \psi$ on a ball around a.*

Compare this statement to the solution (4.3) and Remark 4.3, where the compact support of ψ was a necessary condition. EXERCISE: Proposition 5.3 also holds for distributional data. EXERCISE: The proposition also holds for $(p, q+1)$ input data, for the existence of a (p,q)-form g.

Proof. Let us first assume that $N = 1$. Then $\psi = \psi_1(z_1)\, d\bar{z}_1$ always satisfies $\bar{\partial}\psi = 0$. Fix point $a \in \Omega$ and let φ be a smooth cut-off function which is $\equiv 1$ near a and has compact support in Ω. Then $(\varphi\psi_1)(z_1)\, d\bar{z}_1$ has compact support and the solution from (4.3) goes through to show that

$$\bar{\partial} g = (\varphi\psi_1)(z_1)\, d\bar{z}_1 = \psi_1(z_1)\, d\bar{z}_1$$

on a small neighborhood of a.

For general N, we proceed inductively on the number of coordinates that appear in the expression for ψ. That is, let k be the largest coordinate index so that $\psi_j \equiv 0$ for all $j > k$. The base case is $k = 0$, and there is nothing to show. Assume we can solve $\bar{\partial} g = \psi$ on a neighborhood of $a \in \Omega$ for any $\bar{\partial}$-closed $(0,1)$-form ψ depending only on $d\bar{z}_1, \ldots, d\bar{z}_{k-1}$. Now let

$$\psi = \sum_{j=1}^{k} \psi_j \, d\bar{z}_j$$

satisfy $\bar{\partial}\psi = 0$. This implies that each ψ_j is holomorphic in variables $z_{k+1}, \ldots, z_N$, and $\frac{\partial \psi_j}{\partial \bar{z}_m} = \frac{\partial \psi_m}{\partial \bar{z}_j}$ for all $j, m \le k$. Let $\varphi : \Omega \to [0,1]$ be a smooth cut-off function which is $\equiv 1$ near a and supported on a ball B centered at a. Let g_k be defined by (4.3) with input function $\varphi\psi_k$, so that

$$g_k(z) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{(\varphi\psi_k)(z_1, \ldots, z_{k-1}, \zeta, z_{k+1}, \ldots, z_N)}{\zeta - z_k} \, d\bar{\zeta} \wedge d\zeta.$$

The 1-variable argument implies that

$$\frac{\partial g_k}{\partial \bar{z}_k} = \psi_k$$

on a small ball around a. Moreover, the definition of g_k implies that it is holomorphic in $z_{k+1}, \ldots, z_N$ for z near a. Consequently,

$$\psi - \bar{\partial} g_k = \sum_{j=1}^{k-1} \psi_j \, d\bar{z}_j - \sum_{j=1}^{k-1} \frac{\partial g_k}{\partial \bar{z}_j} \, d\bar{z}_j.$$

The right-hand-side is $\bar{\partial}$-closed, so we may apply the induction hypothesis to write

$$\psi - \bar{\partial} g_k = \bar{\partial} \hat{g}$$

on a small neighborhood of a in Ω. In other words, $\psi = \bar{\partial}(\hat{g} + g_k)$, and the proof is complete. ∎

Proposition 5.4. *A closed and positive* $(1,1)$*-current* T *can always be locally expressed as*

$$T = dd^c u$$

for a plurisubharmonic function u.

Proof. See [Dem, Chapter III §1.18]. The proof combines the usual Poincaré Lemma for d with the solution for the $\bar{\partial}$ equation of Proposition 5.3. (Demailly gives a proof of a more general statement in [Dem, Chapter I (3.29)].

In detail, as T is closed, the Poincaré Lemma allows us to write $T = dS$ for a 1-current S, and note that $S = S^{(1,0)} + S^{(0,1)}$, when arranged by bidegree. And $\bar{\partial} S^{(0,1)} = 0$ because T has no $(0,2)$ part. So $S^{(0,1)} = \bar{\partial} g$, locally, for some function g in the sense of distributions. Positivity of T implies that $\overline{S^{(1,0)}} = S^{(0,1)}$. Therefore

$$S = \overline{\bar{\partial} g} + \bar{\partial} g = \partial \bar{g} + \bar{\partial} g$$

so that

$$T = dS = \bar{\partial}\partial\bar{g} + \partial\bar{\partial} g = \partial\bar{\partial}(g - \bar{g}) = -\pi i\, dd^c(g - \bar{g}).$$

Set $u = \pi i(g - \bar{g}) = -2\pi \operatorname{Im} g$. The positivity of T implies that u is plurisubharmonic. □

The function u of Proposition 5.4 is called a *(local) potential function for* T.

6. Examples from dynamics

Here we present the construction of some positive (1,1)-currents on projective space $\mathbb{P}^N$ which are not currents of integration and not smooth (1,1)-forms. We start with polynomials in dimension $N = 1$ of degree $d > 1$.

EXERCISE: What is the dynamical behavior of a map in degree 1, i.e., of an automorphism of $\mathbb{P}^N$? (Throughout this section and later, we will always assume the degree is > 1.)

6.1. **Escape rate in dimension 1.** The *escape-rate function* for a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$ is defined by

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)| \tag{6.1}$$

where $\log^+ = \max\{\log, 0\}$. (This was introduced in Example 0.3.) The *filled Julia set* of f is its set of points with bounded orbit,

$$K(f) = \{z \in \mathbb{C} : \sup_n |f^n(z)| < \infty\}.$$

In particular, note that all periodic points of f in $\mathbb{C}$ lie in $K(f)$. The *Julia set* $J(f)$ is the boundary of $K(f)$. (The Julia set has many characterizations; one of these is given below in §6.3.)

Note that $K(f)$ is "filled" in the sense that its complement must be connected; if $|f^n(z)|$ were unbounded at a point in a bounded component of $\mathbb{C} \setminus K(f)$, this would violate the Maximum Principle. Note also that a point z has unbounded orbit if and only if $f^n(z) \to \infty$, because $|f(z)| \geq C|z|^d > 2|z|$ for some constant $C > 0$ and all $|z|$ large. In other words, the complement $\mathbb{C} \setminus K(f)$ is the *basin of attraction* for the fixed point of f at ∞.

Theorem 6.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. The escape-rate function G_f is continuous and subharmonic on $\mathbb{C}$. It is a potential function for the equilibrium measure μ_f on $K(f)$ and* $\operatorname{supp} \mu_f = J(f)$. *The measure μ_f satisfies*

$$\frac{1}{d} f^* \mu_f = \mu_f.$$

Remark 6.2. The equilibrium measure is a potential-theory notion. Here, we are saying that G_f is the *Green's function* for the domain $\mathbb{C} \setminus K(f)$ with logarithmic pole at ∞, and

$$\mu_f = \frac{1}{2\pi} \Delta G_f = dd^c G_f$$

in the sense of distributions. Consult [Ra] for background.

Remark 6.3. The pullback of a measure by f has not been defined, because f is not a submersion. But we can still make sense of it by duality with integration over the fibers. That is, we set

$$\langle f^* \nu, \varphi \rangle := \int_{\mathbb{C}} \sum_{f(x)=y} \varphi(x)\, d\nu_f(y)$$

for all test functions $\mathcal{D}(\mathbb{C})$ and measure ν on $\mathbb{C}$. The preimages by f should be counted with multiplicity.

EXERCISE: the pull-back invariance $\frac{1}{d} f^* \mu_f = \mu_f$ in Theorem 6.1 implies that

$$\mu_f(f(A)) = \mu(A)$$

whenever $f|A$ is one-to-one. It implies the weaker notion of invariance that

$$f_* \mu_f = \mu_f$$

meaning that $\mu_f(f^{-1}(A)) = \mu_f(A)$ for all Borel sets $A \subset \mathbb{C}$.

Proof of Theorem 6.1. For each $n \geq 1$, the function $g_n(z) = d^{-n} \log^+ |f^n(z)|$ is continuous and subharmonic on $\mathbb{C}$, being the maximum of two subharmonic functions $d^{-n} \log |f^n(z)|$ and 0. These functions g_n converge uniformly to G_f on $\mathbb{C}$. Indeed, we first observe that there is a radius R and a constant $C > 1$ so that

$$C^{-1}|z|^d \leq |f(z)| \leq C|z|^d$$

for all $z \in \mathbb{C}$ with $|z| \geq R$. Therefore, we can find another constant C so that

$$\left|\log^+ |f(z)| - d\, \log^+ |z|\right| \leq C$$

for all $z \in \mathbb{C}$. Write

$$g_n(z) = \log^+ |z| + \sum_{j=1}^{n} \frac{1}{d^j} \left(\log^+ |f^j(z)| - d\, \log^+ |f^{j-1}(z)|\right)$$

for each $n \geq 1$. Then, for any $m > n$, we have

$$|g_m(z) - g_n(z)| = \left| \sum_{j=n+1}^{m} \frac{1}{d^j} \left(\log^+ |f^j(z)| - d\, \log^+ |f^{j-1}(z)|\right) \right| \leq \sum_{j=n+1}^{m} \frac{C}{d^j}.$$

This upper bound can be made as small as desired, taking m and n large, the sequence is Cauchy.

It follows that G_f is continuous and subharmonic on $\mathbb{C}$, being a uniform limit of such functions. Note that G_f is, by definition, $\equiv 0$ on the set $K(f)$ of points with bounded orbits. It is clearly ≥ 0 at all points of $\mathbb{C}$, and it grows like $\log |z|$ near ∞. We also know that G_f is harmonic on $\mathbb{C} \setminus K(f)$, as a uniform limit of harmonic functions $\log |f^n(z)|$. These properties characterize G_f as the Green's function for the domain $\mathbb{C} \setminus K(f)$ with (logarithmic) pole at ∞. In particular, this implies that the measure

$$\mu_f := dd^c G_f = \frac{1}{2\pi} \Delta G_f \, (dx \wedge dy)$$

defined in the sense of distributions, is the equilibrium measure on $K(f)$.

Note that $\operatorname{supp} \mu_f \subset J(f) = \partial K(f)$ from the properties of G_f. But, in fact, we have that $\operatorname{supp} \mu_f = J(f)$ by the Maximum Principle for harmonic functions: As $G_f \geq 0$ everywhere, 0 on $K(f)$, and positive near ∞, we know that $G_f > 0$ on $\mathbb{C} \setminus K(f)$, applying Maximum Principle to $-G_f$. In fact, this implies that G_f is positive on its maximal (connected) domain where it is harmonic, so it must fail to be harmonic on a neighborhood of every point in $J(f)$.

It remains to prove the invariance property, i.e., that $\frac{1}{d} f^* \mu_f = \mu_f$. As

$$f : \mathbb{C} \setminus f^{-1}(\operatorname{Crit}(f)) \to \mathbb{C} \setminus \operatorname{Crit}(f)$$

is a submersion, where $\operatorname{Crit}(f)$ is the set of critical points of f, we know that

$$\frac{1}{d} f^* \mu_f = \frac{1}{d} f^*(dd^c G_f) = \frac{1}{d} dd^c (G_f \circ f) = dd^c G_f = \mu_f$$

where defined. As single points have μ_f-measure 0 (because the potential G_f is continuous), the relation holds on all of $\mathbb{C}$. □

Example 6.4. Let $f(z) = z^2$. Then $K(f)$ is the closed unit disk $\overline{\mathbb{D}}$, and $G_f(z) = \log^+ |z|$, and μ_f is the (normalized) Lebesgue measure on the unit circle.

6.2. **Holomorphic maps on $\mathbb{P}^N$.** Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be holomorphic of degree $d > 1$. Then we can express f in homogeneous coordinates as

$$f = (F_0 : F_1 : \cdots : F_N)$$

where each F_j is a homogeneous polynomial of degree d in $N + 1$ variables with $\{F_0 = \cdots = F_N = 0\} = \emptyset$. The map f is finite with topological degree d^N. As $d > 1$, the map f has a nonempty critical locus $\mathrm{Crit}(f) = \{\det Df = 0\}$, which is a hypersurface in $\mathbb{P}^N$ of degree $(N + 1)(d - 1)$.

Let

$$\tau : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N \tag{6.2}$$

be the tautological projection. We can define an escape-rate function for f on $\mathbb{C}^{N+1}$. Let $F = (F_0, \ldots, F_N)$ be a homogeneous presentation of f. We set

$$G_F(z) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(z)\|.$$

Any other choice for F would be of the form cF where $c \in \mathbb{C}^*$, so that $G_{cF}(z) = G_F(z) + \frac{1}{d-1} \log |c|$.

Note that $\log \|z\|$ is a continuous and plurisubharmonic function on $\mathbb{C}^{N+1} \setminus \{0\}$. As in dimension 1 and the proof of Theorem 6.1, this sequence of continuous and plurisubharmonic functions will converge uniformly on $\mathbb{C}^{N+1} \setminus \{0\}$. Indeed, note that $F^{-1}(0) = 0$ and $F(\alpha z) = \alpha^d F(z)$ for all $\alpha \in \mathbb{C}$, so we can find a constant $C > 1$ so that

$$C^{-1}\|z\|^d \leq \|F(z)\| \leq C\|z\|^d$$

for all $z \in \mathbb{C}^{N+1}$, so that

$$\big| \log \|F(z)\| - d \log \|z\| \big| \leq \log C$$

uniformly. Then, we express

$$G_n(z) := \frac{1}{d^n} \log \|F^n(z)\|,$$

as a telescoping sum, exactly as in the proof of Theorem 6.1. It follows that G_F is continuous and plurisubharmonic. Note that

$$G_F(\alpha z) = G_F(z) + \log |\alpha|$$

for all $\alpha \in \mathbb{C}^*$. The function G_F uniquely determines a positive (1,1)-current T_f on $\mathbb{P}^N$ by

$$\tau^* T_f = dd^c G_F.$$

In local coordinates on $\mathbb{P}^N$, we have $T_f = dd^c(G_F \circ \sigma)$ for any (locally defined) holomorphic section σ of τ. This is independent of the section, because any pair σ_1 and σ_2 will satisfy $\sigma_1 = \alpha\sigma_2$ for some $\alpha \in \mathcal{O}^*(U)$ (i.e., a nonvanishing holomorphic function on U) so that $G_F \circ \sigma_1 = G_F \circ \sigma_2 + \log |\alpha|$ and $dd^c(\log |\alpha|) = 0$.

This T_f is called the *dynamical Green current* for f. We shall see in the next section that it satisfies the same pull-back invariance property that we had in dimension 1 for polynomials; namely,

$$\frac{1}{d} f^* T_f = T_f$$

once we can make sense of f^* in this expression.

Example 6.5. A polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d > 1$ extends holomorphically to a map on $\mathbb{P}^1$ that can be written in homogeneous coordinates as

$$(z_0 : z_1) \mapsto (f(z_0/z_1) z_1^d, z_1^d).$$

Choosing this presentation for F on $\mathbb{C}^2$, we have

$$G_f(z) = G_F(z, 1),$$

so that $T_f = \mu_f$.

6.3. **The Julia set and T_f.** Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be holomorphic of degree $d > 1$. By definition, the *Fatou set* of f is the largest open set in $\mathbb{P}^N$ on which the sequence of iterates $\{f^n\}$ forms a normal family. The *Julia set* $J(f)$ is the complement in $\mathbb{P}^N$ of the Fatou set. When f is a polynomial on $\mathbb{C}$, viewed as a holomorphic map on $\mathbb{P}^1$, this definition coincides with the definition given in §6.1, as a consequence of Montel's Theorem on normal families.

Example 6.6. For the polynomial $f(z) = z^2$, defining a holomorphic map on $\mathbb{P}^1$ by $f(x : y) = (x^2 : y^2)$, its Julia set is the unit circle. For the rational function

$$f(z) = \frac{(z^2+1)^2}{4z(z^2-1)},$$

the Julia set is all of $\mathbb{P}^1$, because this f is a quotient of $[2] : E \to E$ on the elliptic curve $E = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ which has a dense set of repelling periodic points and derivatives of the iterates will blow up everywhere.

