2
Subharmonic and plurisubharmonic functions

In this chapter we begin our study of plurisubharmonic functions; they will play the central part throughout the remaining chapters. First, however, we present a comprehensive account of some fundamental properties of harmonic, pluriharmonic, and subharmonic functions. We also give some examples of applications of these functions in complex analysis. This is followed by a short exposition of some fundamental properties of plurisubharmonic functions. The analogies between plurisubharmonicity, pseudoconvexity, and convexity (of functions and sets, respectively) are explored in the exercises closing the chapter. For further study of subharmonic functions the reader can consult Helms (1969), Landkof (1972), Wermer (1974), Hayman and Kennedy (1976), Doob (1984), and Hayman (1989). As far as general properties of plurisubharmonic functions are concerned, Lelong's monograph (1969) can be regarded as the main reference. Substantial parts of various books on complex analysis in several variables (Vladimirov 1966; Hörmander 1973; Krantz 1982; Lelong and Gruman 1986) are devoted to plurisubharmonic functions.

Throughout the chapter we shall be assuming that

$$\boxed{m \geq 2 \quad \text{and} \quad \Omega \neq \emptyset.}$$

2.1 INTEGRAL AVERAGES

Integral means or averages play a prominent role in the theory of subharmonic functions. It should not come as a surprise that they are useful in investigation of differential properties of functions; after all, the usual derivatives result from evaluating limits of averages. However, being related to various differential operators, the integral averages can offer a way of studying some differential-like properties of functions which are not differentiable. And it is this feature that makes integral means so important. In this section we define two basic integral averages and explain their relationship. Further properties of integral averages will be shown in the following sections.

Let $A = [a_{ij}]$ be an $m \times n$ rectangular matrix, where $m \geq n$. We define the *modulus* of A by the following formula

$$|A| = \left(\sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq m} \left(\det[a_{i_k j}]\right)^2 \right)^{\frac{1}{2}} .$$

If $m = n$, then $|A| = |\det A|$. (Algebraically, the modulus of A is the Euclidean norm of the standard exterior product of the columns of A.) If $L: \mathbf{R}^n \longrightarrow \mathbf{R}^m$ is a linear mapping, $|L|$ will denote the modulus of the matrix representing L (with respect to the canonical bases).

Let $M \subset \mathbf{R}^m$ be a k-dimensional submanifold. A mapping $\psi: D \longrightarrow M$ is a local parametrization of M if D is an open subset of $\mathbf{R}^k$, $\operatorname{rank}_a \psi = k$ at each point $a \in D$, ψ is injective, and its range is an open subset of M. Let $\mathcal{A}$ denote the set of all local parametrizations of M. Let Σ be the σ-algebra of all sets $S \subset M$ such that $\psi^{-1}(S)$ is Lebesgue measurable for all $\psi \in \mathcal{A}$. Clearly, Σ contains the Borel subsets of M. The *surface area measure* σ for M is the unique measure on Σ with the following property: if $\psi \in \mathcal{A}$, $S \in \Sigma$, and S is contained in the range of ψ, then

$$\sigma(S) = \int_{\psi^{-1}(S)} |d_x \psi| d\lambda(x) ,$$

where λ denotes the *Lebesgue measure* in $\mathbf{R}^k$. Clearly, $\sigma(S)$ is independent of the choice of the local parametrization.

Integrals, with respect to σ, of Σ-measurable functions on M are called *surface integrals*. Let $\psi: D \longrightarrow M$ be a member of $\mathcal{A}$. If $f: M \longrightarrow \mathbf{R}$ is a Σ-measurable function and

$$\sigma\{x \in M \setminus \psi(D) : f(x) \neq 0\} = 0 ,$$

then the surface integral of f over M is given by the formula

$$\int_M f(x) d\sigma(x) = \int_D (f \circ \psi)(t) |d_t \psi| d\lambda(t) .$$

The letters λ and σ will be used throughout the book to denote the Lebesgue measure and the surface measure in any dimension and on any surface; the context will always clarify their domains of definition.

Since the surface integrals over Euclidean spheres are particularly important, we now recall the basic properties of spherical coordinates.

For any n, we will denote by p the mapping

$$p: \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R} \times \mathbf{R}^n ,$$
$$p(\alpha, v) = (\cos\alpha, \sin\alpha \cdot v) .$$

Note that if $\|v\| = 1$, the mapping $\Phi: [0, 2\pi] \times [0, \infty) \longrightarrow \mathbf{R}^{n+1}$, given by the formula $\Phi(\alpha, r) = rp(\alpha, v)$, is the standard polar coordinate system for the plane in $\mathbf{R}^{n+1}$ generated by $e_1 = (1, 0, \ldots, 0)$ and $(0, v)$.

Define by induction the spherical coordinate system φ_m for the unit sphere $\partial B(0,1)$ in $\mathbf{R}^m$:

$$\varphi_2(\alpha_1) = p(\alpha_1, 1),$$

where $\alpha_1 \in [0, 2\pi]$; and

$$\varphi_{m+1}(\alpha_1, \ldots, \alpha_m) = p(\alpha_1, \varphi_m(\alpha_2, \ldots, \alpha_m)),$$

where $\alpha_1, \ldots, \alpha_{m-1} \in [0, \pi]$ and $\alpha_m \in [0, 2\pi]$.

We shall use the following notation. If A and B are matrices with the same number of rows, then $[A|B]$ will denote the augmented matrix obtained by writing A and B side by side.

Proposition 2.1.1 *For any suitable α, we have*

$$|d_\alpha \varphi_m| = \big|[\varphi_m(\alpha) | d_\alpha \varphi_m]\big|, \tag{2.1.1}$$

where $d_\alpha \varphi_m$ is identified with its matrix representation (with respect to the canonical bases) and $\varphi_m(\alpha)$ in the second matrix is regarded as a column vector. Moreover,

$$|d_\alpha \varphi_{m+1}| = \sin^{m-1} \alpha_1 |d_\alpha \varphi_m|. \tag{2.1.2}$$

Proof The proposition can be proved by induction. It is clear that (2.1.1) is true for $m = 2$. Also, (2.1.2) for $m+1$ follows from (2.1.1) for m. Indeed,

$$d\varphi_{m+1} = \left[\begin{array}{c|c} -\sin\alpha_1 & 0 \\ \hline \cos\alpha_1 \varphi_m & \sin\alpha_1 d\varphi_m \end{array}\right],$$

and hence

$$\begin{aligned} |d\varphi_{m+1}|^2 &= \sin^{2m} \alpha_1 |d\varphi_m|^2 + \cos^2 \alpha_1 \sin^{2m-2} \alpha_1 \big|[\varphi_m | d\varphi_m]\big| \\ &= \sin^{2m-2} \alpha_1 |d\varphi_m|^2 \end{aligned}$$

by (2.1.1) for m. Moreover, (2.1.1) for m follows from (2.1.1) for $m-1$ and (2.1.2) for m (for $m \geq 3$). Indeed,

$$[\varphi_m | d\varphi_m] = \left[\begin{array}{c|c|c} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \hline \sin\alpha_1 \varphi_{m-1} & \cos\alpha_1 \varphi_{m-1} & \sin\alpha_1 d\varphi_{m-1} \end{array}\right]$$

and thus, by expanding the determinant with respect to the first row, we obtain

$$\begin{aligned} \big|[\varphi_m | d\varphi_m]\big| &= \cos^2 \alpha_1 \sin^{m-2} \alpha_1 \big|[\varphi_{m-1} | d\varphi_{m-1}]\big| + \sin^m \alpha_1 \big|[\varphi_{m-1} | d\varphi_{m-1}]\big| \\ &= \sin^{m-2} \alpha_1 \big|[\varphi_{m-1} | d\varphi_{m-1}]\big|. \end{aligned}$$

Hence, by (2.1.1) for $m-1$ and (2.1.2) for m, we have

$$\big|[\varphi_m | d\varphi_m]\big| = |d\varphi_m|. \quad \blacksquare$$

The *spherical coordinate system* for the sphere $\partial B(a, R) \subset \mathbf{R}^m$ is thus given by $a + R\varphi_m$, and for the ball $\bar{B}(a, R)$ by $\Phi_m(r, \alpha) = a + r\varphi_m(\alpha)$, where $r \in [0, R]$.

Corollary 2.1.2

$$|d_{(r,\alpha)} \Phi_m| = r^{m-1} |d_\alpha \varphi_m|.$$

Proof $d\Phi_m = [\varphi_m | r d\varphi_m]$, and so the corollary follows from (2.1.1). ∎

Note that the definition of φ_m and the relationship between Φ_m and φ_m imply that, if $B(0, R) \subset \mathbf{R}^m$,

$$\int_{\partial B(0,R)} f(x) d\sigma(x) = R^{m-1} \int_{\partial B(0,1)} f(Rx) d\sigma(x) \tag{2.1.3}$$

if f is a measurable function on $\partial B(0, R)$, and

$$\int_{B(0,R)} f(x) d\lambda(x) = \int_0^R \left(\int_{\partial B(0,r)} f(x) d\sigma(x) \right) dr \tag{2.1.4}$$

if f is a measurable function on $B(0, R)$. This implies that if we set

$$\begin{aligned} s_m &= \sigma(\partial B(0,1)), \\ b_m &= \lambda(B(0,1)), \end{aligned}$$

then

$$\begin{aligned} m b_m &= s_m, \\ \sigma(\partial B(a, R)) &= R^{m-1} s_m, \\ \lambda(B(a, R)) &= R^m b_m. \end{aligned} \tag{2.1.5}$$

The main integral averages that we are going to use are defined as follows. Let $B(a,R)$ be an open ball in $\mathbf{R}^m$. We set

$$\mathbf{L}(u;a,R) = \frac{1}{s_m R^{m-1}} \int\limits_{\partial B(a,R)} u(x)d\sigma(x)$$

and

$$\mathbf{A}(u;a,R) = \frac{1}{b_m R^m} \int\limits_{B(a,R)} u(x)d\lambda(x)\,,$$

where u is a measurable function on $\partial B(a,R)$ and $B(a,R)$ respectively. In view of (2.1.4) and (2.1.5), we have the following identity.

Corollary 2.1.3

$$\mathbf{A}(u;a,R) = \frac{m}{R^m}\int_0^R r^{m-1}\mathbf{L}(u;a,r)dr\,. \qquad \blacksquare$$

We also have some useful continuity properties.

Corollary 2.1.4 *If $u: \ \Omega \longrightarrow \mathbf{R}$ is a function defined on an open set $\Omega \subset \mathbf{R}^m$, then, for $a \in \Omega$, we have*

$$\limsup_{R\to 0} \mathbf{L}(u;a,R) \le \limsup_{x\to a} u(x)\,,$$
$$\limsup_{R\to 0} \mathbf{A}(u;a,R) \le \limsup_{x\to a} u(x)\,.$$

If u is continuous at a, then

$$\lim_{R\to 0} \mathbf{L}(u;a,R) = \lim_{R\to 0} \mathbf{A}(u;a,R) = u(a)\,.$$

Proof It is enough to note that

$$\limsup_{x\to a} u(x) = \lim_{R\to 0}\left(\sup\{u(x):\ \|x-a\| \le R\}\right). \qquad \blacksquare$$

2.2 HARMONIC FUNCTIONS

A $\mathcal{C}^2$-function $u: \ \Omega \longrightarrow \mathbf{R}$ defined on an open set $\Omega \subset \mathbf{R}^m$ is said to be *harmonic* in Ω if it satisfies the Laplace equation:

$$\Delta u = \sum_{j=1}^m \frac{\partial^2 u}{\partial x_j^2} \equiv 0 \quad \text{in} \quad \Omega\,.$$

The family of all harmonic functions in Ω will be denoted by $\mathcal{H}(\Omega)$.

Consider the following problem. Let $u \in \mathcal{C}^2(\mathbf{R}^m \setminus \{0\})$ be of the form $u(x) = v(\|x\|)$ for $x \in \mathbf{R}^m \setminus \{0\}$ and a function $v \in \mathcal{C}^2(\mathbf{R})$. We would like to characterize the functions v for which u is harmonic.

Note that if $r = \|x\|$, then

$$\Delta u = v''(r) + \frac{m-1}{r}v'(r). \tag{2.2.1}$$

Therefore $u \in \mathcal{H}(\mathbf{R}^m \setminus \{0\})$ if and only if

$$r^{m-1}v''(r) + (m-1)r^{m-2}v'(r) = 0$$

or, in other words, if $\left(r^{m-1}v'(r)\right)' = 0$. Therefore the function u is harmonic if and only if v is of the form

$$v(r) = Ag(r) + B\,,$$

where $A, B \in \mathbf{R}$ are constants and

$$g(r) = \begin{cases} -\log r & (m=2) \\ r^{2-m} & (m>2)\,. \end{cases} \tag{2.2.2}$$

It can be shown (see Section 4.1) that, in the sense of distributions,

$$\Delta g\left(\|x\|\right) = -s_m \max\{1, m-2\}\delta_0\,, \tag{2.2.3}$$

where δ_0 is the Dirac δ-function, i.e. the distribution defined by the formula $\delta_0(\varphi) = \varphi(0)$ for any test function φ in $\mathbf{R}^m$. For this reason the function $u(x) = -\left(s_m \max\{1, m-2\}\right)^{-1} g\left(\|x\|\right)$ (or $g\left(\|x\|\right)$) is called the *fundamental solution* for the Laplacian in $\mathbf{R}^m$.

The function g will be used to solve the following problem. Let Ω be a bounded open subset of $\mathbf{R}^m$, and let $f \in \mathcal{C}(\partial\Omega)$. The *classical Dirichlet problem* is that of finding a function $u \in \mathcal{H}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $u|\partial\Omega = f$. For our purposes it will suffice to show that the classical Dirichlet problem has a unique solution when Ω is an open ball. However, before proving this statement, we shall need a more detailed characterization of harmonic functions.