Theorem 6.7. *Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be holomorphic of degree $d > 1$ and T_f its dynamical Green current. Then*

$$\operatorname{supp} T_f = J(f).$$

Proof. Suppose that U lies in the Fatou set of f, so the sequence of iterates $\{f^n\}$ forms a normal family on U. We aim to show that $T_f|U = 0$. Shrinking U if necessary, and passing to a subsequence, we see that f_{n_k} converges uniformly on U to a holomorphic map $g : U \to \mathbb{P}^N$. Shrinking U further, we may assume that both U and the image $g(U)$ lie in a bounded subset K of an affine chart, say where $z_0 \neq 0$. In particular, for large enough n_k, we can write

$$F^{n_k} = (F_{0,n_k}, \dots, F_{N,n_k}) = F_{0,n_k}\left(1, \frac{F_{1,n_k}}{F_{0,n_k}}, \dots, \frac{F_{N,n_k}}{F_{0,n_k}}\right)$$

on $\mathbb{C}^{N+1}$, with the ratios $\left|\frac{F_{j,n_k}}{F_{0,n_k}}\right|$ uniformly bounded on U. It follows that

$$G_F(z) = \lim_{n_k\to\infty} \frac{1}{d^{n_k}} \left(\log |F_{0,n_k}| + \log \left\| \left(1, \frac{F_{1,n_k}}{F_{0,n_k}}, \dots, \frac{F_{N,n_k}}{F_{0,n_k}}\right) \right\| \right)$$

for $z \in U$. The first term is pluriharmonic, and the last term is uniformly bounded so disappears in the limit. In other words, G_F is pluriharmonic on $\tau^{-1}(U)$, where τ is the projection (6.2).

For the converse, we assume T_f vanishes on U. Shrinking U if needed, we let $\sigma : U \to \mathbb{C}^{N+1} \setminus \{0\}$ be a holomorphic section of τ. Then $G_F \circ \sigma$ is pluriharmonic and is the real part of a holomorphic function h on U. Let

$$\sigma' = e^{-h}\sigma,$$

so that

$$G_F \circ \sigma' = G_F \circ \sigma + \log |e^{-h}| \equiv 0.$$

But note that the compact set $\{G_F = 0\}$ is invariant for F, as $G_F(F(z)) = d\, G_F(z)$. It follows that all iterates $F^n(\sigma')$ lie in this compact set. By Montel's theorem, the uniform boundedness of the sequence $\{F^n \circ \sigma' : U \to \mathbb{C}^{N+1} \setminus \{0\}$ implies that that this is a normal family. It follows that $\{f^n\}$ is normal on U. □

Remark 6.8. Suppose U is in the Fatou set of f in $\mathbb{P}^N$. In the proof of Theorem 6.7, we found a holomorphic section σ' of π over U with image lying in the boundary of the *filled Julia set*

$$K_F = \left\{ z \in \mathbb{C}^{N+1} : \sup_n \|F^n(z)\| < \infty \right\}.$$

The section was determined only up to scalar multiple of the form e^{ic} for $c \in \mathbb{R}$; in other words, up to a multiple of norm 1. In particular, there is an S^1-family of such holomorphic sections, and these define *a foliation of the boundary of K_F over the Fatou set.* See [HP].

7. More dynamics: pullback-invariance

In Theorem 6.1, we saw that the equilibrium measure μ_f on the filled Julia set $K(f)$ satisfies a pullback relation, as $\frac{1}{d} f^* \mu_f = \mu_f$, for a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree d. Here we show the same relation holds for the Green current T_f of a holomorphic map $f : \mathbb{P}^N \to \mathbb{P}^N$.

First, suppose that $F : X \to Y$ is a finite map between complex manifolds X and Y of the same dimension. For any closed and positive (1,1)-current $T = dd^c u$ on Y, we set

$$F^*(dd^c u) := dd^c(u \circ F) \tag{7.1}$$

on X. Where F is a submersion, this agrees with the notion of pullback we have already defined. When u is locally bounded, there is no mass of T on the critical locus of F by Theorem 8.4, so this definition works well in that case too, extending $F^*(dd^c u)$ trivially across any "bad" subvariety.

In particular, if $f : \mathbb{P}^N \to \mathbb{P}^N$ is holomorphic and of degree $d > 1$, then

$$\tau^*\left(\frac{1}{d}f^*T_f\right) = \frac{1}{d}(f\circ\tau)^*T_f = \frac{1}{d}(\tau\circ F)^*T_f = \frac{1}{d}F^*(dd^cG_F) = dd^cG_F = \tau^*T_f$$

for the projection τ of (6.2), so that

$$\frac{1}{d}f^*T_f = T_f.$$

The restriction

$$f : \mathbb{P}^N \setminus f^{-1}(f(\mathrm{Crit}(f)) \longrightarrow \mathbb{P}^N \setminus f(\mathrm{Crit}(f))$$

is a submersion, and the same holds for F on τ^{-1} of these sets. But T_f has continuous potentials, so it can be extended trivially to the critical locus without causing any trouble.

Let ω_0 be the Fubini-Study form on $\mathbb{P}^N$. It can be defined by

$$\tau^*\omega_0 = dd^c \log \|z\|$$

for $\tau : \mathbb{C}^{N+1}\setminus\{0\} \to \mathbb{P}^N$. That is, for a holomorphic section $\sigma : U \to \mathbb{C}^{N+1}\setminus\{0\}$ over a neighborhood U in $\mathbb{P}^N$, we define $\omega_0 = dd^c \log\|\sigma\|$. In dimension $N = 1$, it is given in local coordinates by

$$\omega_0 = \frac{1}{2\pi}\frac{i\,dz\wedge d\bar{z}}{(1+|z|^2)^2}$$

and defines the unit-area spherical measure. We can compute that

$$\int_{\mathbb{P}^N} \omega_0^{\wedge N} = 1.$$

Theorem 7.1. *Suppose T is any closed and positive $(1,1)$-current on $\mathbb{P}^N$ with bounded potentials for which $\langle T, (\omega_0)^{\wedge(N-1)}\rangle = 1$. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d > 1$. Then*

$$\frac{1}{d^n}(f^n)^*T \longrightarrow T_f$$

in the sense of distributions.

Proof. From Lemma 7.3, below, there exists a bounded function $g : \mathbb{P}^N \to \mathbb{R}$ so that

$$T - T_f = dd^c g.$$

Then

$$f^*(T - T_f) = dd^c(g\circ f)$$

so that

$$\frac{1}{d^n}(f^n)^*T - T_f = \frac{1}{d^n}\,dd^c(g\circ f^n)$$

converges to the 0 current because $\frac{1}{d^n} g \circ f^n$ converges uniformly to the 0 function. □

Remark 7.2. The conclusion of Theorem 7.1 can fail if we replace T with an arbitrary closed and positive $(1,1)$-current with unbounded potentials. For example, take $T = \mu = \delta_0$ and $f(z) = z^2$ on $\mathbb{P}^1$. Then $\frac{1}{d^n}(f^n)^*\mu = \mu$ for all n, while T_f (which is the measure defined above as μ_f) is the Lebesgue measure on the unit circle.

We have seen from Proposition 5.4 that a closed and positive $(1,1)$-current T on a complex manifold can be expressed *locally* as $dd^c u$ for a plurisubharmonic function u. The proof of Theorem 7.1 required a global version:

Lemma 7.3. *If T is a closed and positive (1,1)-current on $\mathbb{P}^N$ defining the same cohomology class as ω_0 in $H^2(\mathbb{P}^N, \mathbb{R})$, then there exists an L^1 function g on $\mathbb{P}^N$ such that*

$$T - \omega_0 = dd^c g$$

in the sense of distributions.

Proof sketch. A proof can be found in [HP, Lemma 5.3] (and probably many other places), written in the language of sheaf cohomology. We choose an open covering of $\mathbb{P}^N$ by neighborhoods U_i on which we can write $T = dd^c u_i$ and $\omega_0 = dd^c v_i$ for plurisubharmonic functions u_i and v_i. The functions $h_{ij} = (u_i - v_i) - (u_j - v_j)$ define pluriharmonic functions on the intersections $U_i \cap U_j$ and a cohomology class in $H^1(\mathbb{P}^N, \mathcal{PH})$ for the sheaf of pluriharmonic functions. The short exact sequence $0 \to \mathbb{R} \to \mathcal{O} \to \mathcal{PH} \to 0$, defined by taking the real part of a holomorphic function (with kernel being functions having constant imaginary part), gives rise to a long exact sequence of cohomology groups. Then $[\{h_{ij}\}]$ will be sent to zero in $H^2(\mathbb{P}^N, \mathbb{R})$, being the class of $T - \omega_0$, and so – after refining the open covering if necessary – we see that $h_{ij} = h_i - h_j$ for pluriharmonic functions defined on U_i and U_j respectively. Then we let $g = (u_i - v_i) - h_i$. □

Remark 7.4. Another proof of Lemma 7.3 appears in [FS, Theorem 5.9], using a global solution to the $\bar{\partial}$-equation (4.2). Here is the idea: we start with T on $\mathbb{P}^N$ and consider the positive (1,1)-current defined by τ^*T on $\mathbb{C}^{N+1} \setminus \{0\}$. This will extend to a closed and positive (1,1)-current on all of $\mathbb{C}^{N+1}$ by the Skoda-El Mir Theorem 8.5. Then we can solve

$$\tau^*T = dd^c U$$

on all of $\mathbb{C}^{N+1}$, because we can globally solve the $\bar{\partial}$ equation here. The function U is uniquely determined up to the addition of a pluriharmonic function, and it turns out there is a unique choice which is logarithmically homogeneous, so that there exists $c \geq 0$ with $U(\alpha z) = U(z) + c \log|\alpha|$ for all $\alpha \in \mathbb{C}$. Then $g = U - c \log \|z\|$ descends to a well-defined function on $\mathbb{P}^N$, and

$$c = \left\langle T, \omega_0^{\wedge(N-1)} \right\rangle = \int_{\mathbb{P}^N} \omega_0^{\wedge N} = 1.$$

8. Mass and intersections of positive currents

For two currents T_1 and T_2 on a domain $\Omega \subset \mathbb{C}^N$, we would like to make sense of $T_1 \wedge T_2$. If T_2 is a smooth form β, then this is well-defined as

$$\langle T_1 \wedge \beta, \alpha \rangle := \langle T_1, \beta \wedge \alpha \rangle.$$

But what about more general pairs of currents?

8.1. **Mass of a positive current.** Recall that a positive (p, p)-current T on $\Omega \subset \mathbb{C}^N$ can be expressed as

$$T = i^{p^2} \sum_{|I|=|J|=p} T_{I,J}\, dz^I \wedge d\bar{z}^J$$

with distributional coefficients such that

$$T_{I,I} \geq 0 \quad \text{and} \quad \overline{T_{I,J}} = T_{J,I}.$$

The *mass measure* of T is the positive measure

$$\|T\| = \sum_{I,J} |T_{I,J}|.$$

Or, alternatively, we have

$$\|T\|(U) = \sup\{|\langle T, \alpha \rangle| : \alpha \in \mathcal{D}^{(N-p,N-p)}(U), \|\alpha\|_{C^0} \leq 1\}$$

on open sets $U \subset \Omega$. We say that a positive current has *finite mass* on an open $U \subset \Omega$ if $\|T\|(U) < \infty$.

Some authors prefer to work with the *trace measure* associated to T, namely

$$\sigma_T := T \wedge \omega^{N-p},$$

where

$$\omega = \sum_j i\, dz_j \wedge d\bar{z}_j.$$

(On a complex Kähler manifold, we use a Kähler form ω for this definition.) In other words, we let

$$\langle \sigma_T, \varphi \rangle := \langle T, \varphi\, \omega^{\wedge(N-p)} \rangle$$

for all functions $\varphi \in \mathcal{D}(\Omega)$. We set

$$\|T\|_K := \int_K \sigma_T$$

for compact sets $K \subset \Omega$.

EXERCISE: There is a constant $C > 1$ so that

$$C^{-1}\|T\|_K \leq \|T\|(K) \leq C\|T\|_K$$

for all compact sets $K \subset \Omega$.

Lemma 8.1. *For any pair of compact sets $K_1 \subset K_2 \subset \Omega$ with K_1 in the interior of K_2, there is a constant C so that*

$$\|dd^c u\|_{K_1} \leq C\,\|u\|_{L^1(K_2)}$$

for all plurisubharmonic functions $u : \Omega \to \mathbb{R} \cup \{-\infty\}$.

Proof. Choose a smooth cut-off function $\varphi \equiv 1$ on K_1 with $\operatorname{supp}\varphi \subset K_2$. Then

$$\begin{aligned}\|dd^c u\|_{K_1} &= \int_{K_1} dd^c u \wedge \omega^{N-1} \\ &\leq \int_{K_2} \varphi\, dd^c u \wedge \omega^{N-1} \\ &= \int_{K_2} u\, dd^c \varphi \wedge \omega^{N-1} \\ &\leq C\,\|u\|_{L^1(K_2)},\end{aligned}$$

where C is the L^∞ norm of the smooth function g satisfying

$$dd^c \varphi \wedge \omega^{N-1} = g\,\omega^N.$$

□

8.2. **Chern-Levine-Nirenberg–Bedford-Taylor.** Let T be a closed and positive (p,p)-current on a domain $\Omega \subset \mathbb{C}^N$, and suppose that $u : \Omega \to \mathbb{R}$ is a *locally bounded* plurisubharmonic function. The current uT is then well-defined, because we can multiply each of the measure coefficients $T_{I,J}$ by the function u. We define

$$(dd^c u) \wedge T := dd^c(uT).$$

Lemma 8.2. *The $(p+1, p+1)$-current $(dd^c u) \wedge T$ is closed and positive.*

Proof. Closed is clear. For positivity, it is useful to approximate u with smooth convolutions $u_\varepsilon = u * \psi_\varepsilon$ as in the proof of Theorem 2.5. These psh functions are C^∞ and descend to u as $\varepsilon \to 0$. By Dominated Convergence, we see that

$$u_\varepsilon T \to uT$$

weakly. This implies that

$$dd^c(u_\varepsilon T) \to dd^c(uT)$$

because dd^c is continuous. But $u_\varepsilon \in C^\infty$, so $dd^c(u^\varepsilon T) = (dd^c u_\varepsilon) \wedge T$ is already meaningful. Its positivity implies that the limit current is also positive. □

It follows that we can inductively define

$$dd^c u_1 \wedge dd^c u_2 \wedge \cdots dd^c u_m \wedge T$$

for any collection of bounded plurisubharmonic functions u_j and any closed and positive (p,p)-current T. In particular, the **Monge-Ampère measure** associated to u is defined by

$$(dd^c u)^N := dd^c u \wedge \cdots \wedge dd^c u.$$

Theorem 8.3. *For any pair of compact sets $K_1 \subset K_2 \subset \Omega$ with K_1 in the interior of K_2, there is a constant C so that*

$$\|dd^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge T\|_{K_1} \leq C\, \|u_1\|_{L^\infty(K_2)} \|u_2\|_{L^\infty(K_2)} \cdots \|u_m\|_{L^\infty(K_2)} \|T\|_{K_2}$$

for any collection $u_1, \ldots, u_m$ of bounded psh functions and any closed and positive (p,p)-current T, and

$$\|(-v)\, dd^c u_1 \wedge dd^c u_2 \wedge \cdots \wedge dd^c u_m\|_{K_1} \leq C\, \|v\|_{L^1(K_2)} \|u_1\|_{L^\infty(K_2)} \|u_2\|_{L^\infty(K_2)} \cdots \|u_m\|_{L^\infty(K_2)}$$

if $v : \Omega \to \mathbb{R} \cup \{-\infty\}$ is any psh function with $v \leq 0$.

The first inequality of Theorem 8.3 is known as the *Chern-Levine-Nirenberg inequality*; see [Dem, Theorem 3.3]. The second inequality is a strengthening which, I believe, was orginally due to Bedford and Taylor [BT]; see [Dem, Proposition 3.11].

Proof. The proof of the first inequality is similar to the proof of Lemma 8.1. Choose a smooth cut-off function $\varphi \equiv 1$ on K_1 with $\operatorname{supp} \varphi \subset K_2$. Let u be a bounded plurisubharmonic function and let T be a closed and positive (p,p)-current. Then

$$\begin{aligned} \|dd^c u \wedge T\|_{K_1} &= \int_{K_1} dd^c u \wedge T \wedge \omega^{N-p-1} \\ &\leq \int_{K_2} \varphi\, dd^c u \wedge T \wedge \omega^{N-p-1} \\ &= \int_{K_2} u\, dd^c \varphi \wedge T \wedge \omega^{N-p-1} \\ &\leq C\, \|u\|_{L^\infty(K_2)} \|T\|_{K_2}. \end{aligned}$$

The first inequality then follows from an induction argument. (To make the induction work, we should start by choosing a collection of m cut-off functions φ_j supported in K_2 so that $\varphi_j \equiv 1$ on $\operatorname{supp} \varphi_{j-1}$, and note that $m \leq N$.)

The second inequality requires more. For simplicity, we assume that K_1 lies in the unit ball B of $\mathbb{C}^N$ which is compactly contained in the interior of K_2 (and the general case will follow by choosing finite covers by balls). Assume in addition that there is an $M_j > 0$ so that

$$-M_j \leq u_j \leq -1$$

on B for all j. And further, assume for now that $m \leq N-1$.

Let $\psi(z) = \|z\|^2 - 1$ so that $B = \{\psi < 0\}$ and $dd^c \psi = \omega$. For each j, let $u_j^\psi = \max\{u_j, R_j \psi\}$ where $R_j > 0$ is chosen large enough that $u_j^\psi = u_j$ on a neighborhood of K_1 but equal to $R_j \psi$ outside a ball B_r of some fixed radius $r < 1$. (Note that R_j

and r are determined by M_j and the compact set K_1.) Let φ be a cut-off function which is $\equiv 1$ on the ball B_r of radius r and supported in B. We have

$$\begin{aligned}
\|(-v)dd^cu_1\wedge\cdots\wedge dd^cu_m\|_{K_1} &= \int_{K_1}(-v)dd^cu_1\wedge\cdots\wedge dd^cu_m\wedge\omega^{N-m} \\
&= \int_{K_1}(-v)dd^cu_1^\psi\wedge\cdots\wedge dd^cu_m^\psi\wedge\omega^{N-m-1}\wedge dd^c(\varphi\psi) \\
&\leq \int_B(-v)dd^cu_1^\psi\wedge\cdots\wedge dd^cu_m^\psi\wedge\omega^{N-m-1}\wedge dd^c(\varphi\psi) \\
&\quad + \int_{B\setminus B_r} v\, dd^cu_1^\psi\wedge\cdots\wedge dd^cu_m^\psi\wedge\omega^{N-m-1}\wedge dd^c(\varphi\psi) \\
&= -\int_B(\varphi\psi)\, dd^cv\wedge dd^cu_1^\psi\wedge\cdots\wedge dd^cu_m^\psi\wedge\omega^{N-m-1} \\
&\quad + R_1\cdots R_m\int_{B\setminus B_r} v\,\omega^{N-1}\wedge dd^c(\varphi\psi)
\end{aligned}$$

The first inequality follows because the measure and the function $-v$ are positive on the closure of the ball B_r. The first integral of the last line is bounded by combining the first inequality of the theorem, to obtain a bound involving the L^∞ norms of the u_j's and trace norm $\|dd^cv\|_B$, and Lemma 8.1 to replace $\|dd^cv\|_B$ with $\|v\|_{L^1(K_2)}$. The second integral is bounded in terms of the R_j's, and so the L^∞ norms of the u_j's again, and the L^1 norm of v. □

Theorem 8.4. *If u is a bounded psh function on Ω, then $T = dd^cu$ puts no mass on pluripolar sets, nor does any wedge power $T^{\wedge m}$ of T.*

Proof. This is immediate from the second statement of Theorem 8.3, taking the pluripolar set $K_1 = \{v = -\infty\} \cap \overline{B(a,r)}$ for a small ball in Ω, with $u_1 = \cdots = u_m = u$. □

8.3. **Skoda - El Mir extension theorem.**

Theorem 8.5. *Let Ω be a domain in $\mathbb{C}^N$. Let $E \subset \Omega$ be a closed and complete pluripolar set, and suppose that T is a positive closed (p,p)-current on $\Omega \setminus E$. Assume that T has finite mass in a neighborhood of each point $e \in E$. Then the trivial extension of T is a closed positive (p,p)-current on Ω.*

A proof is given in [Dem, Chapter III Theorem 2.3].

8.4. **Continuity.** The Monge-Ampère operation $u \mapsto (dd^cu)^N$ is *not* continuous with respect to the weak topology, not even if we assume the input functions are continuous. Examples of Lelong, presented in [Kl, §3.8], show that there are continuous functions u on open subsets $\Omega \subset \mathbb{C}^2$ and sequences of continuous functions $u_n \to u$ in $L^1_{loc}(\Omega)$ for which $(dd^cu_n)^2 \equiv 0$ for all n while $(dd^cu)^2$ is a nonzero measure supported on all of Ω. Nevertheless, under additional hypotheses we can guarantee continuity. For example:

Proposition 8.6. *Suppose that $u_1, \ldots, u_m$ are continuous and bounded plurisubharmonic functions on $\Omega \subset \mathbb{C}^N$, and suppose that $\{u_j^n\}_n$ is a sequence of psh functions converging locally uniformly to u_j, for each j. Then*

$$u_j^n \, T_n \longrightarrow u_j \, T$$

weakly, for any sequence of closed positive currents $T_n \to T$ weakly. Consequently,

$$dd^c u_1^n \wedge \cdots \wedge dd^c u_m^n \wedge T_n \longrightarrow dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T$$

weakly.

Proof. Let $u_{j,\varepsilon} = u_j * \chi_\varepsilon$ be a smooth convolution for $\varepsilon > 0$. Then

$$u_j^n \, T_n - u_j \, T = (u_j^n - u_j) \, T_n + (u_j - u_{j,\varepsilon}) \, T_n + u_{j,\varepsilon} \, (T_n - T) + (u_{j,\varepsilon} - u_j) \, T.$$

Fix a compact set $K \subset \Omega$. The weak convergence of T_n implies that $\|T_n\|_K$ is uniformly bounded. Indeed, if $\varphi \geq 0$ is a smooth function $\equiv 1$ on K, we can use the positivity of T_n to see that

$$\|T_n\|_K \leq \int_\Omega \varphi \, T_n \wedge \omega^{N-p} \to \int_\Omega \varphi \, T \wedge \omega^{N-p}.$$

Therefore

$$\|(u_j^n - u_j)T_n\|_K \leq \|u_j^n - u_j\|_{L^\infty(K)} \|T_n\|_K$$

converges to 0 as $n \to \infty$, and we can make both

$$\|(u_j - u_{j,\varepsilon})T_n\|_K \quad \text{and} \quad \|(u_j - u_{j,\varepsilon})T\|_K$$

uniformly small by choosing ε small enough. The smoothness of $u_{j,\varepsilon}$ is used to show that $u_{j,\varepsilon}(T_n - T) \to 0$ weakly with n. The conclusion of the proposition follows by induction and the weak continuity of dd^c. □

A stronger version of Proposition 8.6 was proved by Bedford and Taylor (see also [Dem, Theorem 3.7]) at the expense of requiring that $T_n = T$ for all n:

Theorem 8.7. [BT] *Suppose that $u_1, \ldots, u_m$ are locally bounded psh functions on Ω, and suppose that u_j^n is a sequence of psh functions that are decreasing to u_j and converging pointwise. Then*

$$dd^c u_1^n \wedge \cdots \wedge dd^c u_m^n \wedge T \longrightarrow dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T$$

weakly, for any closed and positive current T.

Corollary 8.8. *For locally bounded psh functions u_j and for closed and positive current T, the wedge product*

$$dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge T$$

is symmetric in $(u_1, \ldots, u_m)$.

Proof. It is clear for smooth functions u_j. Then apply Theorem 8.7 to the convolutions of u_j with smoothing kernel $\psi_{1/n}$ as $n \to \infty$ (recall the definition from §1.4 and how it is used in §2.5). □

9. Currents of Integration and Theorem of the support

9.1. **Current of integration on a smooth hypersurface.** As a key example, let $f : \Omega \to \mathbb{C}^N$ be a holomorphic function with gradient $(\partial f/\partial z_1, \ldots, \partial f/\partial z_N)$ everywhere nonzero along $Z = \{f(z) = 0\}$, so Z is a smooth analytic hypersurface in Ω. Then the function

$$u(z) = \log |f(z)|$$

is plurisubharmonic on Ω and

$$T = dd^c u$$

is the current of integration $[Z]$. Indeed, we have already seen that $\mathrm{supp}(dd^c|f(z)|) \subset Z$. We can choose local coordinates $(z_1, \ldots, z_N)$ on a neighborhood U where $f(z) = g(z)z_1$ for a nonvanishing holomorphic function g on U. Thus

$$dd^c \log |f(z)| = dd^c|z_1| = \pi^* \left(dd^c \log |z|\right) = \pi^* \delta_0$$

for the projection $\pi : U \to \mathbb{C}$ defined by $(z_1, \ldots, z_N) \mapsto z_1$. And we know that the pullback of δ_0 by a projection is a current of integration.

9.2. **Current of integration on general analytic subvariety.** Now suppose that Z is any analytic subvariety of Ω of codimension p, possibly singular. To define a current of integration along Z, we need to know that the integral

$$\int_{Z^{reg}} \alpha$$

converges for all $\alpha \in \mathcal{D}^{N-p,N-p}(\Omega)$ where the regular part of Z is $Z^{reg} = Z \setminus Z^{sing}$.

Lemma 9.1. *The current of integration $[Z^{reg}]$ on $\Omega \setminus Z^{sing}$ has finite mass in a neighborhood of each point $z_0 \in Z^{sing}$.*

Proof sketch. Suppose that Z has codimension p, so $[Z^{reg}]$ is a closed and positive (p,p)-current on $\Omega \setminus Z^{sing}$. In a neighborhood of z_0, we can choose coordinates $(w_1, \ldots, w_N)$ so that the projection to $(w_1, \ldots, w_{N-p})$ is a branched covering map, ramified over the image of the singular locus. See [Dem, Chapter II §2 and §4]. Note that this branch locus has Lebesgue measure 0. Elsewhere, the projection is a covering map of finite degree, say m. It follows that the mass of $[Z^{reg}]$ near z_0 is bounded by the mass of the constant function m. □

Theorem 8.5 then implies that

$$\langle [Z], \alpha \rangle := \int_{Z^{reg}} \alpha$$

is a well-defined closed and positive (p,p)-current on Ω, by extending it trivially across Z^{sing}.

9.3. **Theorem of the Support.**

Theorem 9.2. *Suppose that $Z \subset \Omega$ is an irreducible analytic subset of codimension p. Suppose that T is a closed and positive (p,p)-current with* $\operatorname{supp} T \subset Z$*. Then*

$$T = c\,[Z]$$

for a constant $c \geq 0$.

Proof. First suppose that the support of T lies in an analytic subset $A \subset Z$ of codimension $> p$ in Ω. We aim to show that $T = 0$. Where A is smooth, we can choose local coordinates so that $A = \{w_1 = \ldots = w_m = 0\}$ for some $m > p$. Write

$$T = i^{p^2} \sum_{|I|=|J|=p} T_{I,J}\, dw^I \wedge d\bar{w}^J.$$

Note that $w_j T = 0$ for all $j \leq m$, because T is supported where w_j vanishes, so

$$dw_j \wedge T = d(w_j T) = 0$$

for all $j \leq m$. It follows that $\left\langle T_{I,J}\, dw^I \wedge d\bar{w}^J, \alpha \right\rangle = 0$ for any $(N-p, N-p)$-form α, because every such α will have terms involving $dw_1, \ldots, dw_m$ with indices in I. Therefore $T_{I,J} = 0$ for each I, J, so that $T = 0$. For non-smooth A, we work inductively on dimension.

Now suppose that Z is smooth with $\operatorname{supp} T \subset Z$, and we locally write Z as $\{w_1 = \cdots = w_p = 0\}$. Now $w_j T = 0$ for all $j \leq p$, so that $dw_j \wedge T = 0$ and $d\bar{w}_j \wedge T = 0$. Pairing T with $(N-p, N-p)$-forms involving the terms $dw_1, \ldots, dw_p$ or $d\bar{w}_1, \ldots, d\bar{w}_p$ must vanish, so we have

$$T = i^{p^2} T_{I,I} dw^I \wedge d\bar{w}^I$$

for $I = (1, \ldots, p)$ and a positive distribution $T_{I,I}$ on Z. But $dT = 0$ so

$$\frac{\partial}{\partial w_j} T_{I,I} = \frac{\partial}{\partial \bar{w}_j} T_{I,I} = 0$$

for $j > p$, so $T_{I,I}$ is constant.

For singular Z, we use the first argument to see that T must vanish along the singularities. □

Proposition 9.3. *Suppose $f : \Omega \to \mathbb{C}$ is a holomorphic function with $f \not\equiv 0$. Let $u = \log|f|$ and $Z = \{f = 0\}$.*

$$[Z] = dd^c u$$

on Ω.

Proof. We know that $[Z] = dd^c u$ holds where Z is smooth. But neither $[Z]$ (by definition) nor $dd^c u$ (by Theorem 9.2) assigns mass to the singular set, so the equality holds on all of Ω. □

9.4. **Wedging with a current of integration.** Suppose now that $[Z]$ is the current of integration along an analytic subvariety of codimension p in $\Omega \subset \mathbb{C}^N$. We saw in the previous section that, at least when u is a locally bounded plurisubharmonic function, the wedge

$$dd^c u \wedge [Z] = dd^c(u[Z])$$

is well defined. But what is this current actually?

Note that $u\,[Z]$ is defined by

$$\langle u\,[Z], \alpha \rangle = \int_{Z^{reg}} u\,\alpha$$

for smooth $(N-p, N-p)$-forms α with compact support in Ω. If u is smooth and if Z is smooth, then we can choose coordinates so that $Z = \{z_1 = \cdots = z_p = 0\}$, and then

$$dd^c(u\,[Z]) = (dd^c)_Z(u|_Z) \wedge [Z]$$

where $(dd^c)_Z$ means that we only differentiate with respect to the variables $(z_{p+1}, \ldots, z_N)$. Indeed, as in the proof of Theorem 9.2, we have $dz_j \wedge [Z] = d\bar{z}_i \wedge [Z] = 0$ for each $j \leq p$. For continuous u, we can approximate with smooth functions by convolutions to see that we can again simply restrict u to Z so that

$$dd^c u \wedge [Z] = (dd^c)_Z(u|_Z) \wedge [Z]$$

by Proposition 8.6.

Chapter III §4 of [Dem] treats cases where u is unbounded, and in particular, shows that

$$[Z_1] \wedge \cdots \wedge [Z_m]$$

is well defined for collections of analytic subvarieties Z_i, as long as the intersections of the Z_i's have the "expected" codimension. In this case, the result is a current of integration along the components of the intersection (counted with appropriate multiplicities). See, for example, [Dem, Proposition 4.12].

10. Slicing

The "restriction" of a current to a subvariety is not always well-defined, but Federer introduced the *slicing* of a current along the fibers of a submersion, at least if the current has good properties [**?**]. It is similar in spirit to Rokhlin's theory of conditional measures, or disintegration [**?**].