By the *classical Green function* of the unit ball with pole at $y \in B(0,1)$ we mean the function $\mathbf{G}$ defined for all $x \in \bar{B}(0,1)$ by the formula

$$\mathbf{G}(x,y) = \begin{cases} g(\|x-y\|) - g\left(\|y\| \, \|x - \frac{y}{\|y\|^2}\|\right) & (y \neq 0) \\ g(\|x\|) - g(1) & (y = 0). \end{cases}$$

It is clear that

$$\mathbf{G}(\cdot, y) \in \mathcal{H}(B(0,1) \setminus \{y\}) \cap \mathcal{C}(\bar{B}(0,1) \setminus \{y\}).$$

Furthermore,

$$\mathbf{G}(x,y) = 0 \quad \text{if} \quad \|x\| = 1. \tag{2.2.4}$$

To see that (2.2.4) is true, observe that for $x \in \partial B(0,1)$ and $y \in B(0,1) \setminus \{0\}$

$$\|y\| \left\| x - \frac{y}{\|y\|^2} \right\| = \|x - y\|. \tag{2.2.5}$$

Indeed,

$$\left\| x - \frac{y}{\|y\|^2} \right\|^2 = 1 + \frac{1}{\|y\|^2} - \frac{2}{\|y\|^2} \sum_{j=1}^m x_j y_j$$
$$= \frac{1}{\|y\|^2} \|x - y\|^2.$$

The case where $y = 0$ is trivial.

Before stating the next result, we recall some basic facts about integration of differential forms on $(m-1)$-dimensional submanifolds in $\mathbf{R}^m$. Suppose that H is a *hypersurface* in $\mathbf{R}^m$, i.e. an $(m-1)$-dimensional submanifold of $\mathbf{R}^m$ of the form $H = f^{-1}(0)$, where $U \subset \mathbf{R}^m$ is open, $f \in \mathcal{C}^\infty(U)$, and $d_x f \neq 0$ for each $x \in H$. Observe that $N : H \longrightarrow \mathbf{R}^m$ given by the formula

$$x \longmapsto \frac{d_x f}{|d_x f|}$$

is a unit normal field on H. That is, $N(x)$ is orthogonal to the tangent space to H at $x \in H$ and $\|N(x)\| = 1$ for each $x \in H$. If H is connected and N' is a continuous unit normal field on H, then either $N \equiv N'$ in H or $N \equiv -N'$ in H. Therefore a connected hypersurface can be given a sense of orientation in only two ways. Note that N can be expressed locally in terms of local parametrizations of H. To be more precise, if D is a domain in $\mathbf{R}^{m-1}$ and $\psi : D \longrightarrow H$ is a local parametrization, then — after permuting the coordinates in $\mathbf{R}^{m-1}$, if necessary —

$$N(\psi(t)) = \left[\frac{(-1)^{j+1} \det\left(d_t(\psi_1, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_m)\right)}{|d_t \psi|} \right]_{j=1,\ldots,m}$$

for $t \in D$.

Define

$$\sigma_j = (-1)^{j+1} dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_m$$

for $j = 1, \ldots, m$. If $F = (F_1, \ldots, F_m) \in \mathcal{C}^1(H, \mathbf{R}^m)$, then

$$\int_H \langle F, N \rangle d\sigma = \int_H \sum_{j=1}^m F_j \sigma_j.$$

(See also Section 2.1.) Indeed, if $*$ denotes the pull-back operation for differential forms, then

$$\psi^*(\sigma_j) = (-1)^{j+1} \det\left(d_t(\psi_1, \ldots, \psi_{j-1}, \psi_{j+1}, \ldots, \psi_m)\right),$$

which implies the previous formula.

Let D be a domain in $\mathbf{R}^m$ such that ∂D is a connected hypersurface. We shall say that the boundary of D has the *natural orientation* induced from D if ∂D is equipped with a smooth unit normal field N such that $N(x)$ is pointing outwards at each $x \in \partial D$.

Lemma 2.2.1 *If u is a harmonic function in a neighbourhood of $\bar{B}(0,1) \subset \mathbf{R}^m$, then*

$$u(y) = \frac{1}{s_m \max\{1, m-2\}} \int_{\partial B(0,1)} \sum_{j=1}^m u \frac{\partial}{\partial x_j} \mathbf{G}(\cdot, y) \sigma_j \qquad (y \in B(0,1)),$$

where

$$\sigma_j = (-1)^{j+1} dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_m$$

and $\partial B(0,1)$ has the natural orientation induced from $B(0,1)$.

Proof Fix $y \in B(0,1)$, and let $G(x) = \mathbf{G}(x,y) / (\max\{1, m-2\})$. Consider the differential form

$$\omega = \sum_{j=1}^m \left(u \frac{\partial G}{\partial x_j} - G \frac{\partial u}{\partial x_j} \right) \sigma_j.$$

Take $\varepsilon > 0$ such that $\bar{B}(y,\varepsilon) \subset B(0,1)$ and $D_\varepsilon = B(0,1) \setminus \bar{B}(y,\varepsilon)$. Suppose that the orientation of ∂D_ε is induced from D_ε. Note that ω is closed in a neighbourhood of $\bar{D}_\varepsilon$. Indeed,

$$d\omega = (u\Delta G - G\Delta u)dx_1 \wedge \ldots \wedge dx_m = 0$$

because u and G are harmonic. By Stokes' theorem,

$$\int_{\partial D_\varepsilon} \omega = 0 . \tag{2.2.6}$$

Let $H(x) = G(x) - \big(g(\|x-y\|)/\max\{1, m-2\}\big)$ for $x \in \bar{D}_\varepsilon$. As $G|\partial B(0,1) \equiv 0$, (2.2.6) can be rewritten as

$$\int_{\partial B(0,1)} \sum_{j=1}^{m} u \frac{\partial G}{\partial x_j} \sigma_j = \int_{\partial B(y,\varepsilon)} \omega . \tag{2.2.7}$$

To finish the proof, it is enough to show that the right-hand side of (2.2.7) converges to $s_m u(y)$ as $\varepsilon \longrightarrow 0$. We have

$$\int_{\partial B(y,\varepsilon)} \omega =$$

$$\int_{\partial B(y,\varepsilon)} \sum_{j=1}^{m} \left(u \frac{\partial H}{\partial x_j} - H \frac{\partial u}{\partial x_j} \right) \sigma_j + \varepsilon^{1-m} \int_{\partial B(y,\varepsilon)} u(x) \sum_{j=1}^{n} \frac{x_j - y_j}{\varepsilon} \sigma_j$$

$$- \frac{g(\varepsilon)}{\max\{1, m-2\}} \int_{\partial B(y,\varepsilon)} \sum_{j=1}^{m} \frac{\partial u}{\partial x_j} \sigma_j = A_\varepsilon + B_\varepsilon + C_\varepsilon .$$

Clearly, $\lim_{\varepsilon \to 0} A_\varepsilon = \lim_{\varepsilon \to 0} C_\varepsilon = 0$. Since $N(x) = \frac{x_j - y_j}{\varepsilon}$ is a unit normal vector field on $\partial B(y,\varepsilon)$,

$$B_\varepsilon = \varepsilon^{1-m} \int_{\partial B(y,\varepsilon)} u(x) d\sigma(x) = s_m \mathbf{L}(u; y, \varepsilon) .$$

In view of Corollary 2.1.4, $\lim_{\varepsilon \to 0} B_\varepsilon = s_m u(y)$. ∎

Define

$$\mathbf{P}(x,y) = \frac{\|x\|^2 - \|y\|^2}{\|x-y\|^m}$$

for $x, y \in \mathbf{R}^m$ such that $x \neq y$. The function $(x,y) \longmapsto \mathbf{P}(x,y)/\big(s_m\|x\|\big)$ is called the *Poisson kernel* in $\mathbf{R}^m$. The function $\mathbf{P}$ is closely tied to the classical Green function of the unit ball.

Proposition 2.2.2 *If $x \in \partial B(0,1) \subset \mathbf{R}^m$ and $y \in B(0,1) \subset \mathbf{R}^m$, then*

$$\big(d_x \mathbf{G}(\cdot, y)\big)(x) = \max\{1, m-2\} \mathbf{P}(x,y) . \tag{2.2.8}$$

In particular, $\mathbf{P}(x, \cdot) \in \mathcal{H}\big(B(0,1)\big)$.

Proof Let $c = 1/\max\{1, m-2\}$. We have

$$c d_x \big[x \longmapsto g\big(\|x-y\|\big)\big] = \frac{x-y}{\|x-y\|^m} .$$

Therefore, if $y \neq 0$, we have

$$c\big(d_x \mathbf{G}(\cdot, y)\big) = \frac{x-y}{\|x-y\|^m} - \|y\|^{2-m} \frac{x - (y/\|y\|^2)}{\big\|x - (y/\|y\|^2)\big\|^m} .$$

In view of (2.2.5),

$$c\big(d_x \mathbf{G}(\cdot, y)\big) = \frac{x-y}{\|x-y\|^m} - \frac{\|y\|^2 x - y}{\|x-y\|^m} = \frac{x(1 - \|y\|^2)}{\|x-y\|^m} . \tag{2.2.9}$$

Obviously, the same holds if $y = 0$. Now — on evaluating the scalar product of the right-hand side of (2.2.9) and $x \in \partial B(0,1)$ — we obtain (2.2.8). The second conclusion of the proposition follows from the fact that $\Delta \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} \Delta$, and that $x \longmapsto g\big(\|x-z\|\big)$ is harmonic in $\mathbf{R}^m \setminus \{z\}$ for any fixed $z \in \mathbf{R}^m$. ∎

Now we have the necessary ingredients to derive the following result.

Theorem 2.2.3 *If u is a harmonic function in a neighbourhood of the closed ball $\bar{B}(a,r) \subset \mathbf{R}^m$, then, for each $y \in B(a,r)$,*

$$\begin{aligned} u(y) &= \frac{1}{s_m r} \int_{\partial B(a,r)} \mathbf{P}(x-a, y-a) u(x) d\sigma(x) \\ &= r^{m-2} \mathbf{L}\big(\mathbf{P}(x-a, y-a) u(x); a, r\big) , \end{aligned} \tag{2.2.10}$$

where the integral average is evaluated with respect to x. In particular, it follows that harmonic functions are of class C^∞.

Proof If $a = 0$ and $r = 1$, the above formula is a direct consequence of Lemma 2.2.1 and Proposition 2.2.2. (Note that the outward normal to $\partial B(0,1)$ at $x \in \partial B(0,1)$ is x itself.) The general case follows from (2.1.3). ∎

The formula (2.2.10) is called the *Poisson integral formula.* The Poisson formula allows us to prove a number of interesting properties of harmonic functions. First, we show several equivalent characterizations of harmonicity.

Theorem 2.2.4 *Let u be a real-valued continuous function on an open set $\Omega \subset \mathbf{R}^m$. The following conditions are equivalent:*
(i) $u \in \mathcal{H}(\Omega)$;
(ii) if $\bar{B}(a,r) \subset \Omega$, then, for each $y \in B(a,r)$,

$$u(y) = r^{m-2}\mathbf{L}\big(\mathbf{P}(x-a, y-a)u(x); a, r\big);$$

(iii) if $\bar{B}(a,r) \subset \Omega$, then

$$u(a) = \mathbf{L}(u; a, r);$$

(iv) if $\bar{B}(a,r) \subset \Omega$, then

$$u(a) = \mathbf{A}(u; a, r).$$

Proof The implication (i)$\Longrightarrow$(ii) coincides with Theorem 2.2.3; (ii)$\Longrightarrow$(iii) follows upon substituting $y = a$ in the Poisson integral formula. Corollary 2.1.3 implies that (iii)$\Longrightarrow$(iv).

It is convenient to interrupt the proof and analyse some important consequences of the part of Theorem 2.2.4 we have already proved. The next result is called the *maximum principle* for harmonic functions.

Theorem 2.2.5 *Suppose that $u \in \mathcal{H}(\Omega) \cap \mathcal{C}(\bar{\Omega})$, where Ω is a bounded domain in $\mathbf{R}^m$. Then either u is constant or it satisfies the inequality*

$$u(x) < \sup_{y \in \partial\Omega} u(y) \qquad (x \in \Omega). \tag{2.2.11}$$

Proof Let α denote the supremum in (2.2.11). Set $A = \Omega \cap u^{-1}(\alpha)$. Of course, A is closed in Ω. We shall prove that if $A \neq \emptyset$, then $A = \Omega$. To achieve this, it is enough to show that A is open and to use the connectedness of Ω. Suppose that $a \in A$, $r > 0$, and $\bar{B}(a,r) \subset \Omega$. If there were a point $b \in B(a,r) \setminus A$, the function u would be strictly less than $u(a) = \alpha$ in a neighbourhood of b; by condition (iv) in Theorem 2.2.4, this would imply that

$$u(a) = \mathbf{A}(u; a, r) < \mathbf{A}(\alpha; a, r) = \alpha,$$

which is impossible. Hence $B(a,r) \subset A$. Therefore we can conclude that A is open, and so $A = \Omega$. ∎

Note that we have actually proved that condition (iv) of Theorem 2.2.4 implies the maximum principle.

Now we can solve the classical Dirichlet problem for open balls in $\mathbf{R}^m$.

Theorem 2.2.6 *Let $f \in \mathcal{C}\big(\partial B(a,r)\big)$, where $a \in \mathbf{R}^m$ and $r > 0$. Define*

$$v(y) = \begin{cases} f(y) & \big(y \in \partial B(a,r)\big) \\ r^{m-2}\mathbf{L}\big(\mathbf{P}(x-a, y-a)f(x); a, r\big) & \big(y \in B(a,r)\big). \end{cases}$$

Then v is the unique function in $\mathcal{H}\big(B(a,r)\big) \cap \mathcal{C}\big(\bar{B}(a,r)\big)$ whose restriction to $\partial B(a,r)$ is f.