10.1. **Over $\mathbb{R}$.** Suppose that $\pi : X \to Y$ is a submersion of C^∞ manifolds with $\dim X = m$ and $\dim Y = n$. Suppose that T is a p-current on X with $p \le m-n$ and such that, local coordinates,

$$T = \sum_{|I|=p} f_I \, dx^I$$

with each coefficient f_I in $L^1_{loc}(X)$. For each $y \in Y$, let $X_y = \pi^{-1}(y)$. Note that $f_I|_{X_y}$ is in $L^1_{loc}(X_y)$ for almost every y, and we define

$$T_y := \sum_{|I|=p} f_I|_{X_y} \, (i_y)^* dx^I$$

where $i_y : X_y \hookrightarrow X$ is the inclusion. The currents T_y are called the *slices* of T with respect to π. By Fubini, we see that the currents T_y are uniquely determined (for a.e. $y \in Y$) by the property that

$$\int_Y \left(\int_{X_y} T_y \wedge (i_y)^* \alpha \right) \omega = \int_X T \wedge \alpha \wedge \pi^* \omega \tag{10.1}$$

for every $\alpha \in \mathcal{D}^{m-n-p}(X)$ and $\omega \in \mathcal{D}^n(Y)$. Indeed, we choose local coordinates $(x_1, \ldots, x_m)$ on X where it looks like a product $U \times V$ of neighborhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^{m-n}$ with π the projection to U, and (via a partition of unity) we work with smooth test forms supported in these neighborhoods. Assume first that $p = 0$ and $T \equiv 1$. The pullback $\pi^*\omega$ involves only a wedge of terms $dx_1, \ldots, dx_n$, so only the terms of α involving $dx_{n+1}, \ldots, dx_m$ appear in $\alpha \wedge \pi^*\omega$, and the same for $i_y^*\alpha$. Then you apply Fubini to see that

$$\int_{U \times V} \alpha \wedge \pi^* \omega = \int_U \left(\int_V i_y^* \alpha \right) \omega(y)$$

The general case is similar.

Note that the currents T_y have *the same degree as the original T.* By convention, if $p > m-n$, we set the slices T_y to be 0.

Slicing commutes with the d operation. That is $(dT)_y = d(T_y)$ for almost every $y \in Y$.

More generally, we can define slices for currents where T and dT have coefficients of order 0, and the slices are characterized by (10.1). Such currents are called *locally normal*. In fact, Federer introduced slicing for a slightly more general class of currents, called *locally flat*; these can be expressed locally as $S + dR$ for currents S and R with locally L^1 coefficients. (These generalizations make sense from the definition above for currents with L^1_{loc} coefficients and from the commuting of d with slicing; again the key property is the characterization (10.1).) In particular, we can define slices for closed and positive currents on complex manifolds.

Remark 10.1. A current T on X is not uniquely determined by its slices. For example, let $T = \pi^*\beta$ for any smooth and nonzero p-form β on Y. Here we assume that $0 < p \leq \min\{n, m-n\}$. Then $\pi^*\beta \wedge \pi^*\omega = \pi^*(\beta \wedge \omega) = 0$ for every smooth n-form ω on Y, because $\beta \wedge \omega$ has degree larger than the dimension of Y. It follows that $\int_X T \wedge \alpha \wedge \pi^*\omega = 0$ for any choice of α and ω, and therefore $T_y = 0$ for all y.

10.2. **Compare to disintegration.** Let ψ_ε be approximate identity functions on the unit ball in $\mathbb{R}^n$ as introduced in §1.4. The following is immediate from the construction of slices and (10.1):

Proposition 10.2. *Let $\pi : X \to Y$ be a submersion as above and T a locally flat p-current on X with $p \leq m-n$. Then for almost every $y \in Y$, the slice T_y satisfies*

$$\langle T_y, i_y^*\alpha\rangle = \lim_{\varepsilon\to 0}\langle T,\ \alpha \wedge \pi^*(\psi_\varepsilon(y+\cdot)\, dx_1 \wedge \cdots \wedge dx_n)\rangle$$

for every $\alpha \in \mathcal{D}^{m-n-p}(X)$, where $(x_1, \ldots, x_n)$ are local coordinates on Y near y.

In Rokhlin's theory of conditional measures [**?**], one considers a probability measure μ on the total space X and defines a family of probability measures $\{\mu_y\}_{y\in Y}$ by

$$\mu_y(E \cap X_y) := \lim_{\varepsilon\to 0} \frac{\mu(E \cap \pi^{-1}(B(y,\varepsilon))}{\pi_*\mu(B(y,\varepsilon))}$$

for $\pi_*\mu$-almost every $y \in Y$ and all measurable sets E in X. The characterizing property of these *conditional measures* is that

$$\mu(E) = \int_Y \mu_y(E \cap X_y)\, \pi_*\mu(y)$$

for all measurable E in X.

10.3. **Special cases, complex setting.** Suppose that u is a plurisubharmonic function on $\Omega \subset \mathbb{C}^N$, and set $T = dd^c u$. For an affine subspace $A \subset \mathbb{C}^N$, the function $u|_A$ is either $\equiv -\infty$ or it is again psh. If $\pi : \Omega \to Y$ is a submersion of complex manifolds with $m = \dim X > n = \dim Y$, then we have

$$\int_\Omega (dd^c u) \wedge \alpha \wedge \pi^*\omega = \int_Y \left(\int_{\Omega_y} dd^c(u|_{\Omega_y}) \wedge (i_y)^*\alpha \right) \omega(y)$$

for all $\alpha \in \mathcal{D}^{(m-n-1,m-n-1)}(\Omega)$ and $\omega \in \mathcal{D}^{(n,n)}(Y)$. The currents

$$T_y = dd^c(u|_{\Omega_y})$$

are the *slices* of T, and these are defined for almost every $y \in Y$. Note that T_y (when defined) is a closed and positive $(1,1)$-current. If the fibers are 1-dimensional, then T_y is a positive measure on the fiber for a.e. y.

Suppose that $Z \subset X$ is an analytic subvariety of codimension p, $\pi : X^m \to Y^n$ is a holomorphic submersion with $p \le m - n$, and $Z \cap X_y$ has codimension p for general points $y \in Y$. Then we can slice $[Z]$ with respect to π: we have

$$[Z]_y = [Z \cap X_y]$$

for almost every $y \in Y$. Note that $[Z]_y = 0$ when $[Z \cap X_y]$ has codimension $\neq p$ in X_y.

10.4. **Slicing and wedging.** In general, it might be difficult to know exactly what the slice currents are. The following is helpful in certain settings. See [BB, Proposition 4.3].

Proposition 10.3. *Suppose that $\pi : X^m \to Y^n$ is a holomorphic submersion and $u_0, \dots, u_r$ are locally bounded plurisubharmonic functions on X, with $r \le m - n$. Suppose that T is a closed and positive (p,p)-current so that $p \le m - n - r$. Then the slices are given by*

$$(u_0 \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_r \wedge T)_y = u_0|_{X_y} \wedge dd^c(u_1|_{X_y}) \wedge \cdots dd^c(u_r|_{X_y}) \wedge T_y$$

for almost every $y \in Y$.

Proof. The proof proceeds inductively. We need only show that

$$u_y T_y = (uT)_y$$

for almost every y and any locally bounded psh function u. If u is smooth, then this is clear by the characterization of slices T_y. Indeed, we have

$$\begin{aligned} \int_Y \left(\int_{X_y} u_y T_y \wedge i_y^* \alpha \right) \omega(y) &= \int_Y \left(\int_{X_y} T_y \wedge i_y^*(u\alpha) \right) \omega(y) \\ &= \int_X T \wedge (u\alpha) \wedge \pi^* \omega \\ &= \int_X u\, T \wedge \alpha \wedge \pi^* \omega \end{aligned}$$

for all α and ω. Note further that

$$\left| \int_K T \wedge (u\alpha) \wedge \pi^* \omega \right| \le C_{\alpha,\omega} \|u\|_K \|T\|_K$$

on compact sets $K \subset X$. So the operator

$$u \mapsto \int_K u\, T \wedge \alpha \wedge \pi^* \omega$$

extends continuously from C^∞ to L^∞. □

11. Dynamics: maximal measures and bifurcation currents

Now we put some of these general concepts (wedging, slicing) to work in some dynamical constructions.

11.1. **Maximal measure.** Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d > 1$. In Section 6, we constructed the Green current T_f, a closed and positive $(1,1)$-current on $\mathbb{P}^N$ with continuous potentials. In Section 7, we characterized T_f as the unique closed and positive $(1,1)$-current T_f with bounded potentials satisfying

$$\frac{1}{d} f^* T_f = T_f.$$

Now that we know how to take wedge products of such currents, we can make sense of

$$\mu_f := T_f^{\wedge N}.$$

This is a probability measure on $\mathbb{P}^N$ and satisfies

$$\frac{1}{d^N} f^* \mu_f = \mu_f. \tag{11.1}$$

And it follows from Proposition 8.6 that

$$\frac{1}{d^{nN}} (f^n)^*(T^{\wedge N}) \longrightarrow \mu_f$$

weakly as $n \to \infty$ for any closed and positive $(1,1)$-current T on $\mathbb{P}^N$ with bounded potentials, because the potentials will converge uniformly (as we saw in Theorem 7.1). Thus, this measure is characterized by the pullback relation, at least among powers of currents with bounded potentials.

In fact, it was proved by Lyubich [Ly] and independently by Freire, Lopes, and Mañé [Ma, FLM] in dimension $N = 1$, and by Briend and Duval [BD] in all dimensions, that μ_f is the unique probability measure satisfying (11.1) that does not charge the *exceptional set* $E \subset \mathbb{P}^N$ for f. The exceptional set is the largest (proper) algebraic subvariety in $\mathbb{P}^N$ for which $f^{-1}(E) = E$; it is empty for a general choice of f. Moreover, they proved that the measure μ_f is the *unique measure of maximal entropy* (equal to $N \log d$) for f.

11.2. **Green current in families.** Now let

$$\hat{f} : \Lambda \times \mathbb{P}^N \to \Lambda \times \mathbb{P}^N$$

be a *holomorphic family* of maps on $\mathbb{P}^1$ of degree $d > 1$, parameterized by a (connected) complex manifold Λ. That is $\hat{f}(\lambda, z) = (\lambda, f_\lambda(z))$ for a holomorphic map $(\lambda, z) \mapsto f_\lambda(z)$, and we assume that the map has degree $d > 1$ for some fixed $\lambda_0 \in \Lambda$

(and therefore will have the same degree for all λ). Let ω_0 denote the Fubini-Study form on $\mathbb{P}^N$. Let $\hat{\omega}_0 = p^*\omega_0$ for the projection $p : \Lambda \times \mathbb{P}^N \to \mathbb{P}^N$. We let

$$\hat{T} = \lim_{n\to\infty} \frac{1}{d^n}(\hat{f}^n)^*\hat{\omega}_0.$$

Note that the potentials for $\frac{1}{d^n}(\hat{f}^n)^*\hat{\omega}_0$ converge locally uniformly in $\Lambda \times \mathbb{P}^N$ to the (continuous) potential for $\hat{T}$. Indeed, restricted to each $\{\lambda_0\} \times \mathbb{P}^N$, this is just the convergence of potentials that we have already seen in §6.2, working with $\frac{1}{d^n}\log \|F^n_{\lambda_0}\|$ in $\mathbb{C}^{N+1} \setminus \{0\}$. Restricting to a small neighborhood of λ_0 in Λ, we can choose the homogeneous polynomial lifts F_λ to have coefficients holomorphic in λ, and we can find a uniform constant C for the estimates in §6.2. Consequently, the convergence is uniform on this small neighborhood of λ_0. In particular, the current $\hat{T}$ has continuous potentials, as we have

$$\hat{T} = dd^c\, G_{F_\lambda}(\sigma(z))$$

locally, where dd^c is acting in both variables (λ, z) and σ is any (locally defined) holomorphic section of $\tau : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$.

In particular, the slices $\hat{T}_\lambda$ of $\hat{T}$, as defined in Section 10, coincide with the Green currents T_{f_λ} for each $\lambda \in \Lambda$. But as observed in Remark 10.1, the current $\hat{T}$ is not uniquely determined by the slices. We have the following:

Lemma 11.1. *The current $\hat{T}$ vanishes along the periodic points of $\hat{f}$ in $\Lambda \times \mathbb{P}^N$.*

Proof. Fix a small neighborhood $U \subset \Lambda$ and suppose that $p(\lambda)$ parameterizes a periodic point of f_λ in $\mathbb{P}^N$. For simplicity, let us assume that it is a fixed point. Then for each λ, note that the complex line in $\mathbb{C}^{N+1}$ over $p(\lambda)$ is fixed for F_λ, and the F_λ action on this line is given by $\zeta \mapsto c_\lambda \zeta^d$ for $c_\lambda \in \mathbb{C}^*$ depending holomorphically in λ. In particular, we can holomorphically lift p to $\tilde{p}$ to parameterize a fixed point of F_λ. Then $G_{F_\lambda}(\tilde{p}(\lambda)) \equiv 0$ so that $\hat{T}|_p = 0$. □

11.3. **The bifurcation current and measure.** Recall that the critical locus $\mathrm{Crit}(f)$ of one map f on $\mathbb{P}^N$ is an algebraic hypersurface; for a holomorphic family of maps defined by

$$\hat{f} : \Lambda \times \mathbb{P}^N \to \Lambda \times \mathbb{P}^N,$$

the critical locus $\mathrm{Crit}(\hat{f})$ is an analytic hypersurface in $\Lambda \times \mathbb{P}^N$. We let

$$T_{\mathrm{bif}} := \pi_*\left(\hat{T}^{\wedge N} \wedge [\mathrm{Crit}(\hat{f})]\right), \tag{11.2}$$

where $\pi : \Lambda \times \mathbb{P}^1 \to \Lambda$ is the projection to the parameter space. Note that T_{bif} is a closed and positive (1,1)-current on Λ, as the projection of a closed and positive $(N+1, N+1)$-current.

The current T_{bif} was introduced in my thesis [De1, De2] in dimension $N = 1$ for the purpose of studying stability properties of the family $\hat{f}$, and the notions were extended to higher dimensions by Bassanelli and Berteloot [BB].

Theorem 11.2. *T_{bif} has locally bounded (in fact continous) potentials.*

Proof. For $N = 1$, this is almost immediate from the definition. Note that each map on $\mathbb{P}^1$ of degree $d > 1$ has exactly $2d - 2$ critical points $\{c_1, \ldots, c_{2d-2}\}$, counted with multiplicity. Over a small neighborhood U in Λ, we can choose homogeneous F_λ with coefficients holomorphic in $\lambda \in U$, and we can choose a holomorphic section σ of $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ defined in a neighborhood of these critical points for all $\lambda \in U$. Then the function

$$\lambda \mapsto \sum_{i=1}^{2d-2} G_{F_\lambda}(\sigma(c_i(\lambda)))$$

is a potential for T_{bif} on U, and this function is continuous.

For general $N \geq 1$, it is convenient to work with globally-defined continuous potentials on $\mathbb{P}^N$. That is, we fix a small neighborhood U in Λ where we can define holomorphically varying lifts F_λ of f_λ, and we set

$$g(\lambda, z) = G_{F_\lambda}(\tilde{z}) - \log \|\tilde{z}\|$$

where $\tilde{z}$ is any choice of point in $\tau^{-1}(z)$ in $\mathbb{C}^{N+1} \setminus \{0\}$. Note that the function g depends on the choice of F_λ but is well-defined and continuous on $U \times \mathbb{P}^N$ and

$$dd^c\, g = \hat{T} - \hat{\omega}_0$$

on $U \times \mathbb{P}^N$. Let g_λ denote the restriction of g to $\{\lambda\} \times \mathbb{P}^N$, so that $dd^c\, g_\lambda = T_{f_\lambda} - \omega_0$ in $\mathbb{P}^N$. Define

$$\mathcal{L}(\lambda) := \sum_{j=0}^{N-1} \int_{\mathbb{P}^N} g_\lambda\, T_{f_\lambda}^j \wedge \omega_0^{N-j-1} \wedge [\mathrm{Crit}(f_\lambda)]$$

on the open set $U \subset \Lambda$.

EXERCISE: the function $\mathcal{L}$ is continuous in λ.

Now let $\ell = \dim \Lambda$ and fix $\alpha \in \mathcal{D}^{(\ell-1,\ell-1)}(U)$. We compute

$$\begin{aligned}
\langle dd^c \mathcal{L}, \alpha \rangle &= \int_U \mathcal{L}\, dd^c \alpha \\
&= \sum_{j=0}^{N-1} \int_U \left(\int_{\mathbb{P}^N} g_{f_\lambda}\, T_{f_\lambda}^j \wedge \omega_0^{N-j-1} \wedge [\mathrm{Crit}(f_\lambda)] \right) dd^c \alpha \\
&= \sum_{j=0}^{N-1} \int_{U \times \mathbb{P}^N} g_f\, \hat{T}^j \wedge \hat{\omega}_0^{N-j-1} \wedge [\mathrm{Crit}(\hat{f})] \wedge \pi^*(dd^c\, \alpha)
\end{aligned}$$

from Proposition 10.3. Integrating by parts, we then have

$$\begin{aligned}\langle dd^c\mathcal{L}, \alpha\rangle &= \sum_{j=0}^{N-1}\int_{U\times\mathbb{P}^N} dd^c\, g_f \wedge \hat{T}^j \wedge \hat{\omega}_0^{N-j-1} \wedge [\mathrm{Crit}(\hat{f})] \wedge \pi^*\alpha \\ &= \sum_{j=0}^{N-1}\int_{U\times\mathbb{P}^N} \left(\hat{T} - \hat{\omega}_0\right) \wedge \hat{T}^j \wedge \hat{\omega}_0^{N-j-1} \wedge [\mathrm{Crit}(\hat{f})] \wedge \pi^*\alpha \\ &= \int_{U\times\mathbb{P}^N} \hat{T}^N \wedge [\mathrm{Crit}(\hat{f})] \wedge \pi^*\alpha - \int_{U\times\mathbb{P}^N} \hat{\omega}_0^N \wedge [\mathrm{Crit}(\hat{f})] \wedge \pi^*\alpha \\ &= \left\langle \pi_*\left(\hat{T}^N \wedge [\mathrm{Crit}(\hat{f})]\right),\, \alpha\right\rangle - \left\langle \pi_*\left(\omega_0^N \wedge [\mathrm{Crit}(\hat{f})]\right),\, \alpha\right\rangle.\end{aligned}$$

The first term in the last line is exactly the value of T_{bif} acting on α. Recall that T_{bif} is closed and positive. The final term is the *negative* of the value of a closed and positive current $T_0 := \pi_*\left(\omega_0^N \wedge [\mathrm{Crit}(\hat{f})]\right)$ acting on α. ... FINISH WRITING. □

As a consequence of Theorem 11.2, we can define the *bifurcation measure*, originally introduced by Bassanelli and Berteloot [BB]:

$$\mu_{\mathrm{bif}} := (T_{\mathrm{bif}})^{\wedge(\dim\Lambda)} \tag{11.3}$$

This measure provides refined information on the bifurcations in the family of maps and has been a central object of study for about 20 years now; see, for example, [Ga, Du, BBD, BeBi, Be]. It was shown in [BB] that μ_{bif} is well-defined *and nonzero* on the moduli space M_d of all maps on $\mathbb{P}^1$ of degree d.

12. Canonical height functions

Height functions are important tools in arithmetic geometry and number theory, with which we measure complexity of algebraic/rational solutions to polynomial equations. In dynamics, we can use them to measure arithmetic complexity in orbits. Here we introduce the canonical height functions associated to morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$, and we illustrate how the height functions are linked to (pluri)potential theory. Helpful background and details can be found in [Si2] and [BG].

12.1. **Logarithmic Weil height.** We start with a notion of height for a number in $\overline{\mathbb{Q}}$. For $\alpha \in \mathbb{Q}^*$ written as $\pm m/n$ for positive integers $(m, n) = 1$, we set

$$h(\alpha) = \log\max\{m, n\}.$$

By convention, we set $h(0) = 0$. Then $h(\alpha) = 0$ for $\alpha \in \mathbb{Q}$ if and only if $\alpha \in \{0, 1, -1\}$. The height h grows with the number of digits of m or n. For $\alpha \in \overline{\mathbb{Q}}$, this definition can be extended by

$$h(\alpha) = \frac{1}{\deg \alpha}\left(\log|a_0| + \sum_{P_\alpha(x)=0} \log^+|x|\right)$$

where $P_\alpha(x) \in \mathbb{Z}[x]$ is the minimal polynomial for α with leading coefficient a_0 and degree $\deg\alpha$. There are a number of alternative but equivalent expressions for this height:

$$\begin{aligned} h(\alpha) &= \frac{1}{\deg P_\alpha}\int_0^1 P_\alpha(e^{2\pi i t})\,dt \\ &= \frac{1}{\deg \alpha}\left(\sum_p \sum_{P_\alpha(x)=0} \log^+|x|_p + \sum_{P_\alpha(x)=0}\log^+|x|\right) \\ &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K} n_v \log^+|\alpha|_v. \end{aligned}$$

The first expression is called the (normalized) Mahler measure of P_α. The first sum in the second expression is over all primes p of $\mathbb{Q}$, where we have fixed an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$, with $|x|_p$ the p-adic absolute value satisfying $|p|_p = 1/p$. In the third line, we let K be any number field containing α and M_K the set of all "places" of K (i.e., equivalence classes of nontrivial absolute values, with representatives chosen to coincide with the standard p-adic or archimedean absolute values on $\mathbb{Q}$), so that

$$\prod_{v\in M_K} |\alpha|_v^{n_v} = 1$$

for every $\alpha \in K^*$, where $n_v = [K_v : \mathbb{Q}_v]$.

Note that, over $\mathbb{C}$, the function $\log^+|\cdot|$ satisfies

$$dd^c \log^+|z| = m_{S^1}$$

the Lebesgue measure on the unit circle, normalized to have total mass 1. Over $\mathbb{C}_p$ (a smallest complete and algebraically closed field containing $\mathbb{Q}_p$), the function $z \mapsto \log^+|z|_p$ is locally constant, so it should be harmonic by any reasonable definition. In fact, the function extends naturally to the Berkovich analytification $\mathbb{A}_p^{1,an}$ over this field $\mathbb{C}_p$, and there it becomes a potential for δ_G, the delta measure supported on the Gauss point (which plays the role of a non-archimedean "boundary" to the closed unit disk in $\mathbb{C}_p$). In particular, we can think of the Weil height function h as associated to the collection of probability measures $\{m_{S^1}\} \cup \{\delta_{G,p}\}_p$, one for each place of $\mathbb{Q}$, with the potential functions $\log^+|\cdot|_p$ on the spaces $\mathbb{A}_p^{1,an}$. We could replace $\log^+|\cdot|$ with other choices of subharmonic functions at finitely many places and define "reasonable" height functions on $\mathbb{P}^1(\overline{\mathbb{Q}})$.

Now suppose $\alpha = (\alpha_0 : \cdots : \alpha_N) \in \mathbb{P}^N(K)$ for a number field K. Set

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \log \max\{|\alpha_0|_v, \ldots, |\alpha_N|_v\}.$$

The value of h is independent of the choice of presentation of α (as long as the coordinates lie in K) by the product formula, and it is independent of the choice of K so extends to $\mathbb{P}^N(\overline{\mathbb{Q}})$.

Note that, with the norm

$$\|(z_0, \ldots, z_N)\| = \max\{|z_0|, \ldots, |z_N|\}$$

on $\mathbb{C}^{N+1}$, the function $U(z) = \log \|z\|$ is plurisubharmonic so that $dd^c U$ induces a closed and positive $(1,1)$-current on $\mathbb{P}^N(\mathbb{C})$. This U is called a local height of h at the archimedean place. (Or perhaps more precisely, we choose a locally-defined section σ of the projection $\mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$, and declare $U \circ \sigma$ to be a local height.) Replacing this norm with the standard Euclidean norm at the archimedean place defines what is called the Arakelov height (instead of the "naive" or "standard" height) on $\mathbb{P}^N(\overline{\mathbb{Q}})$.

12.2. **Call-Silverman canonical height on $\mathbb{P}^N$.** [CS]

Proposition 12.1. *Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over K, of degree $d > 1$. There exist constants $C_1, C_2 > 0$ (depending on f) so that*

$$d\, h(\alpha) - C_1 \le h(f(\alpha)) \le d\, h(\alpha) + C_2$$

for all $\alpha \in \mathbb{P}^N(\overline{\mathbb{Q}})$.

See [Si2, Theorem 3.11] for proof details. We choose a homogeneous presentation of $F = (f_0, \ldots, f_N)$ defined over K. The first observation is that, just as we saw over $\mathbb{C}$ in §6.2, there is a constant $C_v > 1$ so that

$$C_v^{-1} \|\alpha\|_v^d \le \|F(\alpha)\|_v \le C_v \|\alpha\|_v^d$$

for all $\alpha \in K^{N+1}$, at each place v of K. But to obtain the result of the proposition, we need to know that the $\log C_v$ are summable over all v. In fact, we have $C_v = 1$ for all but finitely many v. The upper bound comes from simple triangle inequalities, noting that the coefficients of F have absolute value 1 at all but finitely many places. The proof of the lower bound comes from observing that there is a degree $e > 0$ so that the monomials X_j^e lie in the ideal $(f_0(X_0, \ldots, X_N), \ldots, f_N(X_0, \ldots, X_N))$ for each j. In other words, there are polynomials defined over K so that

$$X_j^e = \sum_k g_{j,k} f_k.$$

The coefficients of the f_j's and the $g_{j,k}$'s will have absolute value 1 at all but finitely many places, and the (non-archimedean) triangle inequality gives the desired result.

It follows from Proposition 12.1 that

$$\hat{h}_f(\alpha) := \lim_{n\to\infty} \frac{1}{d^n} h(f^n(\alpha)) \tag{12.1}$$

is well defined on $\mathbb{P}^N(\overline{\mathbb{Q}})$ and there exists a constant C (depending on f) so that

$$|\hat{h}_f(\alpha) - h(\alpha)| \leq C \tag{12.2}$$

for all $\alpha \in \mathbb{P}^N(\overline{\mathbb{Q}})$. In fact, the proof of Proposition 12.1 implies that the limit of the local heights also exists on $\mathbb{A}^{N+1}$ for each place v of K, so we have

$$\hat{h}_f(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K} n_v G_{F,v}(\tilde{\alpha}) \tag{12.3}$$

for escape-rate functions $G_{F,v}$ on $\mathbb{A}^{N+1}$ and any choice of lift $\tilde{\alpha} \in \mathbb{A}^{N+1}(K)$ of α.

Example 12.2. For $f(z) = (z_0^d, \ldots, z_N^d)$, we have $\hat{h}_f = h$.

12.3. Preperiodic points.

Theorem 12.3. *Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism defined over a number field K. Then $\hat{h}_f(\alpha) = 0$ if and only if α is preperiodic.*

Proof. This follows from the definition above of $\hat{h}_f$ and was observed by Call and Silverman [CS, Theorem 1.1], as a consequence of the Northcott property of the Weil height: for every $D, B > 0$, the set

$$\{\alpha \in \mathbb{P}^N(\overline{\mathbb{Q}}) : \deg \alpha \leq D,\ h(\alpha) \leq B\} \tag{12.4}$$

is finite. In fact, Northcott proved this property for the purpose of showing the set of rational preperiodic points of f is finite [**?**].

To the proof: one implication is immediate. If α has a finite orbit, then the sequence of height values $\{h(f^n(\alpha))\}_n$ is bounded, so $\hat{h}_f(\alpha) = 0$. On the other hand, if $\hat{h}_f(\alpha) = 0$, then $\hat{h}_f(f^n(\alpha)) = 0$ for all n, so that (12.2) implies that the sequence $\{f^n(\alpha)\}$ has bounded height. But there is a uniform bound on the degree of these points over $\mathbb{Q}$, so Northcott implies that the sequence is finite. □

13. Adelically metrized line bundles

13.1. Line bundles on $\mathbb{P}^N$. Recall that, up to isomorphism, the line bundles on $\mathbb{P}^N$ are $\mathcal{O}(m)$ for $m \in \mathbb{Z}$, defined by transition functions $\varphi_{m,ij}(z_0 : \cdots : z_N) = \left(\frac{z_i}{z_j}\right)^m$ on charts $U_i = \{z_i \neq 0\}$.

Recall: a line bundle $\mathcal{L}$ on complex manifold X is given by local biholomorphic charts $\varphi_U : \mathcal{L}|_U \to U \times \mathbb{C}$ and a collection of holomorphic transition functions $\varphi_{UV} : U \cap V \to \mathbb{C}^*$ satisfying $\varphi_V \circ \varphi_U^{-1}(z, \zeta) = (z, \varphi_{UV} \cdot \zeta)$. The tensor product $\mathcal{L} \otimes \mathcal{M}$ will have transition functions given by the product $\varphi_{UV}^{\mathcal{L}} \varphi_{UV}^{\mathcal{M}}$.

Example 13.1. $\mathcal{O}(-1)$ is our tautological $\tau : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ with the zeros filled in, or rather, $\mathbb{C}^{N+1}$ blown up at the origin. Indeed, over U_i we can work in coordinates

$$\zeta_i \cdot \left(\frac{z_0}{z_i}, \dots, 1, \dots, \frac{z_N}{z_i}\right)$$

on $\mathbb{C}^{N+1}$, for $\zeta_i \in \mathbb{C}$, and note that $\zeta_j = \varphi_{-1,ij}(z)\zeta_i = \frac{z_j}{z_i}\zeta_i$.

Recall: a morphism $f : X \to Y$ can naturally pull back a line bundle $\mathcal{L}$ from Y to X by defining new transition functions $\varphi^X_{UV}(x) = \varphi^Y_{UV}(f(x))$ on the intersections of charts from $f^{-1}(U)$ to $f^{-1}(V)$.

Proposition 13.2. *If $f : \mathbb{P}^N \to \mathbb{P}^N$ is a morphism of degree d, then $f^*\mathcal{O}(m) \simeq \mathcal{O}(dm)$.*

The isomorphism is not canonical.

13.2. **Hermitian metrics.** Recall: a Hermitian metric on a line bundle $\mathcal{L}$ over X is given by a collection of continuous/smooth functions $\{h_U : U \to \mathbb{R}_{>0}\}_U$ such that

$$h_U(z) = h_V(z)\,|\varphi_{UV}(z)|$$

on the overlaps. The magnitude of a vector $\zeta \in \mathbb{C}$ over a point z in this metric is given by

$$\|(z, \zeta)\|_h = h_U(z)\,|\zeta|$$

in the local coordinates over U.

Example 13.3. On $\mathcal{O}(-1)$, consider the metric defined by

$$h_i(z) = \max\left\{\left|\frac{z_0}{z_i}\right|, \dots, \left|\frac{z_N}{z_i}\right|\right\}$$

over $U_i = \{z_i \neq 0\}$, with the usual transition functions $\varphi_{-1,ij} = z_j/z_i$. This is the metric induced by the max norm

$$\|(z_0, \dots, z_N)\|_{\max} = \max\{|z_0|, \dots, |z_N|\}$$

on $\mathbb{C}^{N+1}$. Indeed, using the coordinates from Example 13.1, we see that

$$\|(z, \zeta_i)\|_h = \max\left\{\left|\frac{z_0}{z_i}\right|, \dots, \left|\frac{z_N}{z_i}\right|\right\} \cdot |\zeta_i| = \left\|\zeta_i \cdot \left(\frac{z_0}{z_i}, \dots, 1, \dots, \frac{z_N}{z_i}\right)\right\|_{\max}$$

If $f : X \to Y$ is holomorphic, then we can pull back a Hermitian metric on $\mathcal{L}$ over Y to a metric on $f^*\mathcal{L}$ over X. Indeed, we set

$$h_{f^{-1}U}(x) = h_U(f(x)).$$

Note that a metric on $\mathcal{O}(-1)$ on $\mathbb{P}^N$ (or on any $\mathcal{O}(m)$ with $m \neq 0$) induces metrics on all the $\mathcal{O}(m)$ (though not canonically). Given $\{h_i\}$ for $\mathcal{O}(-1)$, we can set

$$h_{m,i} = h_i^{-m}.$$

Note that

$$\frac{h_{m,i}}{h_{m,j}} = \left(\frac{h_j}{h_i}\right)^m = \left|\frac{z_i}{z_j}\right|^m = |\varphi_{m,ij}|$$

on $U_i \cap U_j$. Given a metric $\|\cdot\|$ on $\mathcal{O}(m)$, we denote this induced metric on $\mathcal{O}(m')$ by

$$\|(z,\zeta)\|^{m'/m}.$$

Example 13.4. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d > 1$, and let $F : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ be a homogeneous lift. On $\mathcal{O}(-1)$, we can define a continuous Hermitian metric by

$$h_i(z) = \exp G_F(z_0/z_i, \ldots, z_N/z_i)$$

over $U_i = \{z_i \neq 0\}$, with the usual transition functions $\varphi_{-1,ij} = z_i/z_j$. This is a metric induced by the escape rate function, where $\{G_F \leq 0\}$ defines the closed unit ball. The induced metric on $\mathcal{O}(1)$ is given by

$$\|(z,\zeta)\|_F = \exp(-G_F(z_0/z_i, \ldots, z_N/z_i))\,|\zeta|$$

over U_i.