Proof The uniqueness part follows from the maximum principle. Furthermore, $v \in \mathcal{H}\big(B(a,r)\big)$ by Proposition 2.2.2. Therefore it suffices to show that for any $y_0 \in \partial B(a,r)$

$$\lim_{\substack{y \to y_0 \\ y \in B(a,r)}} v(y) = f(y_0).$$

Take $\varepsilon > 0$ and $y_0 \in \partial B(a,r)$. Choose $\delta > 0$ such that if $x \in \partial B(a,r)$ and $\|x - y_0\| \leq \delta$, then $\|f(x) - f(y_0)\| \leq \varepsilon$. Define

$$h(x,y) = \mathbf{P}(x-a, y-a)[f(x) - f(y_0)].$$

If $y \in B(a,r)$, then, by Theorem 2.2.3 applied to the constant function $v(y_0)$, we have

$$\begin{aligned} v(y) - v(y_0) &= \frac{1}{s_m r} \int_{\partial B(a,r)} h(x,y)\,d\sigma(x) \\ &= \frac{1}{s_m r}\left(\int_A h(x,y)\,d\sigma(x) + \int_B h(x,y)\,d\sigma(x) \right), \end{aligned}$$

where $A = \partial B(a,r) \cap \bar{B}(y_0, \delta)$ and $B = \partial B(a,r) \setminus A$. Then, by Theorem 2.2.3,

$$\frac{1}{s_m r}\left| \int_A h(x,y)\,d\sigma(x) \right| \leq \frac{\varepsilon}{s_m r} \int_{\partial B(a,r)} \mathbf{P}(x-a, y-a)\,d\sigma(x) = \varepsilon.$$

Moreover, if $x \in B$ and $\|y - y_0\| \leq \delta/2$, then

$$\mathbf{P}(x-a, y-a) \leq \frac{r^2 - \|y-a\|^2}{(\delta/2)^m}.$$

Since $\|y - a\| \longrightarrow r$ as $y \longrightarrow y_0$, we conclude that

$$\int_B \mathbf{P}(x-a, y-a)\,d\sigma(x) \longrightarrow 0$$

as $y \longrightarrow y_0$. Consequently,

$$\left| \int_B h(x,y)\,d\sigma(x) \right| \leq 2 \sup_{z \in \partial B(a,r)} |f(z)| \int_B \mathbf{P}(x-a, y-a)\,d\sigma(x) < \varepsilon,$$

provided that $\|y-y_0\|$ is sufficiently small. Therefore $v(y)$ converges to $v(y_0)$ as y approaches y_0. ∎

Now we can finish the proof of Theorem 2.2.4.

Proof (*Implication* (iv)$\Longrightarrow$(i)) Suppose that $u \in \mathcal{C}(\Omega)$, and that (iv) is satisfied. Let $a \in \Omega$ and $r > 0$ be such that $\bar{B}(a,r) \subset \Omega$, and let v be the solution to the Dirichlet problem

$$v \in \mathcal{H}(B(a,r)) \cap \mathcal{C}(\bar{B}(a,r)),$$
$$v = u \quad \text{on} \quad \partial B(a,r).$$

Since (i)$\Longrightarrow$(iv), the functions $\pm(u-v)$ satisfy (iv) in $B(a,r)$ and hence they satisfy the maximum principle there. (See the remark following Theorem 2.2.5.) As $\pm(u-v) = 0$ on $\partial B(a,r)$, it follows that $u = v$ in $B(a,r)$ and hence $u \in \mathcal{H}(B(a,r))$. Since a was chosen arbitrarily, this means that u is harmonic. ∎

Corollary 2.2.7 *If $u \in \mathcal{H}(\mathbf{R}^m)$ and u is bounded from above (or below), then u is constant.*

Proof Without loss of generality we may suppose that u is negative. Take $x \in \mathbf{R}^m$. By Theorem 2.2.4 (iv), for any $n \in \mathbf{N}$,

$$b_m(n+\|x\|)^m u(x) = \int_{B(x,n+\|x\|)} u(y)d\lambda(y) \le \int_{B(0,n)} u(y)d\lambda(y) = b_m n^m u(0).$$

Thus

$$u(x) \le \left(\frac{n}{n+\|x\|}\right)^m u(0) \quad \text{for} \quad n = 1,2,\ldots.$$

Hence $u(x) \le u(0)$. Similarly, $u(x) \ge u(0)$. ∎

Let u be a complex function on a set S. In what follows, $\|u\|_S$ will denote the supremum of $|u|$ on S, i.e.

$$\|u\|_S = \sup\{|u(x)| : x \in S\}.$$

We need the following auxiliary result.

Lemma 2.2.8 *Let Ω be a neighbourhood of a ball $\bar{B}(a,r) \subset \mathbf{R}^m$. There exists a constant M such that, for each $u \in \mathcal{H}(\Omega)$,*

$$|u(y)-u(z)| \le M\|u\|_\Omega \|y-z\|$$

and

$$\left|\frac{\partial u}{\partial x_j}(y) - \frac{\partial u}{\partial x_j}(z)\right| \le M\|u\|_\Omega \|y-z\|$$

for $y,z \in B(a,r)$ and $j = 1,\ldots,m$.

Proof Without loss of generality we may suppose that $a = 0$. Choose $R > r$ such that $\bar{B}(0,R) \subset \Omega$, and define

$$K = \partial B(0,R) \times \bar{B}(0,r) \subset \mathbf{R}^{2m},$$
$$p(w,y,z) = \max_{m<j\le 2m}\left\{|\mathbf{P}(w,y)-\mathbf{P}(w,z)|, \left|\frac{\partial \mathbf{P}}{\partial x_j}(w,y) - \frac{\partial \mathbf{P}}{\partial x_j}(w,z)\right|\right\},$$

where $(w,y),(w,z) \in K$, and

$$L = \max_{m<j,k\le 2m}\left\{\left\|\frac{\partial \mathbf{P}}{\partial x_j}\right\|_K, \left\|\frac{\partial^2 \mathbf{P}}{\partial x_j \partial x_k}\right\|_K\right\}.$$

By the Poisson integral formula (2.2.10),

$$\frac{\partial u}{\partial x_j}(y) = \frac{1}{s_m R}\int_{\partial B(0,R)} \frac{\partial \mathbf{P}}{\partial x_{m+j}}(w,y)u(w)d\sigma(w) \tag{2.2.12}$$

for $y \in B(0,R)$ and $j = 1,\ldots,m$. By the mean value theorem applied to $\mathbf{P}(w,\cdot)$ and $\frac{\partial \mathbf{P}}{\partial x_j}(w,\cdot)$, we have

$$p(w,y,z) \le \sqrt{m}L\|y-z\| \tag{2.2.13}$$

for $(w,y),(w,z) \in K$. By the Poisson formula (2.2.10), and the formulae (2.2.12) and (2.2.13), we have, for all $y,z \in B(0,r)$,

$$\max_{1\le j\le m}\left\{|u(y)-u(z)|, \left|\frac{\partial u}{\partial x_j}(y) - \frac{\partial u}{\partial x_j}(z)\right|\right\} \le R^{m-2}\sqrt{m}L\|y-z\| \cdot \|u\|_{\partial B(0,R)}.$$

As $\|u\|_{\partial B(0,R)} \le \|u\|_\Omega$, the result follows. ∎

The following estimates are known as *Harnack's inequalities.*

Lemma 2.2.9 *Let u be a positive harmonic function in a neighbourhood of $\bar{B}(a,R)$ in $\mathbf{R}^m$, and let $r \in (0,R)$. If $y \in \bar{B}(a,r)$, then*

$$\frac{R^{m-2}(R^2-\|y-a\|^2)}{(R+r)^m}u(a) \le u(y) \le \frac{R^{m-2}(R^2-\|y-a\|^2)}{(R-r)^m}u(a).$$

Proof Obviously, if $x \in \partial B(a,R)$, then

$$R - r \le \|x-y\| \le R + r.$$

Therefore, for all $x \in \partial B(a,R)$,

$$\frac{R^{m-2}(R^2-\|y-a\|^2)}{(R+r)^m}u(x) \le R^{m-2}\mathbf{P}(x-a,y-a)u(x)$$
$$\le \frac{R^{m-2}(R^2-\|y-a\|^2)}{(R-r)^m}u(x).$$

Upon evaluating the spherical averages $\mathbf{L}$ of the above expressions, we obtain the required estimates by Theorem 2.2.4. ∎

Now we are in a position to prove *Harnack's theorem.*

Theorem 2.2.10 *Let Ω be an open connected subset of $\mathbf{R}^m$, and let $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{H}(\Omega)$.*
(i) If the sequence $\{u_j\}$ is locally uniformly convergent to a function $u : \Omega \longrightarrow \mathbf{R}$, then $u \in \mathcal{H}(\Omega)$. Furthermore, the sequence $\left\{\frac{\partial u_j}{\partial x_i}\right\}_{j\in\mathbf{N}}$ is locally uniformly convergent to $\frac{\partial u}{\partial x_i}$ for $i = 1,\dots,m$.
(ii) If $u_1 \le u_2 \le u_3 \le \dots$ and $u = \lim_{j\to\infty} u_j$, then either $u \in \mathcal{H}(\Omega)$ or $u \equiv +\infty$.

Proof The first conclusion in (i) is an immediate consequence of Theorem 2.2.4 (iii). To prove the second part of (i) we can use Lemma 2.2.8. Take $a \in \Omega$ and choose $r > 0$ so that $\bar{B}(a,r) \subset \Omega$. As the space $\mathcal{C}(\bar{B}(a,r))$ with the supremum norm is complete, Lemma 2.2.8 implies that $\left\{\frac{\partial u_j}{\partial x_i}\right\}_{j\in\mathbf{N}}$ is a Cauchy sequence in $\mathcal{C}(\bar{B}(a,r))$ and hence it is convergent uniformly on $\bar{B}(a,r)$. Since a was chosen arbitrarily, it means that $\left\{\frac{\partial u_j}{\partial x_i}\right\}_{j\in\mathbf{N}}$ is locally uniformly convergent to a continuous function on Ω, and that function must be $\frac{\partial u}{\partial x_i}$.

Suppose that the sequence $\{u_j\}_{j\in\mathbf{N}}$ is increasing. Define $A = \{x \in \Omega : u(x) < \infty\}$ and $B = \Omega \setminus A$. By Harnack's inequalities (Lemma 2.2.9), both A and B are open. As Ω is connected, either $A = \Omega$ or $B = \Omega$. Suppose that $A = \Omega$. If $a \in \Omega$, $0 < r < R$, and $\bar{B}(a,R) \subset \Omega$, then for $j > k$ and $y \in B(a,r)$,

$$0 < u_j(y) - u_k(y) \le \frac{R^{m-2}(R^2-\|y-a\|^2)}{(R-r)^m}(u_j(a)-u_k(a))$$
$$\le \frac{R^m}{(R-r)^m}(u_j(a)-u_k(a)).$$

Thus $\{u_j\}$ converges to u uniformly on $\bar{B}(a,r)$. Consequently, we can deduce that u is continuous and $\{u_j\}$ converges to u locally uniformly. Therefore, by the first part of the theorem, $u \in \mathcal{H}(\Omega)$. ∎

Recall that a family $\mathcal{F}$ of complex functions on $\Omega \subset \mathbf{R}^m$ is said to be *locally uniformly bounded* if for each $a \in \Omega$ there is a neighbourhood V of a and $M > 0$, such that for each $u \in \mathcal{F}$, $\|u\|_V \le M$. It is said to be *normal* if each sequence in $\mathcal{F}$ has a subsequence that converges locally uniformly to a function in Ω

The next result is known as the *compactness principle.*

Theorem 2.2.11 *Let Ω be an open subset of $\mathbf{R}^m$, and let $\mathcal{F}$ be a locally uniformly bounded family in $\mathcal{H}(\Omega)$. Then $\mathcal{F}$ is normal.*

Proof Lemma 2.2.8 implies that the family $\mathcal{F}$ is equicontinuous, and so the result follows from the classical Ascoli theorem (e.g. Royden 1963). ∎

To finish this section we shall conduct a brief analysis of the relationship between harmonic and holomorphic functions.

If $\Omega \subset \mathbf{C}^n$ is open and $u \in \mathcal{C}^2(\Omega)$ is real valued, then u is said to be *pluriharmonic* in Ω if

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0 \quad \text{in} \quad \Omega,$$

where $j,k = 1,\dots,n$. We shall denote by $\mathcal{PH}(\Omega)$ the family of all such functions. Formula (1.4.5) implies that $u \in \mathcal{C}^2(\Omega)$ is pluriharmonic if and only if the restriction of u to the intersection of Ω and any complex line that meets Ω is harmonic as a function of one complex variable. That is, if $a \in \Omega$ and $b \in \mathbf{C}^n \setminus \{0\}$, then the one complex variable function $\lambda \longmapsto u(a+\lambda b)$, regarded as a function of two real variables, is harmonic in its domain of definition. Equivalently, $u \in \mathcal{PH}(\Omega)$ if and only if $u \circ T \in \mathcal{H}(T^{-1}(\Omega))$ for any affine $\mathbf{C}$-isomorphism $T : \mathbf{C}^n \longrightarrow \mathbf{C}^n$.

Clearly, for $n = 1$, pluriharmonic means the same as harmonic in $\mathbf{R}^2$. When $n > 1$ and $\Omega \subset \mathbf{C}^n \approx \mathbf{R}^{2n}$, then $\mathcal{PH}(\Omega) \subset \mathcal{H}(\Omega)$ and the inclusion is proper.

Example 2.2.12 Suppose that $n > 1$. Define $u(z) = \|z\|^{2-2n}$ for $z \in \mathbf{C}^n \setminus \{0\}$. By (2.2.2), $u \in \mathcal{H}(\mathbf{C}^n \setminus \{0\})$. In view of the considerations preceding (2.2.2), the function $z_1 \longmapsto u(z_1, 0, \dots, 0)$ is not harmonic; thus u is not pluriharmonic. ∎

In view of the Cauchy–Riemann equations, if $f \in \mathcal{O}(\Omega)$, where $\Omega \subset \mathbf{C}^n$, then $\operatorname{Re} f, \operatorname{Im} f \in \mathcal{PH}(\Omega)$. The converse is only partially true; before stating it, let us recall that Ω is star-shaped with respect to $a \in \Omega$ if, for any $b \in \Omega$, we have $a + (b-a)t \in \Omega$ for all $t \in [0,1]$.