The following is now an immediate consequence of the definitions, similar to the proof in §6.2, where we showed that G_F is the limit of the functions G_n, independent of the norm we started with:

Proposition 13.5. *Let $\|\cdot\|$ be the Hermitian metric on $\mathcal{O}(1)$ induced by the max norm on $\mathcal{O}(-1)$. Fixing an isomorphism $f^*\mathcal{O}(1) \simeq \mathcal{O}(d)$, the dynamical metric on $\mathcal{O}(1)$ of Example 13.4 coincides with the limit*

$$\|(z,\zeta)\|_F = \lim_{n\to\infty} \|(f^n)^*(z,\zeta)\|^{1/d^n}$$

for some choice of homogeneous lift F. Up to scale, this is the unique continuous metric on $\mathcal{O}(1)$ satisfying

$$(\eta^* f^* \|\cdot\|_F)^{1/d} = \|\cdot\|_F$$

where η is the isomorphism from $\mathcal{O}(d)$ to $f^*\mathcal{O}(1)$.

The choice of η corresponds to a choice of F; it is determined up to scale. For the uniqueness, note that if $\|\cdot\|$ is any other metric satisfying this relation, then $R(z) = \|\cdot\|/\|\cdot\|_F$ defines a continuous and positive *function* on $\mathbb{P}^N$ for which $R \circ f = f^*R = f^*\|\cdot\|/f^*\|\cdot\|_F = \eta^*f^*\|\cdot\|/\eta^*f^*\|\cdot\|_F = R^d$. But this forces $R = 1$. See [Zh, Theorem 2.2].

13.3. **Adelically.** One repeats the same constructions at each place of the given number field K. Suppose X is a smooth projective algebraic variety defined over a number field K and $\mathcal{L}$ is an ample line bundle on X. One works with a collection of v-adic metrics, one for each place of K, defined on a family of "completed" line bundles $\mathcal{L}_v|U \simeq U \times \overline{K}_v$ in charts. We will say that $\overline{\mathcal{L}} = (\mathcal{L}, \{\|\cdot\|_v\}_{v\in M_K}$ defines an *adelically metrized line bundle* if each metric is continuous and all but finitely many of the metrics $\|\cdot\|_v$ agree with the standard one $|\cdot\|_{\max,v}$ coming from the max norm, and so that so that the ratios $\|\cdot\|_v/\|\cdot\|_{\max,v}$ are bounded for all places v. (For the "correct" definition, we need to impose additional conditions on the metrics in order to work with them; for example, we want them to extend to the Berkovich analytification at each place, but we need not go there in this course.)

13.4. **What is a height function?** Working over a number field K, we will say a function $h : \mathbb{P}^N(\overline{K}) \to \mathbb{R}$ is a *height function* for $O(1)$ if it is given by

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K}\frac{1}{[K':K]}\sum_{K'\hookrightarrow\overline{K}_v} n_v(-\log\|\sigma(\alpha)\|_v)$$

for any number field K' over which α is defined, global section σ defined over K which is non-vanishing at α, and for an adelically metrized line bundle $\overline{\mathcal{L}} = (\mathcal{O}(1), \{\|\cdot\|_v\}_{v\in M_K}$.

Example 13.6. For the standard metric $\|\cdot\|_{\max,v}$, we recover the usual height function. Fix $\alpha = (\alpha_0 : \cdots : \alpha_N) \in \mathbb{P}^N(K')$ for a finite extension K' of K, and assume that $\alpha_i \neq 0$. Choose section $\sigma(z_0 : \cdots : z_N) = z_i$. Then

$$\begin{aligned}
h(\alpha) &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K}\frac{1}{[K':K]}\sum_{K'\hookrightarrow\overline{K}_v} n_v(-\log\|\sigma(\alpha)\|_v)\\
&= \frac{1}{[K':\mathbb{Q}]}\sum_{v'\in M_{K'}} n_{v'}(-\log\|\sigma(\alpha)\|_{\max,v'})\\
&= \frac{1}{[K':\mathbb{Q}]}\sum_{v'\in M_{K'}} n_{v'}(-\log\left(\max\{|\alpha_0/\alpha_0|_{v'},\ldots,|\alpha_N/\alpha_i|_{v'}\}^{-1}|\alpha_i|_{v'})\right)\\
&= \frac{1}{[K':\mathbb{Q}]}\left(\sum_{v'\in M_{K'}} n_{v'}\log\max\{|\alpha_0/\alpha_i|_{v'},\ldots,|\alpha_N/\alpha_i|_{v'}\} - \sum_{v'}\log|\alpha_i|_{v'}\right)
\end{aligned}$$

and the last sum vanishes by the product formula.

For Weil, to be a height function meant to be a real-valued function that differs by a bounded amount from the standard h defined above. But it is more convenient to work with functions that have additional structure, to view heights as sums of their local parts and to assume that these local contributions are continuous.

14. Arithmetic intersections, heights of subvarieties

Here I introduce the notion of arithmetic intersection numbers, though without details, to illustrate how the wedge of positive currents plays a role. Then we consider the geometric setting, working over function fields in characteristic 0, where the heights can be computed differently.

14.1. **Curvature.** Suppose that $\|\cdot\|$ is a continuous metric on an ample line bundle $\mathcal{L}$ on a smooth projective variety X over $\mathbb{C}$. We set

$$c_1(\mathcal{L}, \|\cdot\|) := dd^c(-\log\|\sigma\|)$$

where σ is a (locally-defined) holomorphic section of $\mathcal{L}$. We say that a metrized line bundle is *semipositive* if $c_1(\overline{\mathcal{L}})_\infty$ is a positive $(1,1)$-current on X.

Example 14.1. Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d > 1$ over $\mathbb{C}$. Consider the metric $\|\cdot\|_F$ on $\mathcal{O}(1)$ over $\mathbb{P}^N$ from Example 13.4. We have

$$c_1(\mathcal{O}(1), \|\cdot\|_F) = dd^c(G_F \circ \sigma) = T_f$$

for any locally-defined non-vanishing holomorphic section σ. This metric is semipositive.

For an adelically metrized line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \{\|\cdot\|_v\}_{v\in M_K}$, there is an analogous notion at each place of a number field K, but it requires working over the Berkovich analytification, which we will not do that in this course. We will let $c_1(\overline{\mathcal{L}})_\infty$ denote the curvature distribution at an archimedean place.

14.2. **Arithmetic intersection numbers.** Arakelov intersection theory, Gillet-Soulé, Zhang, ... For the perspective discussed here, see [CL].

For simplicity, we work with $\mathcal{O}(1)$ on $\mathbb{P}^N$. Suppose we have $N+1$ adelic metrics on $\mathcal{L} = \mathcal{O}(1)$; we denote these by $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_N$. Assume that each is continuous and semipositive. Fix global sections $\sigma_0, \ldots, \sigma_N$ that there are no points where all vanish. For example, we can take $\sigma_j = z_j$.

Fix an archimedean place v, which we will denote by ∞. We inductively define a local intersection number by

$$\begin{aligned}\left(\overline{\mathcal{L}}_0 \cdot \ldots \cdot \overline{\mathcal{L}}_N, \sigma_0, \ldots, \sigma_N\right)_\infty &= \int_{\mathbb{P}^N(\mathbb{C})} (-\log\|\sigma_0\|)\, c_1(\overline{\mathcal{L}}_1)_\infty \wedge \cdots \wedge c_1(\overline{\mathcal{L}}_N)_\infty \\ &\quad + \left(\overline{\mathcal{L}}_1|_{(\sigma_0)} \cdot \ldots \cdot \overline{\mathcal{L}}_N|_{(\sigma_0)}, \sigma_1, \ldots, \sigma_N\right)_\infty,\end{aligned}$$

where $\overline{\mathcal{L}}|_{(\sigma_0)}$ means that we restrict the line bundle to the support of the divisor of the section (and remember multiplicities). And we set

$$(\overline{\mathcal{L}}|_x, \sigma)_\infty = -\log\|\sigma(x)\|_\infty$$

at a point x. The semipositivity of the $\overline{\mathcal{L}}_i$ guarantees that the integrals and measures are well defined.

There is an analogous definition at each place of the number field – which I will not get into here – and the arithmetic intersection number is then the sum

$$\overline{\mathcal{L}}_0 \cdot \ldots \cdot \overline{\mathcal{L}}_N = \sum_{v \in M_K} n_v \left(\overline{\mathcal{L}}_0 \cdot \ldots \cdot \overline{\mathcal{L}}_N, \sigma_0, \ldots, \sigma_N \right)_v .$$

The local intersection numbers depend on the choices of sections, but the global definition will not.

14.3. **Heights of subvarieties.** Suppose V is an algebraic subvariety of $\mathbb{P}^N$ defined over a number field K, L is a very ample line bundle on V. Let $\overline{\mathcal{L}}$ be an adelic metric on L, and suppose that $h_{\overline{\mathcal{L}}}$ is the associated height function. $\dim V = m$, then we choose sections $s_0, \ldots, s_m$ of L defined over K with $\operatorname{supp}(s_0) \cap \cdots \cap \operatorname{supp}(s_m) = \emptyset$ and define *the height of* V by

$$h_{\overline{\mathcal{L}}}(V) = \overline{\mathcal{L}}^{m+1}$$

The value of $h_{\overline{\mathcal{L}}}(V)$ is rather mysterious in practice.

Many people have worked (or are working) on research problems centered around understanding the heights of subvarieties.

14.4. **Function field of a curve.** The height machinery described above all makes sense for a general *product formula field* K, where there exists a collection of absolute values indexed by a set M_K and some positive integer weights n_v for which

$$\prod_{v \in M_K} |\alpha|_v^{n_v} = 1$$

for all nonzero α in K. Helpful background can be found in [BG].

Suppose that $K = \mathbb{C}(S)$ is the function field of a smooth projective curve S defined over $\mathbb{C}$. Then an element $\alpha \in K$ is, by definition, a meromorphic function $S \to \mathbb{P}^1$. For each point $s \in S(\mathbb{C})$, we can define an absolute value

$$|\alpha|_s := e^{-\operatorname{ord}_s \alpha}.$$

Letting M_K be the set of all points in S, these satisfy the product formula (with weights $n_s = 1$ for all $s \in S(\mathbb{C})$) because any meromorphic function has the same number of zeroes as poles.

A point $\alpha \in \mathbb{P}^N(K)$ is a tuple $(\alpha_0 : \cdots : \alpha_N)$ and thus defines – on the complement S' of finitely many points of $\mathbb{C}$ – a holomorphic map $\alpha : S' \to \mathbb{P}^N_{\mathbb{C}}$. But by clearing poles/zeroes in a neighborhood of a point, we see that it extends to a holomorphic map $\alpha : S \to \mathbb{P}^N$.

Suppose $N = 1$. Fix $\alpha = (\alpha_0 : \alpha_1)$. We identify α with the map $\alpha_1/\alpha_0 : S \to \mathbb{P}^1$. Then

$$\begin{aligned} h(\alpha) &= \sum_{s \in S(\mathbb{C})} \log\max\{|\alpha_0|_s, |\alpha_1|_s\} \\ &= -\sum_s \min\{0, \operatorname{ord}_s \alpha\} = \deg(\alpha : S \to \mathbb{P}^1) \\ &= \int_S \alpha^*\omega \end{aligned}$$

for the Fubini-Study form ω on $\mathbb{P}^1$ satisfying $\int \omega = 1$. For general N and $\alpha = (\alpha_0 : \cdots : \alpha_N)$ with $\alpha_0 \neq 0$, we have

$$\begin{aligned} h(\alpha) &= \sum_s \log\max\{|\alpha_0|_s, \ldots, |\alpha_N|_s\} \\ &= \sum_s \log\max\{1, |\alpha_1/\alpha_0|_s, \ldots, |\alpha_N/\alpha_0|_s\} \\ &= -\sum_s \min\{0, \operatorname{ord}_s(\alpha_1/\alpha_0), \ldots, \operatorname{ord}_s(\alpha_N/\alpha_0)\} \\ &= \#\text{ intersections of } \alpha(S) \text{ with } \{z_0 = 0\} \text{ in } \mathbb{P}^N_{\mathbb{C}} \\ &= \deg \alpha^*\mathcal{O}(1) \\ &= \int_S \alpha^*\omega \end{aligned} \tag{14.1}$$

The heights of subvarieties $V \subset \mathbb{P}^N$ defined over the function field $K = \mathbb{C}(S)$ can be computed geometrically. Note that V can be "spread out" over S as a subvariety of $\mathcal{V} \subset S \times \mathbb{P}^N$ defined over $\mathbb{C}$; the equations defining V have coefficients in K and so define a *family* of subvarieties $V_s \subset \mathbb{P}^N_{\mathbb{C}}$ as s varies in S. A priori, the subvariety V_s is only defined for s outside of a finite set in S, but we let $\mathcal{V}$ denote its Zariski closure in $S \times \mathbb{P}^N_{\mathbb{C}}$. If V has dimension ℓ then $\mathcal{V}$ has dimension $\ell + 1$ over $\mathbb{C}$. So the current of integration $[\mathcal{V}]$ is a well-defined positive $(N - \ell, N - \ell)$-current on $S \times \mathbb{P}^N$. For the standard Weil height on $\mathbb{P}^N$ on $L = \mathcal{O}(1)$ over this field K, we can set

$$h(V) = \langle [\mathcal{V}], \hat{\omega}^{\ell+1} \rangle = \int_{\mathcal{V}} \hat{\omega}^{\ell+1} \tag{14.2}$$

where $\hat{\omega} = p^*\omega$ for the projection $p : S \times \mathbb{P}^N \to \mathbb{P}^N$, defined over $\mathbb{C}$. See Gubler [**?**] for a comparison of this definition to the definition suggested in §14.3. Note that this definition of $h(V)$ agrees with the formula (14.1) when V is 0-dimensional.

15. Function field heights and complex dynamics

Let S be a smooth projective curve over $\mathbb{C}$. In this section and the next, we prove recent theorems of Gauthier and Vigny about the canonical height over fields such as $K = \mathbb{C}(S)$ [GV]. You will see that all of the analytic machinery we developed this semester will play a role in the proofs.

15.1. **Dynamics over K.** Suppose that $f : \mathbb{P}^N \to \mathbb{P}^N$ is a morphisms of degree $d > 1$ defined over the field $K = \mathbb{C}(S)$. By evaluating the coefficients of f (as elements of K) at points $s \in S(\mathbb{C})$, we obtain a holomorphic family of maps

$$\hat{f} : \Lambda \times \mathbb{P}^N_{\mathbb{C}} \to \Lambda \times \mathbb{P}^N_{\mathbb{C}}$$

for $\Lambda \subset S$ the complement of finitely many points, in the sense of §11.2. Let $\hat{T}_f$ be the dynamical Green current on $\Lambda \times \mathbb{P}^N$ for this family, as defined in §11.2.