Proposition 2.2.13 *Suppose that $\Omega \subset \mathbf{C}^n$ is open and star shaped with respect to one of its own points. If $u \in \mathcal{PH}(\Omega)$, then there exists a function $v \in \mathcal{PH}(\Omega)$ such that $u + iv \in \mathcal{O}(\Omega)$.*

Proof Consider the differential form $\omega = -d^c u$. By (1.5.2), $d\omega = 0$ and so, in view of the Poincaré lemma, there is a $\mathcal{C}^\infty$-function $v: \Omega \longrightarrow \mathbf{R}$ such that $dv = \omega = -d^c u$; therefore $u + iv$ satisfies the Cauchy–Riemann equations. ∎

Consequently, every pluriharmonic function is locally the real part of a holomorphic function.

Note also that if $\Omega \subset \mathbf{C}^n$ is open and f is a holomorphic function in Ω, then $\log|f| \in \mathcal{PH}(\Omega \setminus f^{-1}(0))$. This can be checked by a direct calculation.

2.3 SEMICONTINUITY

Before progressing to a study of subharmonic functions, we have to take a 'topological interlude' and describe some basic properties of upper semicontinuous functions. Let X be a metric space. A function $u: X \longrightarrow [-\infty, \infty)$ is said to be *upper semicontinuous* if for each $c \in \mathbf{R}$ the set $\{x \in X: u(x) < c\}$ is open. A function u is said to be *lower semicontinuous* if $-u$ is upper semicontinuous. Note that u is upper semicontinuous if and only if at each point $a \in X$ we have $\limsup_{x\to a} u(x) = u(a)$, where

$$\limsup_{x\to a} u(x) = \inf_{\varepsilon>0}\left(\sup\{u(y) \ : y \in \bar{B}(a,\varepsilon)\}\right).$$

Proposition 2.3.1 *If K is a compact metric space and u is an upper semicontinuous function on K, then u attains its maximum at a point in K.*

Proof Since $K = \bigcup_{j=1}^\infty U_j$, where $U_j = \{x \in K: u(x) < j\}$ and K is compact, there exists a natural number j_0 such that $u < j_0$ on K; thus $s < \infty$, where $s = \sup\{u(x): x \in K\}$. Let $\{x_j\}_{j\geq 1}$ be a sequence in K such that $u(x_j)$ converges to s as $j \longrightarrow \infty$. Since K is compact, there is a subsequence $\{x_{j_k}\}_{k\geq 1}$ convergent to a point $x_0 \in K$. Therefore $s = \lim_{k\to\infty} u(x_{j_k}) \leq u(x_0) \leq s$ and hence $u(x_0) = s$. ∎

Lemma 2.3.2 *Let X be a separable metric space, and let $\{u_\alpha\}_{\alpha\in A}$ be a family of upper semicontinuous functions. Set $u = \inf_{\alpha\in A} u_\alpha$. Then the function u is upper semicontinuous and there is a countable subset A' of A such that $u = \inf_{\alpha\in A'} u_\alpha$.*

Proof The first statement follows directly from the definition. In order to prove the second one, note that the sets

$$U_\alpha = \{(x,t) \in X \times \mathbf{R}: u_\alpha(x) < t\}$$

are open and constitute an open cover of the set

$$V = \{(x,t) \in X \times \mathbf{R}: u(x) < t\}.$$

By the Lindelöf property, there is a countable subcover, say $\{U_\alpha\}_{\alpha\in A'}$, of $\{U_\alpha\}_{\alpha\in A}$, and then obviously $u = \inf_{\alpha\in A'} u_\alpha$. ∎

Proposition 2.3.3 *Let K be a compact metric space, and let $u: K \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function. Then there exists a sequence $u_1 \geq u_2 \geq u_3 \geq \ldots$ of continuous functions such that $\lim_{j\to\infty} u_j(x) = u(x)$ at each $x \in X$.*

Proof We claim that it is sufficient to show the existence of a family $\mathcal{U} \subset \mathcal{C}(K)$ such that $u = \inf \mathcal{U}$. Indeed, in view of the above lemma, if $\mathcal{U}$ is such a family, then there is a sequence $\{v_j\}_{j\geq 1} \subset \mathcal{U}$ such that $u = \inf_j v_j$, and hence the sequence $u_j = \min\{v_1, v_2, \ldots, v_j\}$ has the required properties.

Let $M \in \mathbf{R}$ be such that $u < M - 1$. For each $x \in K$ and $\varepsilon \in (0,1)$, choose $r_{\varepsilon,x} > 0$ such that $u < u(x) + \varepsilon$ on $B(x, r_{\varepsilon,x})$, and define

$$u_{\varepsilon,x}(y) = \begin{cases} M & (y \in K \setminus B(x, r_{\varepsilon,x})) \\ \operatorname{dist}(y,x)\big(M - u(x) - \varepsilon\big)/r_{\varepsilon,x} + u(x) + \varepsilon & (y \in \bar{B}(x, r_{\varepsilon,x})). \end{cases}$$

Now define $\mathcal{U} = \{u_{\varepsilon,x}\}_{x\in K, \varepsilon\in(0,1)}$. ∎

Let X be a metric space, and let Y be a non-empty subset of X. Let $u: Y \longrightarrow [-\infty, \infty)$ be a function which is locally bounded from above near each point of $\bar{Y}$. That is, for each $a \in \bar{Y}$ there exists a neighbourhood U of a in X, such that $\sup\{u(y): y \in U \cap Y\} < \infty$. We define the *upper semicontinuous regularization* u^* of u by the formula

$$u^*(x) = \limsup_{\substack{y\to x \\ y\in Y}} u(y) = \inf_{\varepsilon>0}\left(\sup\{u(y): y \in \bar{B}(x,\varepsilon) \cap Y\}\right) \qquad (x \in \bar{Y}).$$

Then $u^*: \bar{Y} \longrightarrow [-\infty, \infty)$ is upper semicontinuous and $u^* \geq u$ in Y. Moreover, if $v: \bar{Y} \longrightarrow [-\infty, \infty)$ is upper semicontinuous and $u \leq v$ in Y, then $u^* \leq v$ in $\bar{Y}$.

In particular, a function $u: X \longrightarrow [-\infty, \infty)$ is upper semicontinuous if and only if it coincides with its upper semicontinuous regularization.

Let X be a metric space, and let u be a real valued function on X. Define

$$S(u) = \{(x,t) \in X \times \mathbf{R}: u(x) > t\},$$
$$T(u) = \{(x,t) \in X \times \mathbf{R}: u(x) \geq t\}.$$

Suppose that u is locally bounded from above. It is clear that $T(u^*)$ is closed. Furthermore,

$$T(u^*) = \overline{S(u)}. \qquad (2.3.1)$$

Indeed, if $U \times V \subset X \times \mathbf{R}$ is a neighbourhood of $(x,t) \in T(u^*)$, then $(U \times V) \cap S(u) \neq \emptyset$. Thus $S(u) \subset T(u^*) \subset \overline{S(u)}$, and since $T(u^*)$ is closed, we have (2.3.1).

The first part of the following result is known as *Choquet's lemma.*

Lemma 2.3.4 *Let X be a separable metric space, and let $\{u_\alpha\}_{\alpha\in A}$ be a family of real valued functions on X. Suppose that this family is locally bounded from above. Then there exists a countable subset B of A such that*

$$\left(\sup_{\alpha\in A} u_\alpha\right)^* = \left(\sup_{\beta\in B} u_\beta\right)^* . \tag{2.3.2}$$

Moreover, if the functions u_α are lower semicontinuous, then B can be chosen so that

$$\sup_{\alpha\in A} u_\alpha = \sup_{\beta\in B} u_\beta . \tag{2.3.3}$$

Proof For any $B \subset A$, set $u_B = \sup\{u_\beta : \beta \in B\}$. Observe that for each $B \subset A$,

$$S(u_B) = \bigcup_{\beta\in B} S(u_\beta) .$$

Moreover, $u_A^* = u_B^*$ if and only if $T(u_A^*) = T(u_B^*)$. Therefore (2.3.2) will follow if we prove that there is a countable subset B of A such that

$$\overline{\bigcup_{\alpha\in A} S(u_\alpha)} = \overline{\bigcup_{\beta\in B} S(u_\beta)} .$$

Let $\{b_j\}_{j\in\mathbf{N}}$ be a countable dense subset of $\bigcup_{\alpha\in A} S(u_\alpha)$. For each natural number j there is $\alpha = \alpha(j) \in A$ such that $b_j \in S(u_\alpha)$. It is enough to take $B = \{\alpha(j)\}_{j\in\mathbf{N}}$.

If $\{u_\alpha\}_{\alpha\in A}$ is a family of lower semicontinuous functions, the sets $\{S(u_\alpha)\}$ constitute an open cover of $S(u_A)$. By the Lindelöf property, there is a countable subcover of $\{S(u_\alpha)\}_{\alpha\in A}$, say $\{S(u_\beta)\}_{\beta\in B}$. Then

$$S(u_A) = \bigcup_{\alpha\in A} S(u_\alpha) = \bigcup_{\beta\in B} S(u_\beta) = S(u_B) ,$$

which means that $u_A = u_B$. ∎

2.4 SUBHARMONIC FUNCTIONS

Let Ω be an open subset of $\mathbf{R}^m$, and let $u : \Omega \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of Ω. Such a function u is said to be *subharmonic* in Ω if for every relatively compact open subset G of Ω and every function $h \in \mathcal{H}(G) \cap \mathcal{C}(\bar{G})$, the following implication is true:

$$u \le h \quad \text{on} \quad \partial G \Longrightarrow u \le h \quad \text{on} \quad G .$$

In this case we write $u \in \mathcal{SH}(\Omega)$.

It follows directly from the maximum principle for harmonic functions that every harmonic function is subharmonic.

The following theorem gives us a convenient characterization of subharmonicity in terms of integral means.

Theorem 2.4.1 *Let $u : \Omega \longrightarrow [-\infty, \infty)$ be upper semicontinuous and not identically $-\infty$ on any connected component of Ω. Then the following conditions are equivalent:*

(i) $u \in \mathcal{SH}(\Omega)$;

(ii) *if $\bar{B}(a,r) \subset \Omega$, then $u(x) \le r^{m-2}\mathbf{L}(\mathbf{P}(y-a, x-a)u(y); a, r)$ for all $x \in B(a,r)$;*

(iii) *if $\bar{B}(a,r) \subset \Omega$, then $u(a) \le \mathbf{L}(u; a, r)$;*

(iv) *if $\bar{B}(a,r) \subset \Omega$, then $u(a) \le \mathbf{A}(u; a, r)$.*

Furthermore, subharmonicity is a local property, i.e. $u \in \mathcal{SH}(\Omega)$ if and only if it is subharmonic in a neighbourhood of each point of Ω.

Proof In order to show the implication (i)$\Longrightarrow$(ii), suppose that $u \in \mathcal{SH}(\Omega)$ and $\bar{B}(a,r) \subset \Omega$. By Proposition 2.3.3, there exists a decreasing sequence $\{u_j\}$ of continuous functions on $\partial B(a,r)$ convergent to $u|\partial B(a,r)$. In view of Theorem 2.2.6, there exists a sequence of functions $\{U_j\} \in \mathcal{H}(B(a,r)) \cap \mathcal{C}(\bar{B}(a,r))$ such that $U_j|\partial B(a,r) = u_j$. Hence $u \le U_j$ in $\bar{B}(a,r)$ for all $j \in \mathbf{N}$. Clearly, the sequence $\{U_j\}$ is decreasing. Let $x \in B(a,r)$. Upon applying the Lebesgue monotone convergence theorem, we have

$$\begin{aligned} u(x) \le \lim_{j\to\infty} U_j(x) &= \lim_{j\to\infty} r^{m-2}\mathbf{L}(\mathbf{P}(y-a, x-a)u_j(y); a, r) \\ &= r^{m-2}\mathbf{L}\Big(\mathbf{P}(y-a, x-a) \lim_{j\to\infty} u_j(y); a, r\Big) . \end{aligned}$$

Suppose that (ii) is satisfied. By taking $x = a$ we obtain (iii). The implication (iii)$\Longrightarrow$(iv) follows directly from Corollary 2.1.3.

Before finishing the proof of the theorem, we need the following result, known as the *maximum principle for subharmonic functions.*

Theorem 2.4.2 *If Ω is a bounded connected open subset of $\mathbf{R}^m$, and if $u \in \mathcal{SH}(\Omega)$, then either u is constant or, for each $x \in \Omega$,*

$$u(x) < \sup_{z\in\partial\Omega} \left\{ \limsup_{\substack{y\to z\\ y\in\Omega}} u(y) \right\} . \tag{2.4.1}$$

Proof As above, we can show that u satisfies condition (iv) in Theorem 2.4.1. Without loss of generality we may suppose that the right-hand side of (2.4.1) is less than ∞. Define

$$v(x) = \begin{cases} u(x) & (x \in \Omega) \\ \limsup\limits_{\substack{y \to x \\ y \in \Omega}} u(y) & (x \in \partial\Omega). \end{cases}$$

Then v is upper semicontinuous, and hence it attains its maximum, say M, in $\bar{\Omega}$. Define $A = \{x \in \Omega : u(x) = M\}$. We shall prove that if $A \neq \emptyset$, then $A = \Omega$. Clearly, A is closed in Ω since u is upper semicontinuous. If $a \in A$ and $\bar{B}(a,r) \subset \Omega$, then $B(a,r) \subset A$. Indeed, if this were not the case, a point $b \in B(a,r)$ would exist such that $u(b) < M$. Due to the upper semicontinuity of u, the latter would imply that $u < M$ in a neighbourhood of b. As a result, we would have

$$u(a) \leq \mathbf{A}(u;a,r) < \mathbf{A}(M;a,r) = u(a),$$

which is impossible. Consequently, $B(a,r) \subset A$. Therefore A is also open. As Ω is connected, $A = \Omega$. ∎

Note that if Ω is unbounded, the maximum principle is not true for some functions in $\mathcal{SH}(\Omega)$ (e.g. consider $u(x) = \|x\|^2$ in $\mathbf{R}^m \setminus \bar{B}(0,1)$). Nevertheless, it should be mentioned that the boundedness requirement for Ω can sometimes be relaxed (see Armitage and Gardiner 1984; Gardiner 1985; Gauthier *et al.* 1988).

Proof (*Implication* (iv)$\Longrightarrow$(i)) Suppose that (iv) is satisfied, that G is a relatively compact open subset of Ω, and that $h \in H(G) \cap \mathcal{C}(\bar{G})$ is such that $u \leq h$ on ∂G. By Theorem 2.2.4, the function $u - h$ satisfies (iv) in G. It follows from the proof of the maximum principle for subharmonic functions that $u - h \leq 0$ in G, i.e. $u \leq h$ in G. Hence $u \in \mathcal{SH}(\Omega)$.

The last conclusion of Theorem 2.4.1 follows from the proof of Theorem 2.4.2 and that of the implication (iv)$\Longrightarrow$(i) above. ∎

It is an immediate consequence of the proof of Theorem 2.4.1 that in the definition of subharmonic functions it suffices to consider Euclidean balls as the 'test' sets G. Furthermore, instead of the functions that are harmonic in G and continuous on $\bar{G}$, one can consider the functions that are harmonic in a neighbourhood of $\bar{G}$.

In view of Theorems 2.2.4 and 2.4.1, we have the following characterization of harmonic functions.

Corollary 2.4.3 *$u \in \mathcal{H}(\Omega)$ if and only if $u \in \mathcal{SH}(\Omega)$ and $-u \in \mathcal{SH}(\Omega)$.* ∎

Theorem 2.4.1 implies also that subharmonic functions can sometimes be 'glued' together to give a new subharmonic function.

Corollary 2.4.4 *Let Ω be an open set in $\mathbf{R}^m$, and let ω be a non-empty proper open subset of Ω. If $u \in \mathcal{SH}(\Omega)$, $v \in \mathcal{SH}(\omega)$, and $\limsup_{x \to y} v(x) \leq u(y)$ for each $y \in \partial\omega \cap \Omega$, then the formula*

$$w = \begin{cases} \max\{u, v\} & \text{in } \omega \\ u & \text{in } \Omega \setminus \omega. \end{cases} \tag{2.4.2}$$

defines a subharmonic function in Ω. ∎

The following criterion of subharmonicity is sometimes useful.

Proposition 2.4.5 *Let Ω be an open subset of $\mathbf{R}^m$, and let $u : \Omega \longrightarrow [-\infty, 0]$ be upper semicontinuous. Then $u \in \mathcal{SH}(\Omega)$ if and only if, for each $a \in \Omega$, $r > 0$, and $s \geq 0$ such that $\bar{B}(a, r+2s) \subset \Omega$, we have*

$$u(x) \leq \frac{r^m}{(r+s)^m} \mathbf{A}(u;a,r) \tag{2.4.3}$$

for all x such that $\|x - a\| \leq s$.

Proof The condition is sufficient, as it reduces to condition (iv) in Theorem 2.4.1 upon substituting $s = 0$. Suppose now that $u \in \mathcal{SH}(\Omega)$ and a, r, s are as above. Notice that $B(a,r) \subset B(x, r+s) \subset B(a, r+2s) \subset \Omega$. Therefore, if $\|x - a\| \leq s$, we have

$$u(x) \leq \mathbf{A}(u;x,r+s) \leq \frac{r^m}{(r+s)^m} \mathbf{A}(u;a,r),$$

as required. ∎

Corollary 2.4.6 *If $u \in \mathcal{SH}(\Omega)$, then the Lebesgue measure of the set $E = \{x \in \Omega : u(x) = -\infty\}$ is zero.*

Proof Suppose that $\lambda(E) > 0$. Then there exist $a \in \Omega$ and $r > 0$ such that $\lambda(E \cap \bar{B}(a,r)) > 0$ and $\bar{B}(a,r) \subset \Omega$. Take a small $s > 0$ so that $\bar{B}(a, r+2s) \subset \Omega$. By Proposition 2.4.5, $B(a,s) \subset E$, which is impossible because $u \in \mathcal{SH}(B(a,s))$. ∎

A subset E of $\mathbf{R}^m$ is said to be *polar* if for each point $a \in E$ there is a neighbourhood V of a and a function $u \in \mathcal{SH}(V)$ such that $u(x) = -\infty$ for $x \in E \cap V$. The last corollary could be restated as follows: *polar sets have Lebesgue measure zero.*

We shall be using the standard notation for L^p spaces. If S is a Borel set in $\mathbf{R}^m$ and $1 \leq p < \infty$, then $L^p(S)$ is the class of all measurable functions f on S such that

$$\|f\|_{L^p(\Omega)} = \left(\int_S |f|^p d\lambda \right)^{1/p} < \infty.$$

By $L^p_{\text{loc}}(S)$ we denote the family of all measurable functions f on S such that for each $a \in S$ there is a neighbourhood U of a such that $(f|U \cap S) \in L^p(U \cap S)$.

Corollary 2.4.7 *If $u \in \mathcal{SH}(\Omega)$, then $u \in L^1_{\text{loc}}(\Omega)$.*

Proof It is enough to prove that for any compact set $K \subset \Omega$, $\left(\int_K u d\lambda\right) \in \mathbf{R}$. Clearly, $\int_K u d\lambda < \infty$ as u is bounded above on K. To show that $\int_K u d\lambda > -\infty$, it is enough to consider the case where $K = \bar{B}(a,r)$ (because an arbitrary K can be covered by a finite number of closed balls contained in Ω). In this case, the estimate follows from Corollary 2.4.6 combined with Proposition 2.4.5. ∎

2.5 SUBHARMONICITY AND SMOOTHING

In this section we shall characterize smooth subharmonic functions, and prove that any subharmonic function can be approximated by smooth subharmonic functions. We shall also look at some useful consequences of this approximation property.

Theorem 2.5.1 *Let Ω be an open set in $\mathbf{R}^m$, and let $u \in \mathcal{C}^2(\Omega)$. Then $u \in \mathcal{SH}(\Omega)$ if and only if $\Delta u \geq 0$ in Ω.*

Proof Let $u \in \mathcal{C}^2(\Omega)$ satisfy $\Delta u \geq 0$ in Ω, and let G be a relatively compact open subset of Ω. Let $h \in \mathcal{H}(G) \cap \mathcal{C}(\bar{G})$ be such that $u \leq h$ on ∂G. Consider the function

$$v(x) = u(x) - \varepsilon + \delta \|x\|^2,$$

where $\varepsilon > 0$ and $\delta \in (0, \varepsilon/R)$ with $R = \sup\{\|x\|^2 : x \in \partial G\}$. Then $v < h$ on ∂G. In order to show that $u \leq h$ in G, it is enough to prove that $v \leq h$ in G because ε can be chosen arbitrarily small. Suppose that $w = v - h$, and that, for some $a \in G$, $w(a) > 0$. Then, as $\bar{G}$ is compact, there exists a point $b \in G$ at which w attains its maximum. In particular, the functions (of one real variable)

$$t \longmapsto w(b + te_j)$$

attain a local maximum at $0 \in \mathbf{R}$, where $e_j = (0, \ldots, 0, \overset{(j)}{1}, 0, \ldots, 0)$ and $j = 1, \ldots, m$. Thus

$$\frac{d^2}{dt^2}\big(w(b+te_j)\big)\Big|_{t=0} = \frac{\partial^2 w}{\partial x_j^2}(b) \leq 0 \qquad (j = 1, \ldots, m).$$

Therefore $\Delta w(b) \leq 0$, contradicting the fact that $\Delta w = \Delta v - \Delta h > 0$ in G. Accordingly, $u \leq h$ in G; as G was arbitrary, this shows that $u \in \mathcal{SH}(\Omega)$.

Suppose now that u is subharmonic. If there were a point $a \in \Omega$ such that $\Delta u(a) < 0$, then we would have $\Delta(-u) > 0$ in a neighbourhood of a; thus, by the first part of the proof, $-u$ would be subharmonic in a neighbourhood of a. In view of Corollary 2.4.3, u would be harmonic in a neighbourhood of a, in contradiction to our assumption that $\Delta u(a) < 0$. ∎

In most applications of subharmonic functions it is crucial to be able to use subharmonic functions which are not smooth. To make a good use of the criterion contained in the above theorem, we have to show that there are methods of smoothing subharmonic functions. This goal can be achieved via convolutions. (Another method, using integral averages, is described in the exercises at the end of the chapter.)

Recall that if $u, v \in L^1(\mathbf{R}^m)$, then the *convolution* $u * v$ of u and v is defined by the formula

$$(u * v)(x) = \int_{\mathbf{R}^m} u(x-y)v(y)d\lambda(y).$$

Clearly, $u * v = v * u$. Moreover, the convolution $u * v$ is also well-defined if $u \in L^1_{\text{loc}}(\mathbf{R}^m)$ and $v \in L^1(\mathbf{R}^m)$ has a compact support.

Now we shall show that convolutions are particularly useful in smoothing of functions.

Define $h : \mathbf{R} \longrightarrow \mathbf{R}$ by the formula

$$h(t) = \begin{cases} \exp(-1/t) & (t > 0) \\ 0 & (t \leq 0). \end{cases}$$

Using the fact that for $t > 0$, $h^{(n)}(t) = h(t)P_n(1/t)$ for $n = 1, 2, \ldots$, where P_n is a polynomial, it is easy to check that $h \in \mathcal{C}^\infty(\mathbf{R})$. Set

$$\chi(x) = Ch(1 - \|x\|^2) \qquad (x \in \mathbf{R}^m),$$

where

$$C = \left(\int_{B(0,1)} h(1 - \|x\|^2)d\lambda(x)\right)^{-1}.$$

Obviously, $\chi \in \mathcal{C}^\infty(\mathbf{R}^m)$, $\operatorname{supp}\chi = \bar{B}(0,1)$, and $\int_{\mathbf{R}^m} \chi(x)d\lambda(x) = 1$. Furthermore, as $\chi(x)$ depends on $r = \|x\|$ rather than x, we shall sometimes write $\chi(r)$ instead of $\chi(x)$.

For $\varepsilon > 0$ we define

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^m}\chi\left(\frac{x}{\varepsilon}\right). \tag{2.5.1}$$

The functions χ_ε are referred to as *standard smoothing kernels* for reasons to be explained in the next proposition. Note that

$$\int_{\mathbf{R}^m} \chi_\varepsilon(x)d\lambda(x) = 1 \quad \text{and} \quad \operatorname{supp}\chi_\varepsilon(x) = \bar{B}(0,\varepsilon).$$

If U is an open set in $\mathbf{R}^m$, let $\mathcal{C}_0^\infty(U)$ denote the family of all $\mathcal{C}^\infty$-functions on U whose support is a compact subset of U.

If $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^m)$, then — using continuity of φ at the origin — it is easy to check that

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^m} \chi_\varepsilon(x)\varphi(x)d\lambda(x) = \varphi(0).$$

In particular, this means that $\{\chi_\varepsilon\}$ converges to the Dirac delta function δ_0 in the sense of the weak convergence of distributions.

Let $\Omega \subset \mathbf{R}^m$ be open. If $\Omega \neq \mathbf{R}^m$, we set

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

for $\varepsilon > 0$. If $\Omega = \mathbf{R}^m$, we set $\Omega_\varepsilon = \mathbf{R}^m$ for $\varepsilon > 0$.

Proposition 2.5.2 *Suppose that $u \in L^p_{\text{loc}}(\Omega)$, where $\Omega \subset \mathbf{R}^m$ is open and $1 \le p < \infty$. Then*
(i) $u * \chi_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$ (provided $\Omega_\varepsilon \neq \emptyset$);
(ii) for any compact set $K \subset \Omega$, if $u \in \mathcal{C}(\Omega)$, then $u * \chi_\varepsilon \longrightarrow u$ uniformly on K as $\varepsilon \searrow 0$;
(iii) for any compact set $K \subset \Omega$, $u * \chi_\varepsilon \longrightarrow u$ in $L^p(K)$ as $\varepsilon \searrow 0$.

Proof (i) follows from the fact that

$$u * \chi_\varepsilon(x) = \chi_\varepsilon * u(x) = \int_{\mathbf{R}^m} \chi_\varepsilon(x-y)u(y)d\lambda(y).$$

To prove (ii), take a compact set $K \subset \Omega$ and fix $\varepsilon_0 > 0$ such that $K_{\varepsilon_0} \subset \Omega$, where

$$K_\varepsilon = \{x \in \mathbf{R}^m : \operatorname{dist}(x, K) \le \varepsilon\} \qquad (\varepsilon > 0). \tag{2.5.2}$$

Let $0 < \varepsilon < \varepsilon_0$. We have

$$(u * \chi_\varepsilon - u)(x) = (\chi_\varepsilon * u - u)(x) = \int \chi_\varepsilon(x-y)\big(u(y) - u(x)\big)d\lambda(y).$$

Therefore

$$\|u * \chi_\varepsilon - u\|_K \le \sup_{x\in K} \sup_{y \in \bar{B}(x,\varepsilon)} |u(y) - u(x)|.$$

The right-hand side tends to zero as $\varepsilon \searrow 0$, because u is uniformly continuous on K_{ε_0}.

If U is an open set in $\mathbf{R}^m$, let $\mathcal{C}_0(U)$ denote the family of all continuous functions on U whose support is a compact subset of U.