Over K, recall that the canonical height of a point $\alpha \in \mathbb{P}^N(K)$ is defined by

$$\hat{h}_f(\alpha) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(\alpha)).$$

The limit exists by the same argument that went into the proof of Proposition 12.1, working at each place of the field K; note that all places are non-archimedean. Therefore

$$\hat{h}_f(\alpha) = \lim_{n\to\infty} \frac{1}{d^n} \deg_S (f^n\alpha)^* \mathcal{O}(1)$$

by (14.1), when viewing each iterate $f^n(\alpha)$ as a holomorphic map from S to $\mathbb{P}^N_{\mathbb{C}}$. See my lecture notes [De4] for additional background.

15.2. **Integral formula for canonical height.** Suppose that V is a subvariety of $\mathbb{P}^N$ defined over K over dimension ℓ. We can define

$$(15.1)\qquad \begin{aligned} \hat{h}_f(V) &= \lim_{n\to\infty} \frac{1}{d^{n(\ell+1)}} h((f^n)_* V) \\ &= \lim_{n\to\infty} \frac{1}{d^{n(\ell+1)}} \langle (\hat{f}^n)_*[\mathcal{V}], \hat{\omega}^{\ell+1}\rangle \end{aligned}$$

if this limit exits.

Theorem 15.1. [GV, Theorem B] *For any subvariety V of dimension ℓ in $\mathbb{P}^N$ defined over K, we have*

$$\hat{h}_f(V) = \int_{\Lambda\times\mathbb{P}^N_{\mathbb{C}}} (\hat{T}_f)^{\wedge(\ell+1)} \wedge [\mathcal{V}] < \infty,$$

where $\mathcal{V}$ is the associated $(\ell+1)$-dimensional subvariety of $S \times \mathbb{P}^N$.

To prove this, we should – at least in principle – be able to replace $(\hat{f}^n)_*[\mathcal{V}]$ in the definition of $\hat{h}_f(V)$ with $[\mathcal{V}]$ paired with $(\hat{f}^n)^*\hat{\omega}^{\ell+1}$, but note that $\hat{f}$ is a not a morphism on the compact space $S \times \mathbb{P}^N_{\mathbb{C}}$. Working on the open subset $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$, the smooth form $\hat{\omega}$ is no longer compactly supported. But assuming this push-pull relation still holds, the next challenge is to interchange the integral sign with the limit. Note that the statement of Theorem 15.1 isn't obvious even for $\ell = 0$ and $N = 1$. We devote the rest of the section to proving Theorem 15.1.

15.3. **Bifurcation-like currents.** Let us first explore properties of the current

$$\hat{T}^{\wedge(\ell+1)} \wedge [\mathcal{V}]$$

on the (quasiprojective) complex algebraic variety $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$. In the special case where V is the critical locus of a morphism f defined over $K = \mathbb{C}(S)$, then this current was used to define the bifurcation current T_{bif}; see §11.2. The following proposition will imply, in particular, that dynamical stability is characterized by volume growth of $\hat{f}^n(\text{Crit}(\hat{f}))$ as $n \to \infty$.

Recall from §8.1 that the mass of a closed and positive (p,p)-current on a set K in a complex Kähler manifold X of dimension N is defined by

$$\|T\|_K := \int_K T \wedge \beta^{N-p}$$

for a choice of Kähler form β.

Proposition 15.2. [GV, Proposition 10] *Let T be a closed and positive (p,p)-current on $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$ with $1 \leq p \leq N$. Then the following are equivalent:*

(1) *the measure $\hat{T}^{\wedge(N+1-p)} \wedge T = 0$ on $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$;*
(2) *for all compact sets K in $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$, we have*

$$\|(\hat{f}^n)_* T\|_K = o(d^{n(N+1-p)});$$

as $n \to \infty$; and

(3) *for all compact sets K in $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$, we have*

$$\|(\hat{f}^n)_* T\|_K = O(d^{n(N-p)})$$

as $n \to \infty$.

Proof. Let $\pi : \Lambda \times \mathbb{P}^N_{\mathbb{C}} \to \Lambda$ denote the projection. Note that $\pi \circ \hat{f} = \pi$. Let

$$\beta = \hat{\omega} + \pi^* \beta_S$$

for a smooth and positve volume form β_S on S. We will work with mass norm

$$\|T\|_K := \int_K T \wedge \beta^{N+1-p}.$$

Fix any compact set K_1 in Λ and let U_1 be an open neighborhood of K_1 contained in a compact set K_2. Choose a smooth bump function $\varphi \geq 0$ on Λ so that $\varphi|_{K_1} \equiv 1$

and $\operatorname{supp}\varphi \subset U_1$.

$$\begin{aligned}
\|(\hat{f}^n)_* T\|_{\pi^{-1}(K_1)} &= \int_{\pi^{-1}(K_1)} (\hat{f}^n)_* T \wedge (\hat{\omega} + \pi^*\beta_S)^{(N+1-p)} \\
&\leq \int_{\pi^{-1}(U_1)} (\varphi\circ\pi)\,(\hat{f}^n)_* T \wedge (\hat{\omega} + \pi^*\beta_S)^{(N+1-p)} \qquad (15.2)\\
&= \int_{\pi^{-1}(U_1)} (\hat{f}^n)^*(\varphi\circ\pi)\,T \wedge (\hat{f}^n)^*(\hat{\omega} + \pi^*\beta_S)^{(N+1-p)} \\
&= \int_{\pi^{-1}(U_1)} (\varphi\circ\pi)\,T \wedge ((\hat{f}^n)^*\hat{\omega} + \pi^*\beta_S)^{(N+1-p)} \\
&= \int_{\pi^{-1}(U_1)} (\varphi\circ\pi)\,T \wedge \left((\hat{f}^n)^*\hat{\omega}\right)^{(N+1-p)} \\
&\quad + (N+1-p)\int_{\pi^{-1}(U_1)} (\varphi\circ\pi)\,T \wedge \left((\hat{f}^n)^*\hat{\omega}\right)^{(N-p)} \wedge \pi^*\beta_S
\end{aligned}$$

Choose a continuous potential g_0 on $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$ so that

$$\hat{T} - \hat{\omega} = dd^c g_0.$$

Note that

$$dd^c(g_0 \circ \hat{f}^n) = (\hat{f}^n)^*(\hat{T} - \hat{\omega}) = d^n\,\hat{T} - (\hat{f}^n)^*\hat{\omega}$$

Now we substitute $d^n\,\hat{T} - dd^c(g_0\circ\hat{f}^n)$ for $(\hat{f}^n)^*\hat{\omega}$ in the two integrals above, and then we expand the wedge powers, giving

$$\|(\hat{f}^n)_* T\|_{\pi^{-1}(K_1)} \leq d^{n(N+1-p)} \int_{\pi^{-1}(U_1)} (\varphi\circ\pi)\,T \wedge \hat{T}^{N+1-p} + d^{n(N-p)}\Big(\cdots\Big).$$

We then apply Chern-Levine-Nirenberg Theorem 8.3 to each integral in the $(\cdots)$ expansion. More precisely, we cover $\pi^{-1}(K_1)$ by finitely many polydisk domains where $\hat{T}$ has bounded potentials, and we apply Theorem 8.3 there. We deduce that

$$\|(\hat{f}^n)_* T\|_{\pi^{-1}(K_1)} \leq d^{n(N+1-p)} \int_{\pi^{-1}(U_1)} (\varphi\circ\pi)\,T \wedge \hat{T}^{N+1-p} + \ O\left(d^{n(N-p)}\right)\ \|T\|_{\pi^{-1}(K_2)}.$$

for constants depending on K_1, K_2, $\hat{f}$ and g_0.

Now assume (1). Any compact K is contained in a compact of the form $\pi^{-1}(K_1)$, so (2) and (3) follow immediately. For the converse implication, we assume there is point x in the support of $\hat{T}^{\wedge(N+1-p)} \wedge T$. Then by positivity, we have

$$\int_{\pi^{-1}(U_1)} (\varphi\circ\pi)\,T \wedge \hat{T}^{N+1-p} > 0$$

for any small U_1 containing $\pi(x)$. Choose compact set K containing $\pi^{-1}(U_1)$, and note that $\|(\hat{f}^n)_* T\|_K$ must grow at least as fast as $d^{n(N+1-p)} \int_{\pi^{-1}(U_1)} (\varphi\circ\pi)\,T \wedge \hat{T}^{N+1-p}$.

(Note that the inequality on $\|(\hat{f}^n)_*T\|_{\pi^{-1}(K_1)}$ in (15.2) came only when enlarging the set, so passing to a yet larger set can only increase the integral further.) $\square$

15.4. **Global estimates.** To prove Theorem 15.1, we need to assert some global control over mass norms of the form $\|(\hat{f}^n)_*T\|_K$ when $T = [\mathcal{V}]$, in the sense that we would like to understand the growth of these integrals over all of $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$.

Lemma 15.3. [GV, Lemma 14] *For any nonconstant and proper morphism $\iota : \Lambda \to \mathbb{C}^M$ for $M \geq 1$, there exists a continuous potential g_0 for $\hat{T} - \hat{\omega}$ on $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$, and constants $C, C' > 0$ so that*

$$\left| g_0|_{\{\lambda\}\times\mathbb{P}^N_{\mathbb{C}}} \right| \leq C \log \|\iota(\lambda)\| + C'$$

for all $\lambda \in \Lambda$, where $\|\cdot\|$ is the standard norm on $\mathbb{C}^M$. Moreover, we have

$$dd^c \left(\frac{1}{d^n} \left(g_0 \circ \hat{f}^n \right) \right) = \hat{T} - \frac{1}{d^n} (\hat{f}^n)^* \hat{\omega}$$

and

$$\left| \frac{1}{d^n} \left(g_0 \circ \hat{f}^n \right) \Big|_{\{\lambda\}\times\mathbb{P}^N_{\mathbb{C}}} \right| \leq \frac{C}{d^n} \log \|\iota(\lambda)\| + \frac{C'}{d^n}$$

for all $n \geq 1$.

Proof. Fix a presentation $f = (f_0 : \cdots : f_N)$ of f with coefficients in $K = \mathbb{C}(S)$. Let E be the finite set of points in S where the specialization f_λ fails to be a morphism of $\mathbb{P}^N_{\mathbb{C}}$, and set $\Lambda = S \setminus E$. In particular, the list of coefficients of f detemines a morphism $\psi : \Lambda \to \mathbb{C}^M$ for some large M, providing a presentation F_λ in homogeneous coordinates for each $\lambda \in \Lambda$. Let $\rho : \Lambda \to \mathbb{C}$ be a proper holomorphic function (for example we can choose any meromorphic function ρ on S with pole set equal to $S \setminus \Lambda$). Set

$$\iota = \rho^m \psi$$

for some choice of integer $m > 0$, so that $\iota : \Lambda \to \mathbb{C}^M$ is a proper holomorphic map.

Now the proof for this ι follows from a global version of the estimates in §6.2. We fix a norm, for example the max norm, on $\mathbb{C}^{N+1}$. Recall that for each $\lambda \in \Lambda$, there is a constant $C_\lambda > 1$ so that

$$C_\lambda^{-1} \|z\|^d \leq \|F_\lambda(z)\| \leq C_\lambda \|z\|^d$$

for all $z \in \mathbb{C}^{N+1}$. Over $\mathbb{C}$ and for a fixed λ, the existence of C_λ was straightforward: we use the continuity of F_λ to know that $\|F_\lambda\|$ is bounded away from 0 and ∞ when restricted to the unit sphere $\{\|z\| = 1\}$ and then homogeneity gave the result. To determine the dependence on λ, we dig into the proof of Proposition 12.1. Recall that the upper bound on $\|F_\lambda\|$ was a straightforward triangle inequality estimate using the max of the coefficients of F_λ. The lower bound on $\|F_\lambda\|$ appeals to Hilbert's Nullstellensatz, and it can be expressed in terms of the resultant of F_λ, a polynomial expression in the coefficients that vanishes if and only if the f_i have a common zero. It follows that

$$\log C_\lambda = O(\log \|\iota(\lambda)\|).$$

In particular, defining

$$g_1(z) := \frac{1}{d} \log \|F_\lambda(\tilde{z})\| - \log \|\tilde{z}\|$$

for any lift $\tilde{z}$ of z, we see that

$$|g_1| \leq C_1 \log \|\iota(\lambda)\| + C_1'.$$

Iterating $\hat{f}$, we have

$$g_0 := G_{F_\lambda} - \log \|\cdot\| = \sum_{j=0}^{\infty} \frac{1}{d^j}(g_1 \circ \hat{f}^j) \leq C \log \|\iota(\lambda)\| + C'.$$

The final statement of the lemma follows from the invariance $\frac{1}{d}\hat{f}^*\hat{T} = \hat{T}$.

Note that the growth of $\log \|\iota(\lambda)\|$ for any proper ι is comparable to the growth of $\log \|\iota(\lambda)\|$ for the ι chosen above. This completes the proof. □

Proposition 15.4. *Let $\hat{f} : \Lambda \times \mathbb{P}^N \to \Lambda \times \mathbb{P}^N$ be a holomorphic family defined by a morphism f of degree $d > 1$ over the function field $K = \mathbb{C}(S)$. Then*

$$\|\hat{T}\|_{\Lambda \times \mathbb{P}^N} < \infty$$

and

$$\left| \int_{\Lambda \times \mathbb{P}^N} T \wedge \hat{T}^{N+1-p} \right| < \infty$$

for any closed and positive (p,p)-current T on $S \times \mathbb{P}^N_{\mathbb{C}}$ with $1 \leq p \leq N$. Moreover,

$$\int_{\Lambda \times \mathbb{P}^N} (\hat{f}^n)_* T \wedge \hat{\omega}^{N+1-p} \;=\; d^{n(N+1-p)} \int_{\Lambda \times \mathbb{P}^N} T \wedge \hat{T}^{N+1-p} + \; O\left(d^{n(N-p)}\right)$$

as $n \to \infty$.

Proof. We will work with compactly supported cut-off functions in $\Lambda \times \mathbb{P}^N$ that are in the class DSH, meaning differences of subharmonic functions on Λ. For each $A > 0$, we set

$$\varphi_A(\lambda) = \frac{1}{A}\left(\log \max\{\|\iota(\lambda)\|, e^{2A}\} - \log \max\{\|\iota(\lambda)\|, e^{A}\}\right)$$

for the map $\iota : \Lambda \to \mathbb{C}^M$ of Lemma 15.3. So φ_A is $\equiv 1$ on $\iota^{-1}(B(0, e^A))$ with support contained in $\iota^{-1}(B(0, e^{2A}))$. We write

$$dd^c \varphi_A = \nu_A^+ - \nu_A^-$$

on Λ, and note that there is a constant $C > 0$ so that

$$|\nu_A^\pm| \leq C/A$$

for all A.

As in the proof of Proposition 15.2, let $\pi : \Lambda \times \mathbb{P}^N_{\mathbb{C}} \to \Lambda$ denote the projection. Note that $\pi \circ \hat{f} = \pi$. Let

$$\beta = \hat{\omega} + \pi^* \beta_S$$

for a smooth and positive volume form β_S on S. Let g_0 be the potential from Lemma 15.3. We estimate

$$\begin{aligned} \left\langle \hat{T} - \frac{1}{d^n}(\hat{f}^n)^*\hat{\omega},\ (\varphi_A \circ \pi)\,\beta^N \right\rangle &= \frac{1}{d^n}\left\langle dd^c(g_0 \circ \hat{f}^n),\ (\varphi_A \circ \pi)\,\beta^N \right\rangle \\ &= \frac{1}{d^n}\int_{\Lambda \times \mathbb{P}^N} (g_0 \circ \hat{f}^n)\, dd^c(\varphi_A \circ \pi) \wedge \beta^N \\ &= \frac{1}{d^n}\int_{\Lambda \times \mathbb{P}^N} (g_0 \circ \hat{f}^n)\, \pi^*(\nu_A^+ - \nu_A^-) \wedge \beta^N \end{aligned}$$

so that in absolute value this pairing is dominated by

$$\frac{1}{d^n}\frac{C'}{A}\left(\sup_{\iota^{-1}(B(0,e^{2A}))\times \mathbb{P}^N} |g_0| \right)$$

for some constant C', independent of A. But $A \asymp \log \|\iota(\lambda)\|$ on the support of $\nu_A^{\pm}$, so we can make the bound independent of A. We deduce that

$$\left| \int_{\Lambda \times \mathbb{P}^N} \left(\hat{T} - \frac{1}{d^n}(\hat{f}^n)^*\hat{\omega} \right) \wedge \beta^N \right| \leq \frac{1}{d^n} C'' \tag{15.3}$$

for some constant C'' and all $n \geq 0$. In particular, from the $n = 0$ estimate we have

$$\|\hat{T}\|_{\Lambda \times \mathbb{P}^N} = \int_{\Lambda \times \mathbb{P}^N} \hat{T} \wedge \beta^N < \infty, \tag{15.4}$$

since $\int \hat{\omega} \wedge \beta^N$ is finite, proving the first statement of the proposition. It then follows from (15.3) that $\|d^{-n}(\hat{f}^n)^*\hat{\omega}\|_{\Lambda \times \mathbb{P}^N}$ is uniformly bounded.