Suppose now that K_{ε_0} is as above, and that $u \in L^p_{\text{loc}}(\Omega)$. Take $\eta > 0$. Since $\mathcal{C}_0(\operatorname{int} K_{\varepsilon_0})$ is dense in $L^p(\operatorname{int} K_{\varepsilon_0})$ (see, e.g. Rudin 1974), there exists a function $\varphi \in \mathcal{C}_0(\Omega)$ such that

$$\|u - \varphi\|_{L^p(K_{\varepsilon_0})} < \eta/3. \tag{2.5.3}$$

Moreover, by (ii), we may choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that for $\varepsilon \in (0, \varepsilon_1)$,

$$\|\varphi * \chi_\varepsilon - \varphi\|_{L^p(K)} \le \eta/3. \tag{2.5.4}$$

Note that if $v \in L^p_{\text{loc}}(\Omega)$, then

$$\|v * \chi_\varepsilon\|_{L^p(K)} \le \|v\|_{L^p(K_\varepsilon)}. \tag{2.5.5}$$

Indeed, by Hölder's inequality and Fubini's theorem,

$$\begin{aligned}
\int_K |v * \chi_\varepsilon(x)|^p d\lambda(x) &= \int_K |\chi_\varepsilon * v(x)|^p d\lambda(x) \\
&\le \int_K \left(\int_{\mathbf{R}^m} \chi_\varepsilon(x-y)|v(y)|d\lambda(y) \right)^p d\lambda(x) \\
&\le \int_K \left[\left(\int_{\mathbf{R}^m} \chi_\varepsilon(x-y)|v(y)|^p d\lambda(y) \right) \left(\int_{\mathbf{R}^m} (\chi_\varepsilon(x-y)d\lambda(y) \right)^{p-1} \right] \\
&= \int_{\mathbf{R}^m} \left(|v(y)|^p \int_K \chi_\varepsilon(x-y)d\lambda(x) \right) d\lambda(y) \\
&\le \int_{K_\varepsilon} |v(y)|^p d\lambda(y).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|u * \chi_\varepsilon - u\|_{L^p(K)} &\le \|u * \chi_\varepsilon - \varphi * \chi_\varepsilon\|_{L^p(K)} + \|\varphi * \chi_\varepsilon - \varphi\|_{L^p(K)} \\
&\quad + \|\varphi - u\|_{L^p(K)} \le \|u - \varphi\|_{L^p(K_\varepsilon)} + 2\eta/3 \le \eta
\end{aligned}$$

by (2.5.4), (2.5.5), and (2.5.3). ∎

Before proving the main approximation theorem for subharmonic functions, we need two auxiliary results. The first describes a connection between convolutions and integral averages; the second shows monotonicity of averages of subharmonic functions.

Lemma 2.5.3 *Let $u \in L^1_{\text{loc}}(\Omega)$, $\Omega \subset \mathbf{R}^m$. Then*

$$\big(\mathbf{L}(u;\ \cdot\ ,r) * \chi_\varepsilon\big)(x) = \mathbf{L}(u * \chi_\varepsilon; x, r) \tag{2.5.6}$$

and

$$u * \chi_\varepsilon(x) = \int_0^1 \chi(r) s_m r^{m-1} \mathbf{L}(u; x, \varepsilon r) dr\,, \tag{2.5.7}$$

where $\varepsilon > 0$ is sufficiently small and $\bar{B}(x,r) \subset \Omega_\varepsilon$.

Proof We have

$$\begin{aligned}(\mathbf{L}(u;\cdot,r) * \chi_\varepsilon)(x) &= \int_{B(0,1)} \left[\frac{1}{s_m r^{m-1}} \int_{\partial B(0,r)} u(x - \varepsilon y + w) d\sigma(w)\right] \chi(y) d\lambda(y) \\ &= \int_{B(0,1)} \left[\frac{1}{s_m} \int_{\partial B(0,1)} u(x - \varepsilon y + rw) d\sigma(w)\right] \chi(y) d\lambda(y) \\ &= \frac{1}{s_m} \int_{\partial B(0,1)} \left[\int_{B(0,1)} u(x - \varepsilon y + rw) \chi(y) d\lambda(y)\right] d\sigma(w) \\ &= \mathbf{L}(u * \chi_\varepsilon; x, r)\,.\end{aligned}$$

Moreover,

$$\begin{aligned}u * \chi_\varepsilon(x) &= \int_{B(0,1)} u(x - \varepsilon y) \chi(y) d\lambda(y) \\ &= \int_0^1 \chi(r) \left(\int_{\partial B(0,r)} u(x - \varepsilon y) d\sigma(y)\right) dr \\ &= \int_0^1 \chi(r) s_m r^{m-1} \mathbf{L}(u; x, \varepsilon r) dr\,.\end{aligned}$$

∎

Lemma 2.5.4 *If $u \in \mathcal{SH}\big(B(a,R)\big)$ (where $a \in \mathbf{R}^m$ and $R > 0$), then the function $r \longmapsto \mathbf{L}(u; a, r)$ is increasing in the interval $(0, R)$.*

Proof Let $0 < r_1 < r_2 < R$, and let $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{C}\big(\partial B(a, r_2)\big)$ be a decreasing sequence that converges to $u|\partial B(a, r_2)$; (see Proposition 2.3.3). By Theorem 2.2.6, there exists a sequence of functions

$$\{h_j\}_{j\in\mathbf{N}} \subset \mathcal{H}\big(B(a, r_2)\big) \cap \mathcal{C}\big(\bar{B}(a, r_2)\big)$$

such that $h_j|\partial B(a, r_2) = u_j$ for all $j \in \mathbf{N}$. We have

$$\begin{aligned}\mathbf{L}(u; a, r_1) \le \mathbf{L}(h_j; a, r_1) = h_j(a) &= \mathbf{L}(h_j; a, r_2) \\ &= \mathbf{L}(u_j; a, r_2) \longrightarrow \mathbf{L}(u; a, r_2) \quad \text{as} \quad j \longrightarrow \infty\,.\end{aligned}$$

∎

Now we can prove the *main approximation theorem for subharmonic functions.*

Theorem 2.5.5 *Let Ω be an open set in $\mathbf{R}^m$, and let $u \in \mathcal{SH}(\Omega)$. If $\varepsilon > 0$ is such that $\Omega_\varepsilon \neq \emptyset$, then $u * \chi_\varepsilon \in \mathcal{C}^\infty \cap \mathcal{SH}(\Omega_\varepsilon)$. Moreover, $u * \chi_\varepsilon$ monotonically decreases with decreasing ε and $\lim_{\varepsilon\to 0} u * \chi_\varepsilon(x) = u(x)$ for each $x \in \Omega$.*

Proof Suppose that $\varepsilon_0 > 0$ is chosen so that $\Omega_{\varepsilon_0} \neq \emptyset$. By Proposition 2.5.2 (i), $u*\chi_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$ if $\varepsilon \in (0, \varepsilon_0)$. In view of (2.5.6) and Theorem 2.4.1 (iii), $u*\chi_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon)$. Lemma 2.5.4 and (2.5.7) imply that $u*\chi_\varepsilon$ decreases as $\varepsilon \searrow 0$. From (2.5.7) and Theorem 2.4.1 it follows that $u * \chi_\varepsilon \ge u$. Now take $x \in \Omega$ and $\eta > 0$. Since u is upper semicontinuous, $u < u(x) + \eta$ in $B(x, \varepsilon_1)$ for some $\varepsilon_1 > 0$ such that $x \in \Omega_{\varepsilon_1}$. Thus, if $\varepsilon < \varepsilon_1$, then

$$u*\chi_\varepsilon(x) = \int_{B(0,\varepsilon)} u(x-y)\chi_\varepsilon(y) dy \le \big(u(x)+\eta\big) \int_{B(0,\varepsilon)} \chi_\varepsilon(y) dy = u(x)+\eta\,.$$

∎

Corollary 2.5.6 *If $u, v \in \mathcal{SH}(\Omega)$ and $u = v$ almost everywhere in Ω, then $u = v$ in Ω.*

Proof It is enough to note that $u * \chi_\varepsilon = v * \chi_\varepsilon$. ∎

Corollary 2.5.7 *Let $u \in \mathcal{SH}(\Omega)$, where $\Omega \subset \mathbf{C}$, and let $f : \Omega' \longrightarrow \Omega$ be a holomorphic mapping, where $\Omega' \subset \mathbf{C}$ is connected. Then either $u \circ f \in \mathcal{SH}(\Omega')$ or $u \circ f \equiv -\infty$.*

Proof If $u \in \mathcal{C}^2(\Omega)$, then

$$\Delta(u \circ f) = \big((\Delta u) \circ f\big)|f'|^2\,,$$

and hence $u \circ f$ is subharmonic. The general case follows from Theorem 2.5.5. ∎

Since subharmonic functions are locally integrable, their Laplacians can be evaluated in the sense of distributions. The next theorem shows that the criterion of subharmonicity described in Theorem 2.5.1 can be applied in a much wider context.

Theorem 2.5.8 *If $\Omega \subset \mathbf{R}^m$ is open and $u \in \mathcal{SH}(\Omega)$, then $\Delta u \geq 0$ in the sense of distributions, i.e.*

$$\int_\Omega u(x)\Delta\varphi(x)d\lambda(x) \geq 0 \qquad (2.5.8)$$

for any non-negative test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Conversely, if $v \in L^1_{\mathrm{loc}}(\Omega)$ is such that $\Delta v \geq 0$ in Ω in the sense of distributions, then the function $u = \lim_{\varepsilon\to 0}(v * \chi_\varepsilon)$ is well-defined, subharmonic in Ω, and equal to v almost everywhere in Ω.

Proof Let $u \in \mathcal{SH}(\Omega)$, and let $u_\varepsilon = u * \chi_\varepsilon$ for $\varepsilon > 0$. Take a non-negative function $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Lebesgue's dominated convergence theorem, combined with integration by parts and the main approximation theorem, shows that

$$\int_\Omega u\Delta\varphi = \lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon\Delta\varphi = \lim_{\varepsilon\to 0}\int_\Omega \varphi\Delta u_\varepsilon \geq 0\,,$$

which proves the first conclusion of the theorem.

Suppose now that $v \in L^1_{\mathrm{loc}}(\Omega)$, and that $\Delta v \geq 0$ in Ω in the sense of distributions. Let $v_\varepsilon = v * \chi_\varepsilon$, for $\varepsilon > 0$ such that $\Omega_\varepsilon \neq \emptyset$ (see Proposition 2.5.2). By Fubini's theorem, $\Delta v_\varepsilon \geq 0$ in Ω_ε in the sense of distributions and thus in the usual sense, as v_ε is smooth. Hence $v_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon)$ by Theorem 2.5.1. By Theorem 2.5.5 and Fubini's theorem, for $\varepsilon_2 > \varepsilon_1 > 0$ and $x \in \Omega_{\varepsilon_2}$, one has

$$\begin{aligned} v_{\varepsilon_2}(x) &= \lim_{\delta\to 0}(v * \chi_{\varepsilon_2}) * \chi_\delta(x) = \lim_{\delta\to 0}(v * \chi_\delta) * \chi_{\varepsilon_2}(x) \\ &\geq \lim_{\delta\to 0}(v * \chi_\delta) * \chi_{\varepsilon_1}(x) = \lim_{\delta\to 0}(v * \chi_{\varepsilon_1}) * \chi_\delta(x) = v_{\varepsilon_1}(x)\,. \end{aligned}$$

Furthermore, according to Proposition 2.5.2, v_ε converges to v in $L^1_{\mathrm{loc}}(\Omega)$. Therefore v_ε converges to v almost everywhere in Ω (e.g. Rudin 1974) and $\lim_{\varepsilon\to 0} v_\varepsilon = u \in \mathcal{SH}(\Omega)$ as the limit of a decreasing family of subharmonic functions. ∎

The same argument can be used to obtain the following property.

Theorem 2.5.9 *Let $\Omega \subset \mathbf{R}^m$ be open, and let $v \in L^1_{\mathrm{loc}}(\Omega)$. Suppose that $v(a) \leq \mathbf{L}(v; a, r)$, provided that $\bar{B}(a,r) \subset \Omega$. Then the function $u = \lim_{\varepsilon\to 0}(v * \chi_\varepsilon)$ is well-defined, subharmonic in Ω, and equal to v almost everywhere.*

Proof Let $v_\varepsilon = v * \chi_\varepsilon$, for $\varepsilon > 0$ such that $\Omega_\varepsilon \neq \emptyset$. By (2.5.6) and Theorem 2.4.1, $v_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon)$. The rest of the proof is the same as in the previous theorem. ∎

2.6 FAMILIES OF SUBHARMONIC FUNCTIONS

Now we are going to investigate some important properties of families of subharmonic functions. The following theorem is an immediate consequence of the definition of subharmonic functions.

Theorem 2.6.1 *Let Ω be an open subset of $\mathbf{R}^m$.*

(i) *The family $\mathcal{SH}(\Omega)$ is a convex cone, i.e. if α, β are non-negative numbers and $u, v \in \mathcal{SH}(\Omega)$, then $\alpha u + \beta v \in \mathcal{SH}(\Omega)$.*
(ii) *If Ω is connected and $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{SH}(\Omega)$ is a decreasing sequence, then $u = \lim_{j\to\infty} u_j \in \mathcal{SH}(\Omega)$ or $u \equiv -\infty$.*
(iii) *If $u : \Omega \longrightarrow \mathbf{R}$ is a function, and if $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{SH}(\Omega)$ converges to u uniformly on compact subsets of Ω, then $u \in \mathcal{SH}(\Omega)$.*
(iv) *Let $\{u_\alpha\}_{\alpha\in A} \subset \mathcal{SH}(\Omega)$ be such that its upper envelope $u = \sup_{\alpha\in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization u^* is subharmonic in Ω.* ∎

One can use the notation from Section 2.5 to make the last conclusion of the above theorem a little more precise.

Proposition 2.6.2 *Let $\{u_\alpha\}_{\alpha\in A} \subset \mathcal{SH}(\Omega)$ be such that its upper envelope $u = \sup_{\alpha\in A} u_\alpha$ is locally upper bounded. Then, for each $\varepsilon > 0$ such that $\Omega_\varepsilon \neq \emptyset$, $u * \chi_\varepsilon \in \mathcal{SH}(\Omega_\varepsilon)$. Moreover, $u * \chi_\varepsilon$ monotonically decreases with decreasing ε and $\lim_{\varepsilon\to 0} u * \chi_\varepsilon = u^*$ at each point of Ω. In particular, $u = u^*$ almost everywhere.*

Proof By Theorem 2.5.9, the function $w = (\lim_{\varepsilon\to 0} u * \chi_\varepsilon)$ is subharmonic and equal to u almost everywhere. As in the proof of Theorem 2.5.8, we can see that $u * \chi_{\varepsilon_1} \leq u * \chi_{\varepsilon_2}$ if $0 < \varepsilon_1 < \varepsilon_2$. Therefore, as $u_\alpha \leq u_\alpha * \chi_\varepsilon \leq u * \chi_\varepsilon$, we can conclude that $u \leq w$, and hence $u \leq u^* \leq w$. Since $u = w$ almost everywhere, the result follows. ∎

The next result is similar to Theorem 2.6.1 (iv), but slightly less obvious.

Theorem 2.6.3 *Let the sequence $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{SH}(\Omega)$ be locally uniformly bounded above. Define $u(x) = \limsup_{j\to\infty} u_j(x)$ for $x \in \Omega$. Then in each component of Ω, the upper semicontinuous regularization u^* is either subharmonic or identically $-\infty$. Moreover, $u = u^*$ almost everywhere in Ω.*

Proof Without loss of generality we may suppose that Ω is connected. For $k \geq 1$ define $v_k = \sup_{j\geq k} u_j$. Then, in view of Theorem 2.6.1, $v_k^* \in \mathcal{SH}(\Omega)$ and the sequence $\{v_k^*\}_{k\in\mathbf{N}}$ decreases pointwise to a function v which is either subharmonic or identically $-\infty$. Clearly, $v \geq u$, and thus $v \geq u^* \geq$

u. Also, by Proposition 2.6.2, $v_k^* = v_k$ almost everywhere, and hence $v = u^* = u$ almost everywhere in Ω. Consequently, for any $a \in \Omega$,

$$v(a) \leq \limsup_{R \to 0} \mathbf{A}(v; a, R) = \limsup_{R \to 0} \mathbf{A}(u^*; a, R) \leq u^*(a)$$

by Corollary 2.1.4, and therefore $v = u^*$ everywhere in Ω. ∎

The next property is known as the *Hartogs lemma.*

Theorem 2.6.4 *Let $\{u_j\}_{j \in \mathbf{N}} \subset \mathcal{SH}(\Omega)$ be locally uniformly bounded above in $\Omega \subset \mathbf{R}^m$. Suppose that*

$$\limsup_{j \to \infty} u_j(x) \leq M$$

for each $x \in \Omega$ and some constant M. Then, for each $\varepsilon > 0$ and each compact set $K \subset \Omega$, there exists a natural number j_0 such that, for $j \geq j_0$,

$$\sup_{x \in K} u_j(x) \leq M + \varepsilon .$$

Proof Without loss of generality we may suppose that $u_j \leq 0$ for all j and $M \leq 0$. Using the standard compactness argument we can reduce the problem at hand to having to prove the following local property: for all $\varepsilon > 0$ and $a \in \Omega$, there exist $\delta > 0$ and $j_0 \in \mathbf{N}$ such that for all $j \geq j_0$, we have

$$\sup_{x \in \bar{B}(a,\delta)} u_j(x) \leq M + \varepsilon . \qquad (2.6.1)$$

Fix $\varepsilon > 0$ and $a \in \Omega$. Take $r > 0$ such that $\bar{B}(a,r) \subset \Omega$. If $\delta > 0$ is sufficiently small, $\bar{B}(a, r + 2\delta) \subset \Omega$ and

$$\left(\frac{r}{r+\delta}\right)^m \left(M + \frac{\varepsilon}{2}\right) < M + \varepsilon . \qquad (2.6.2)$$

By Fatou's lemma,

$$\limsup_{j \to \infty} \mathbf{A}(u_j; a, r) \leq \mathbf{A}\left(\limsup_{j \to \infty} u_j; a, r\right) \leq M .$$

Therefore we can choose $j_0 \in \mathbf{N}$ such that, for all $j \geq j_0$, $\mathbf{A}(u_j; a, r) \leq M + \frac{\varepsilon}{2}$. Now (2.6.1) follows from Proposition 2.4.5 and (2.6.2). ∎

Later on, the above result will constitute the vital part of the proof of Hartogs' theorem about separately holomorphic functions.

Here we have yet another application of Fatou's lemma.

Theorem 2.6.5 *Let (T, μ) be a σ-finite measure space, and let Ω be a connected open set in $\mathbf{R}^m$. Suppose that $u : \Omega \times T \longrightarrow [-\infty, \infty)$ is a measurable function such that:*
(i) $x \longmapsto u(x,t)$ is subharmonic in Ω for each $t \in T$;
(ii) there exists $g \in L^1(\mu)$ such that

$$u(x,t) \leq g(t) \qquad (x \in \Omega,\ t \in T) .$$

Then the function

$$U(x) = \int_T u(x,t) d\mu(t) \qquad (x \in \Omega)$$

is either subharmonic in Ω or identically $-\infty$.

Proof Consider the case when $U \not\equiv -\infty$. Let $x_0 \in \Omega$, and let $\{x_n\}_{n \in \mathbf{N}} \subset \Omega$ be a sequence convergent to x_0. By Fatou's lemma, applied to the sequence $\{t \longmapsto u(x_n, t) - g(t)\}_{n \in \mathbf{N}}$,

$$\limsup_{n \to \infty} U(x_n) \leq U(x_0) .$$

Consequently, U is upper semicontinuous. Subharmonicity follows directly from Theorem 2.4.1 and Fubini's theorem. ∎

We shall close this section by proving a very useful application of Theorem 2.6.1 (i), (iv). The result is due to Gardiner (1985); the proof given here can be found in Gardiner and Klimek (1986).

Theorem 2.6.6 *Let Ω be an open subset of $\mathbf{R}^m$.*
(i) Let u, v be harmonic in Ω and $v > 0$. If $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ is convex, then $v\phi(u/v)$ is subharmonic in Ω.
(ii) Let $u \in \mathcal{SH}(\Omega)$, $v \in \mathcal{H}(\Omega)$, and $v > 0$ in Ω. If $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ is convex and increasing, then $v\phi(u/v)$ is subharmonic in Ω. ($\phi(-\infty)$ is interpreted as $\lim_{x \to -\infty} \phi(x)$.)
(iii) Let $u, -v \in \mathcal{SH}(\Omega)$, $u \geq 0$ in Ω, and $v > 0$ in Ω. If $\phi : [0,\infty) \longrightarrow [0,\infty)$ is convex and $\phi(0) = 0$, then $v\phi(u/v) \in \mathcal{SH}(\Omega)$.

Proof Observe that, corresponding to each part of the theorem, ϕ can be written as:
(i) $\phi(x) = \sup\{ax + b :\ a, b \in \mathbf{R}$ and $at + b \leq \phi(t)$ for all $t \in \mathbf{R}\}$;
(ii) $\phi(x) = \sup\{ax + b :\ a \geq 0, b \in \mathbf{R}$ and $at + b \leq \phi(t)$ for all $t \in \mathbf{R}\}$;
(iii) $\phi(x) = \sup\{ax + b :\ a \geq 0, b \leq 0$ and $at + b \leq \phi(t)$ for all $t \in \mathbf{R}\}$, where $x \geq 0$.

In each of these cases, $v[a(u/v)+b] = au+bv \in \mathcal{SH}(\Omega)$ for the appropriate values of a and b. Therefore $v\phi(u/v)$ can be written as $w = \sup_\alpha u_\alpha$, where each $u_\alpha \in \mathcal{SH}(\Omega)$. If we can show that w is upper semicontinuous, the theorem will follow from Theorem 2.6.1 (iv).

For (i), this is trivial, as all functions are continuous. For (ii), it follows from the fact that the function ϕ is continuous and increasing. In the case of (iii), we can proceed as follows. We extend the function $x^{-1}\phi(x)$ to a continuous function on $[0,\infty)$ by defining it at 0 by its right-hand side limit there. This function is increasing. Since u/v is upper semicontinuous, it now follows that so is $(u/v)^{-1}\phi(u/v)$, and therefore $v\phi(u/v)$, the latter being the product of two non-negative upper semicontinuous functions. ∎

Remark 2.6.7 The special cases of (i) and (ii), with $v \equiv 1$, have been known for a long time and are commonly proved using Jensen's inequality.

Corollary 2.6.8 *If $u \in \mathcal{SH}(\Omega)$, then also $e^u \in \mathcal{SH}(\Omega)$. If $u \in \mathcal{SH}(\Omega)$ and $u \geq 0$, then $u^\alpha \in \mathcal{SH}(\Omega)$ for any number $\alpha \geq 1$.*

Proof The functions $t \longmapsto e^t$ and $t \longmapsto t^\alpha$ $(\alpha \geq 1)$ are convex and increasing. ∎

Corollary 2.6.9 *If u_1, u_2 are non-negative functions in $\Omega \subset \mathbf{R}^m$ and $\log u_1, \log u_2 \in \mathcal{SH}(\Omega)$, then $u_1u_2 \in \mathcal{SH}(\Omega)$ and $\log(u_1+u_2) \in \mathcal{SH}(\Omega)$.*

Proof $u_1u_2 = \exp(\log u_1 + \log u_2)$, and so the first conclusion is true. To show the second one, take an open set G such that $\bar{G}$ is a compact subset of Ω. Let $h \in \mathcal{H}(G) \cap \mathcal{C}(\bar{G})$ be such that $\log(u_1+u_2) \leq h$ on ∂G. Then $(u_1+u_2)e^{-h} \leq 1$ on ∂G and it is subharmonic in G (because the functions u_1e^{-h} and u_2e^{-h} are subharmonic by the first part of the proof) and semicontinuous in $\bar{G}$. By the maximum principle, $(u_1+u_2)e^{-h} \leq 1$ in G, and thus $\log(u_1+u_2) \leq h$ in G. ∎

Note that, in general, the product of two subharmonic functions is not subharmonic. (For instance, if $u, v \in \mathcal{C}^2(\Omega)$, then $\Delta(uv) = u\Delta v + v\Delta u + \langle 2\operatorname{grad} u, \operatorname{grad} v\rangle$, where $\langle , \rangle$ denotes the usual inner product in $\mathbf{R}^m$.) Observe also that if $u \geq 0$ is such that $\log u \in \mathcal{SH}(\Omega)$, then for any $\alpha > 0$, $u^\alpha \in \mathcal{SH}(\Omega)$. Indeed, $u^\alpha = \exp(\alpha \log u)$, and thus this property follows from Corollary 2.6.8.

2.7 REMOVABLE SINGULARITIES OF SUBHARMONIC FUNCTIONS

We have already seen that polar sets, being of Lebesgue measure zero, are rather small. Here, we describe another aspect of their 'smallness'.

Theorem 2.7.1 *Let Ω be an open subset of $\mathbf{R}^m$, and let F be a closed subset of Ω of the form $F = \{x \in \Omega : v(x) = -\infty\}$, where $v \in \mathcal{SH}(\Omega)$. If $u \in \mathcal{SH}(\Omega \setminus F)$ is bounded from above, then the function $\tilde{u}$ defined by the formula*

$$\tilde{u}(x) = \begin{cases} u(x) & (x \in \Omega \setminus F) \\ \limsup\limits_{\substack{y \to x \\ y \notin F}} u(y) & (x \in F) \end{cases}$$

is subharmonic in Ω. If u is harmonic and bounded in $\Omega \setminus F$, then $\tilde{u}$ is harmonic in Ω. If Ω is connected, then so is $\Omega \setminus F$.

Proof By restricting our consideration to a smaller set Ω, we may suppose that $v < 0$ in Ω. For $\varepsilon > 0$, define

$$u_\varepsilon = \begin{cases} u + \varepsilon v & \text{on } \Omega \setminus F \\ -\infty & \text{on } F. \end{cases}$$

Clearly, u_ε is subharmonic in Ω and $(\sup_{\varepsilon>0} u_\varepsilon) = u$ in $\Omega \setminus F$. Moreover, $(\sup_{\varepsilon>0} u_\varepsilon)^* = \tilde{u}$ and hence $\tilde{u} \in \mathcal{SH}(\Omega)$ in view of Theorem 2.6.1.

Suppose that $u \in \mathcal{H}(\Omega \setminus F)$ is bounded. By the first part of the proof, both $\tilde{u}$ and $(\widetilde{-u})$ are subharmonic in Ω. Since $\tilde{u} + (\widetilde{-u}) = 0$ almost everywhere in Ω, equality holds everywhere by Corollary 2.5.6. Thus $\tilde{u}$ and $-\tilde{u}$ are both subharmonic, whence $\tilde{u} \in \mathcal{H}(\Omega)$.

If $\Omega \setminus F$ were not connected, it could be written as the union of two non-empty open sets Ω_0 and Ω_1, and the function u defined by the conditions $u|\Omega_0 \equiv 0$, $u|\Omega_1 \equiv 1$ could be extended to a harmonic function $\tilde{u}$ on Ω. Then the sets $F_i = \{x \in \Omega : \tilde{u}(x) = i\}$, where $i = 0, 1$, would be closed and their union would be Ω, contradicting the connectedness of Ω. ∎

It has been shown (e.g. Gardiner 1991*a*) that the boundedness requirement for the function u in Theorem 2.7.1 can be relaxed if certain constraints are imposed on the set F.

Now we shall present two interesting applications of the removable singularity theorem.

A subset E of $\mathbf{R}^m$ is said to be *thin* at a point $a \in \mathbf{R}^m$ if either a is not a limit point of E or there exist a neighbourhood V of a and a function $u \in \mathcal{SH}(V)$ such that

$$\limsup_{\substack{x \to a \\ x \in E \setminus \{a\}}} u(x) < u(a). \tag{2.7.1}$$

Note that in the definition of thin sets one can require that u has values in $[0,1]$ and $u = 0$ on $(E \setminus \{a\}) \cap V$. Indeed, suppose that $u \in \mathcal{SH}(V)$ and (2.7.1) is satisfied. By taking a smaller V and adding a constant to u, if necessary, one may suppose that u is negative in V. Take a constant C such that

$$\limsup_{\substack{x \to a \\ x \in E \setminus \{a\}}} u(x) < C < u(a),$$

and define $u_1 = \frac{1}{|C|}\max\{u, C\} + 1$. There is a neighbourhood $V_1 \subset V$ of a such that $u < C$ in $V_1 \cap (E \setminus \{a\})$. Then $u_1 = 0$ on $V_1 \cap (E \setminus \{a\})$ and

$$\limsup_{\substack{x \to a \\ x \in E \setminus \{a\}}} u_1(x) = 0 < u_1(a),$$

as required.