For $q = N + 1 - p$, note that

$$\hat{T}^q - \left(\frac{1}{d^n}(\hat{f}^n)^*\hat{\omega}\right)^q = \sum_{j=0}^{q-1} dd^c\left(\frac{1}{d^n}(g_0 \circ \hat{f}^n)\right) \wedge \hat{T}^j \wedge \left(\frac{1}{d^n}(\hat{f}^n)^*\hat{\omega}\right)^{q-j-1}$$

for the potential g_0 of Lemma 15.3. Then

$$\frac{1}{d^{nq}}\left\langle (\hat{f}^n)_* T \wedge \left(\hat{T}^q - \hat{\omega}^q\right),\ \varphi_A \circ \pi \right\rangle$$

$$= \left\langle \left(\hat{T}^q - \frac{1}{d^{nq}} (\hat{f}^n)^*(\hat{\omega}^q) \right) \wedge T, \ \varphi_A \circ \pi \right\rangle$$

$$= \frac{1}{d^n} \sum_{j=0}^{q-1} \left\langle dd^c \left(g_0 \circ \hat{f}^n \right) \wedge \hat{T}^j \wedge \left(\frac{1}{d^{n(q-j-1)}} (\hat{f}^n)^* \hat{\omega}^{q-j-1} \right) \wedge T, \ \varphi_A \circ \pi \right\rangle$$

$$= \frac{1}{d^n} \sum_{j=0}^{q-1} \int_{\Lambda \times \mathbb{P}^N} (g_0 \circ \hat{f}^n) \, \hat{T}^j \wedge \left(\frac{1}{d^{n(q-j-1)}} (\hat{f}^n)^* \hat{\omega}^{q-j-1} \right) \wedge T \wedge dd^c(\varphi_A \circ \pi).$$

Recall from (15.4) that the mass norm of $\hat{T}$ is finite and the mass norms $\|d^{-n(q-j-1)}(\hat{f}^n)^*\hat{\omega}^{q-j-1}\|$ are uniformly bounded. Taking absolute values, we use the same estimates as above to bound this by

$$\frac{1}{d^n} \frac{C}{A} \left(\sup_{\iota^{-1}(B(0,e^{2A})) \times \mathbb{P}^N} |g_0| \right) \|T\|_{S \times \mathbb{P}^N} \ \leq \ \frac{C'}{d^n} \|T\|_{S \times \mathbb{P}^N}$$

for some constants $C, C' > 0$ and all n, independent of A. Multiplying by d^{nq} completes the proof of the proposition. □

16. Function field heights and complex dynamics, part II

Let S be a smooth projective curve over $\mathbb{C}$. We continue with the recent theorems of Gauthier and Vigny about the canonical height over fields such as $K = \mathbb{C}(S)$ [GV].

16.1. **Isotriviality.** Suppose that $f : \mathbb{P}^N \to \mathbb{P}^N$ is a morphism of degree $d > 1$ defined over the function field $K = \mathbb{C}(S)$. Let

$$\hat{f} : \Lambda \times \mathbb{P}^N \to \Lambda \times \mathbb{P}^N$$

be the associated holomorphic family of maps on $\mathbb{P}^N$ for $\Lambda \subset S$. We say that f is *isotrivial* if each f_λ is conjugate to a given f_{λ_0} by an automorphism of $\mathbb{P}^N_{\mathbb{C}}$. Or, equivalently, if there exists an automorphism

$$\Phi : \mathbb{P}^N \to \mathbb{P}^N$$

defined over a finite extension K' of K so that $\Phi \circ f \circ \Phi^{-1}$ is defined over the field $\mathbb{C}$ of constants.

16.2. **A Northcott property for geometric height.** The Northcott property (12.4) fails for the standard Weil height on the function field $K = \mathbb{C}(S)$. See details about this height in §14.4; note that all constant functions $\alpha \in \mathbb{C} \subset K$ have height 0. It will also fail for the dynamical canonical height of maps f on $\mathbb{P}^N$ defined over K if the map f is isotrivial, because – after changing coordinates so that the maps f_λ are independent of the parameter λ – all constant points will have constant iterates and thus 0 canonical height.

But importantly, the Northcott property actually holds when working with canonical heights heights on (non-isotrivial) elliptic curves over K; see, for example, [Si1]. And it turns out that versions of the Northcott poperty – for small values of the height – hold for dynamical canonical heights on $\mathbb{P}^N$ for morphisms defined over K. For maps on $\mathbb{P}^1$, the following was proved by Matt Baker in [Ba]:

Theorem 16.1. [Ba] *Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is defined over $K = \mathbb{C}(S)$, of degree $d > 1$ and non-isotrivial. There exists $\varepsilon > 0$ so that the set*

$$\left\{a \in \mathbb{P}^1(K) : \hat{h}_f(a) < \varepsilon\right\}$$

is finite. In particular, $\hat{h}_f(a) = 0$ if and only if a is preperiodic.

See also the work of Chatzidakis and Hrushovski [CH1, CH2].

16.3. **Stability and canonical height.** In [De3], I studied the connection between the vanishing of $\hat{h}_f$ and dynamical stability over $\mathbb{C}$, and I gave a complex-analytic proof of Baker's Theorem 16.1. In particular, the proof showed:

Theorem 16.2. [De3, Theorem 1.1] *Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is defined over $K = \mathbb{C}(S)$, of degree $d > 1$. Fix $a \in \mathbb{P}^1(K)$. The following are equivalent:*

1. *the measure $\hat{T}_f \wedge [\Gamma_a] = 0$ in $\Lambda \times \mathbb{P}^1_{\mathbb{C}}$;*
2. *$\hat{h}_f(a) = 0$; and*
3. *either a is preperiodic for f or the pair (f, a) is isotrivial.*

We say that *the pair (f, a) is isotrivial* if f is isotrivial and the point a is constant (i.e., independent of λ) in the coordinates determined by the automorphism Φ of §16.1.

In [GV], Gauthier and Vigny gave a new proof of Theorem 16.2 and extended it to all dimensions. Note that the equivalence of (1) and (2) is already covered by their integral formula stated above as Theorem 15.1. They showed:

Theorem 16.3. [GV, Theorem A] *Suppose that $f : \mathbb{P}^N \to \mathbb{P}^N$ is defined over $K = \mathbb{C}(S)$, of degree $d > 1$. Fix $a \in \mathbb{P}^N(K)$. The following are equivalent:*

1. *the measure $\hat{T}_f \wedge [\Gamma_a] = 0$ in $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$;*
2. *$\hat{h}_f(a) = 0$; and*
3. *the point a is part of an isotrivial subsystem for f.*

An *isotrivial subsystem* is a proper subvariety $V \subset \mathbb{P}^N$ defined over K so that $f(V) \subset V$ and $f|V$ is isotrivial. That is, over a possibly smaller choice of Λ, and spreading V out to form $\mathcal{V} \subset \Lambda \times \mathbb{P}^N_{\mathbb{C}}$, there exists a family of (complex) isomorphisms

$$\Phi_\lambda : \mathcal{V}_{\lambda_0} \to \mathcal{V}_\lambda$$

so that $\Phi_\lambda \circ f_{\lambda_0} = f_\lambda \circ \Phi_\lambda$ for all $\lambda \in \Lambda$. We say the point a is *part of* the isotrivial subsystem if $a_\lambda = \Phi_\lambda(a_{\lambda_0})$ for all $\lambda \in \Lambda$.

Note that this definition includes (pre-)periodic points as 0-dimensional isotrivial subsystems! That is, if a is a preperiodic point for f defined over K, then

$$V = \bigcup_{n\geq 0} f^n(a)$$

is an algebraic subvariety of $\mathbb{P}^N$ defined over K. The complex subvariety $\mathcal{V}$ is a finite union of sections over Λ. These may have intersections over a finite set of points in Λ, so we exclude those points, and then the fibers of $\mathcal{V}$ are all isomorphic and the dynamics on each finite set $\mathcal{V}_\lambda$ are conjugate.

16.4. **Repelling periodic points.** There is an important step in the proof of Theorem 16.3 that uses much of the machinery we have developed in this course. We formulate it as follows:

Lemma 16.4. [GV, Lemma 18] *Fix a point $a \in \mathbb{P}^N(K)$ and let Γ_a be its graph in $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$. Assume that a is not periodic for f, but suppose there is a point $\lambda_0 \in \Lambda$ so that a_{λ_0} is a repelling periodic point of f_{λ_0} in the support of μ_{λ_0}. Then*

$$\hat{T}_f \wedge [\Gamma_a] \neq 0$$

in $\Lambda \times \mathbb{P}^N_{\mathbb{C}}$.

Proof. Let $x_0 = a_{\lambda_0} \in \operatorname{supp} \mu_{\lambda_0}$. Let p be the period of x_0. Let η denote the parameterization of the nearby repelling periodic points over a neighborhood U of λ_0, and let Γ_η denote its image in $\Lambda \times \mathbb{P}^N$. By hypothesis, x_0 is an isolated point of the intersection of Γ_η with Γ_a.

Recall that $\pi : \Lambda \times \mathbb{P}^N \to \Lambda$ denotes the projection. Shrinking U if necessary, there exists a tubular neighborhood W of Γ_η in $\pi^{-1}(U)$ and a constant $\kappa > 1$ so that

$$d_{\mathbb{P}^N_{\mathbb{C}}}(f^p_\lambda(x), f^p_\lambda(\eta(\lambda))) \geq \kappa\, d_{\mathbb{P}^N_{\mathbb{C}}}(x, \eta(\lambda))$$

for all $x \in W \cap \pi^{-1}(\lambda)$ and all $\lambda \in U$ and for any reasonable choice of distance function $d_{\mathbb{P}^N_{\mathbb{C}}}$ on the fibers. In particular, there exists a nested sequence of tubular neighborhoods $W_n \subset W$ around $\Gamma_\eta \cap \pi^{-1}(U)$, for $n \geq 1$, so that $\hat{f}^{np} : W_n \to W$ is proper and one-to-one. Then for all integers $n \geq 0$,

$$\begin{aligned} d^{npN} \int_{W_n} \hat{T}_f \wedge [\Gamma_a] &= \int_{W_n} (\hat{f}^{np})^*(\hat{T}_f) \wedge [\Gamma_a] \\ &= \int_W \hat{T}_f \wedge (\hat{f}^{np})_*[\Gamma_a] \\ &= \int_W \hat{T}_f \wedge [\hat{f}^{np}(\Gamma_a)]. \end{aligned}$$

On the other hand, passing to a subsequence if necessary, we have

$$\lim_{n\to\infty} \chi_W\, [\hat{f}^{np}(\Gamma_a)] \;=\; c\,[D_{\lambda_0} \cap W]$$

for some $c > 0$, in the weak sense of currents, where χ_W is the indicator function and D_{λ_0} is a holomorphic curve in $\pi^{-1}(\lambda_0)$. Indeed, since $W \cap \Gamma_a \cap \Gamma_\eta = \{x_0\}$, the vertical expansion of $\hat{f}^p$ shows that the limit must be supported in the fiber $\pi^{-1}\{\lambda_0\}$. Passing to a subsequence, the limit current must be supported in a holomorphic curve D_{λ_0}. As the limit current is closed and positive and nonzero, it must be (a scalar multiple of) the current of integration along D_{λ_0} by Theorem 9.2.

Finally, since x_0 is a repelling periodic point, it must lie in the support of the slice current $(\hat{T}_f)|_{\lambda_0} = T_{f_{\lambda_0}}$ when restricted to any holomorphic curve D passing through x_0. (See the proof that $\operatorname{supp} T_f$ is equal to the Julia set.) Therefore

$$\int_W \hat{T}_f \wedge [D_{\lambda_0} \cap W] \; > \; 0$$

so that

$$\int_W \hat{T}_f \wedge [\hat{f}^{np}(\Gamma_a)] \; > \; 0$$

for all sufficiently large n. □

16.5. **Sketch proof of Theorem 16.3.** Now suppose that $f : \mathbb{P}^N \to \mathbb{P}^N$ is a non-isotrivial morphism of degree $d > 1$ defined over the function field $K = \mathbb{C}(S)$. We already know from Theorem 15.1 that points (1) and (2) are equivalent.

Assume (3). If a is preperiodic, then (2) is immediate. If a is not preperiodic, then its orbit must lie in a positive-dimensional isotrivial subsystem V. We consider the restriction $f|V$, and we change coordinates (passing to a finite extension of K as needed) so that the dynamics on V are constant (meaning independent of λ). It follows that the Weil heights of the iterates of a are all 0. So (2) holds.

Now assume (1) and (2), and suppose that a has infinite orbit. Let D be a constant so that

$$|h(\alpha) - \hat{h}_f(\alpha)| \leq D$$

for all $\alpha \in \mathbb{P}^N(K)$. Consider the Zariski closure Z of the union of all curves $\bigcup_{n \geq 0} \Gamma_{f^n(a)}$ in the Chow variety $Ch_D(\Lambda \times \mathbb{P}^N)$ of curves with bidegree $(1, k)$ for $k \leq D$. Note that each $\Gamma \in Z$ is the graph in $\Lambda \in \mathbb{P}^N$ corresponding to some point $b \in \mathbb{P}^N(K)$. As such, the space Z can be identified with a subvariety of $\mathbb{P}^N$ defined over K. By the f-invariance of the orbit of a, it follows that the space Z is also invariant by f, in the sense that $\Gamma_b \in Z$ implies that $\Gamma_{f(b)} \in Z$. Note further that each such b satisfies $\hat{h}_f(b) = 0$, because the degrees of all iterates are bounded by D.

Since the orbit of a is infinite, the space Z is positive-dimensional. Consider the space

$$Z' = \{(\Gamma, x) : \Gamma \in Z, x \in \Gamma\} \subset Ch_D(\Lambda \times \mathbb{P}^N) \times (\Lambda \times \mathbb{P}^N).$$

Let $\mathcal{Z}$ be the image of Z' when projected to the second factor $\Lambda \times \mathbb{P}^N$. Note that $\mathcal{Z}$ has dimension at least 2 and is invariant by $\hat{f}$. Now we consider a general choice of $\lambda \in \Lambda$. The projection $Z' \to \mathcal{Z}$ induces a map from Z to the fiber $\mathcal{Z}_\lambda$. We aim to show that this induced map is an isomorphism conjugating the action of f on curves Γ to $f_\lambda|_{\mathcal{Z}_\lambda}$. This would imply that a is part of an isotrivial system defined by Z.

Note that restriction $f_\lambda|_{\mathcal{Z}_\lambda}$ is a polarized dynamical system of degree d, and so it has an equilibrium measure $\mu_{Z,\lambda}$ with a Zariski dense set of repelling periodic points in its support. [ADD REFERENCES, add details re irreducible components.] Suppose that $\Gamma = \Gamma_b \in Z$ passes through one of these points. By Lemma 16.4 [but need to rewrite it for general polarized systems, not just on $\mathbb{P}^N$], combined with the equivalence of (1) and (2) for b, we see that b must be a periodic point for f. But this implies that induced map $Z \to \mathcal{Z}_\lambda$ is one-to-one over these repelling periodic points. As they are Zariski dense, it follows that the map is birational on these components. With a bit of care, one can argue that the birational map is in fact an isomorphism.

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