The elegant elementary proof of the following important property is due to Ransford (1983).

Theorem 2.7.2 *The interval* $[0,1] \subset \mathbf{C}$ *is not thin at 0.*

Proof In the light of the above comments, it is enough to prove that if u is a subharmonic function in the unit disc D, with values in $[0,1]$, and $u = 0$ on $(0,1)$, then $u(0) = 0$.

Suppose that u is such a function. Define

$$\Delta_n = \{re^{it} : 0 \le r < 1, \ 0 < t < 2\pi/n\}$$

and

$$u_n(z) = \begin{cases} u(z^n) & \text{if } z \in \Delta_n \\ 0 & \text{if } z \in D \setminus \Delta_n, \end{cases}$$

for $n = 1, 2, \ldots$. The function u_n is clearly upper semicontinuous in $D \setminus \{0\}$. Moreover,

$$\limsup_{\substack{z \to 0 \\ z \neq 0}} u_n(z) = \limsup_{\substack{z \to 0 \\ z \neq 0}} u(z) = u(0). \tag{2.7.2}$$

By Corollary 2.5.7, $u_n \in \mathcal{SH}(\Delta_n \setminus \{0\})$. Hence, by Theorem 2.4.1 (iii), $u_n \in \mathcal{SH}(D \setminus \{0\})$. In view of Theorem 2.7.1 (with $v(z) = \log|z|$) and (2.7.3), $u_n \in \mathcal{SH}(D)$. Applying again Theorem 2.4.1 (iii), we obtain the following estimate

$$u(0) = u_n(0) \le \mathbf{L}\left(u_n; 0, \frac{1}{2}\right) \le \frac{1}{n} \qquad (n = 1, 2, \ldots).$$

Therefore $u(0) = 0$. ∎

Another interesting application of the removable singularity theorem is the following proposition.

Proposition 2.7.3 *If* $u \in \mathcal{SH}(\mathbf{R}^2)$ *and* u *is bounded from above, then* u *is constant.*

Proof We can identify $\mathbf{R}^2$ with $\mathbf{C}$.

By the maximum principle (Theorem 2.4.2), applied to u restricted to the unit disc, there is a point a such that $|a| = 1$ and $u(a) \ge u(z)$ if $|z| \le 1$. On the other hand, the function $v(z) = u(1/z)$ extends to a subharmonic function on $\mathbf{C}$ by Theorem 2.7.1; thus for some b from the unit circle, $v(b) \ge v(z)$ if $|z| \le 1$. Therefore for all $z \in \mathbf{C}$, $u(z) \le \max\{u(a), \ u(1/b)\}$, which means that u has a global maximum at a or at $1/b$. Thus, by the maximum principle, u is constant. ∎

Remark 2.7.4 The above result is not true in $\mathbf{R}^m$ for $m > 2$. Indeed, the function

$$u(x) = \begin{cases} -\|x\|^{2-m} & (x \in \mathbf{R}^m \setminus \{0\}) \\ -\infty & (x = 0) \end{cases}$$

is subharmonic in $\mathbf{R}^m$ (by (2.2.2) and Theorem 2.4.1 (iii)) and negative.

2.8 APPLICATIONS TO HOLOMORPHIC FUNCTIONS

It turns out that a number of fundamental theorems concerning holomorphic functions can be deduced from the corresponding properties of subharmonic functions. However, our presentation here is limited to only a few topics related to this book. When combined with some of the exercises at the end of the previous chapter, it furnishes a short introduction to the theory of holomorphic functions of several variables. Narasimhan (1971) provides a broader though elegant and concise introduction to the theory. For a more comprehensive treatment see, for example, Hörmander (1973), Rudin (1980), and Krantz (1982). Suppose throughout the section that Ω is an open subset of $\mathbf{C}^n$.

Theorem 2.8.1 (The identity principle) *If* $f, g \in \mathcal{O}(\Omega)$, Ω *is connected, and* $f = g$ *on a non-empty open subset of* Ω, *then* $f = g$ *in* Ω.

Proof Note that if $h \in \mathcal{O}(\Omega)$ and $h \not\equiv 0$, then $\log|h| \in \mathcal{PH}(\Omega \setminus h^{-1}(0))$ and $\log|h| = -\infty$ on the set $h^{-1}(0)$. Consequently, $\log|h| \in \mathcal{SH}(\Omega)$ by Theorem 2.4.1.

Therefore, if we had $f \not\equiv g$, then the set

$$\{z \in \Omega : \ f(z) = g(z)\} = \{z \in \Omega : \ \log|(f-g)(z)| = -\infty\}$$

would be polar. Thus the result follows from Corollary 2.4.6. ∎

Theorem 2.8.2 (Liouville) *If* $f \in \mathcal{O}(\mathbf{C}^n)$ *and* $|f|$ *is bounded, then* f *is a constant function.*

Proof The result is a direct consequence of Corollary 2.2.7 applied to the pluriharmonic functions $\operatorname{Re} f$, $\operatorname{Im} f$. ∎

Theorem 2.8.3 (The maximum principle) *Let $f: \Omega \longrightarrow \mathbf{C}^m$ be a holomorphic mapping such that $z \longmapsto \|f(z)\|$ attains a local maximum at a point $a \in \Omega$. If Ω is connected, then f is constant.*

Proof First, observe that the function $z \longmapsto \|f(z)\|^2$ is subharmonic in Ω, as

$$\frac{1}{4}\Delta(\|f(z)\|^2) = \sum_{j=1}^{m}\sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}|f_j(z)|^2 = \sum_{j=1}^{m}\sum_{k=1}^{n}\left|\frac{\partial f_j}{\partial z_k}(z)\right|^2 \geq 0.$$

By the maximum principle for subharmonic functions we conclude that this function is constant in a neighbourhood U of a. Thus, in view of the above formula,

$$\left|\frac{\partial f_j}{\partial z_k}\right| \equiv 0 \quad \text{in} \quad U,$$

meaning that f is constant in U, and hence in Ω, according to the identity principle. ∎

Note that if the Euclidean norm in the maximum principle were replaced by the maximum norm, we would get a false statement. For instance, if $f(z,w) = (z,1)$, $|f|$ has a local maximum at $(0,0)$.

The next result is a direct consequence of Harnack's theorem (Theorem 2.2.10).

Theorem 2.8.4 (Weierstrass) *If the sequence $\{f_j\}_{j\in\mathbf{N}} \subset \mathcal{O}(\Omega)$ is locally uniformly convergent to a function $f: \Omega \longrightarrow \mathbf{C}$, then $f \in \mathcal{O}(\Omega)$. Furthermore, the sequence $\left\{\frac{\partial f_j}{\partial z_k}\right\}_{j\in\mathbf{N}}$ is locally uniformly convergent to $\left\{\frac{\partial f}{\partial z_k}\right\}$ for $k = 1, \ldots, n$.* ∎

The compactness principle (Theorem 2.2.11) implies the following property.

Theorem 2.8.5 (Montel) *Any locally uniformly bounded family in $\mathcal{O}(\Omega)$ is normal.* ∎

Next, we have two interesting consequences of the removable singularity theorem (Theorem 2.7.1).

Theorem 2.8.6 (Rado) *If $f: \Omega \longrightarrow \mathbf{C}$ is a continuous function which is holomorphic in $\Omega \setminus f^{-1}(0)$, then $f \in \mathcal{O}(\Omega)$.*

Proof Without loss of generality we may suppose that Ω is connected and $f \not\equiv 0$. Then $v = \log|f| \in \mathcal{SH}(\Omega)$. Moreover, $f^{-1}(0) = \{v = -\infty\}$. By Theorem 2.7.1, continuity of f, and Corollary 2.4.6, $\operatorname{Re} f$ and $\operatorname{Im} f$ are pluriharmonic. Since they are of class $\mathcal{C}^2$ and satisfy the Cauchy–Riemann equations on a dense subset of Ω, they satisfy these equations at each point of Ω. ∎

Similarly, we can show the following extension property.

Theorem 2.8.7 (The Riemann extension theorem) *Suppose that Ω is connected and $g \in \mathcal{O}(\Omega)$ is not identically zero in Ω. If $f \in \mathcal{O}(\Omega \setminus g^{-1}(0))$ and $|f|$ is bounded in $\Omega \setminus g^{-1}(0)$, then f can be extended to a holomorphic function in Ω.* ∎

The last two theorems can be used in the proof of the next result that provides us with another example of the significant difference between **R**- and **C**-differentiable mappings. Before stating the theorem we prove a useful topological lemma. We recall that a continuous mapping $f: X \longrightarrow Y$ between two topological Hausdorff spaces is said to be *proper* if the inverse images of compact sets in Y are compact in X.

Lemma 2.8.8 *Let $f: X \longrightarrow Y$ be a continuous mapping of a locally compact space X into a Hausdorff space Y. If E is a subset of Y such that $f^{-1}(E)$ is compact, then there exists a relatively compact open neighbourhood U of $f^{-1}(E)$ and an open set V in Y such that $f(U) \subset V$ and $f: U \longrightarrow V$ is proper.*

Proof Let B be an open neighbourhood of $f^{-1}(E)$ such that $\bar{B}$ is compact and the interior of $\bar{B}$ is equal to B. Define $V = Y \setminus f(\partial B)$ and $U = f^{-1}(V) \cap B$. Let K be a compact set in V. Put $L = f^{-1}(K) \cap U$. Clearly, $\bar{L} \subset \bar{B}$. It is enough to show that $\bar{L} \subset U$. The set $f^{-1}(K)$ is closed; therefore $\bar{L} \subset f^{-1}(K) \subset f^{-1}(V)$. Moreover, $\bar{L} \cap \partial B = \emptyset$ because $f(\bar{L} \cap \partial B) \subset f(\bar{L}) \cap f(\partial B) \subset V \cap f(\partial B) = \emptyset$. Thus $\bar{L} \subset B$. ∎

Theorem 2.8.9 (Osgood) *If $f: \Omega \longrightarrow \mathbf{C}^n$ is a holomorphic injection, then $f(\Omega)$ is open and $f: \Omega \longrightarrow f(\Omega)$ is biholomorphic.*

Proof Without loss of generality we may suppose that Ω is connected. Define

$$h_f(z) = \det\left[\frac{\partial f_k}{\partial z_j}(z)\right]_{j,k=1,\ldots,n} \qquad (z \in \Omega).$$

Let $A = h_f^{-1}(0)$. In order to prove the theorem, it suffices to show that $A = \emptyset$, and then to use the inverse mapping theorem (Theorem 1.3.1).

First, note that $A \neq \Omega$. Indeed, if this were not so, the (real) rank of the differential $d_z f$ would be equal to some number $k < 2n$ on a non-empty open subset of Ω. Therefore the rank theorem (e.g. Bröcker and Jänich

(1973), or Rudin (1976)) would imply that f is not injective.

The theorem will follow from the inverse mapping theorem if we can show that for each $a \in \Omega$ there is a neighbourhood U of a such that $U \cap A = \emptyset$.

Take $a \in \Omega$. Choose U and V as in the above lemma (with $E = \{f(a)\}$) and set $W = f(U \setminus A)$. By the inverse mapping theorem, $f|(U \setminus A) : U \setminus A \longrightarrow W$ is a biholomorphic mapping and W is an open subset of $\mathbf{C}^n$. Let $g = \left(f|(U \setminus A)\right)^{-1}$ and $h_g(z) = \det[\partial g_k(z)/\partial z_j]_{j,k=1,\ldots,n}$ for $z \in W$. Then $h_g \in \mathcal{O}(W)$ and

$$h_f(g(z))h_g(z) = 1 \qquad (z \in W).$$

Note that the function

$$H(z) = \begin{cases} 1/h_g(z) & (z \in W) \\ 0 & (z \in V \setminus W) \end{cases}$$

is continuous in V. Indeed, let $b_0 \in \partial W \cap V$, and let $b_\nu \in W \cap V$ be such that $b_\nu \longrightarrow b_0$ as $\nu \longrightarrow \infty$. Since $f|U$ is proper, $f^{-1}(\{b_\nu : \nu = 0, 1, \ldots\}) \cap U$ is a compact subset of U that contains $\{g(b_\nu) : \nu = 1, 2, \ldots\}$. Hence every accumulation point of the set $\{g(b_\nu) : \nu = 1, 2, \ldots\}$ belongs to $A \cap U$. Therefore $h_f(g(b_\nu)) \longrightarrow 0$ as $\nu \longrightarrow \infty$.

By Rado's theorem, $H \in \mathcal{O}(V)$. Since U is bounded, we can use the Riemann extension theorem to extend g to a holomorphic mapping $\tilde{g}$ on V (with values in $\bar{U}$). As $f \circ g = \mathrm{id}_W$ and W is dense in V, we have $f \circ \tilde{g} = \mathrm{id}_V$. This shows that $(f|U)^{-1} \in \mathcal{O}(V)$. Consequently, $A \cap U = \emptyset$.∎

We close this section with the Hartogs theorem about separately holomorphic functions. Our presentation is a modification of that given by Hörmander (1973). Let Ω be an open set in $\mathbf{C}^n$, where $n \geq 2$. A function $f : \Omega \longrightarrow \mathbf{C}$ is said to be *separately holomorphic* if it is holomorphic with respect to each variable separately when the other variables are fixed.

Theorem 2.8.10 *Let Ω be an open set in $\mathbf{C}^n$, where $n \geq 2$. If a function $f : \Omega \longrightarrow \mathbf{C}$ is separately holomorphic, then it is holomorphic.*

Clearly, separately $\mathcal{C}^\infty$ functions do not enjoy a similar property. For instance, $g : \mathbf{R}^2 \longrightarrow \mathbf{R}$ defined by the formula

$$g(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & ((x, y) \neq (0, 0)) \\ 0 & ((x, y) = (0, 0)) \end{cases}$$

is separately $\mathcal{C}^\infty$ and is not even continuous at the origin.

It can be shown that the separately harmonic functions are harmonic, and that separately real analytic functions which satisfy some additional

conditions are real analytic (Lelong 1961; Siciak 1969). Arsove (1966) proved that if a separately subharmonic function has a locally integrable majorant, it is subharmonic. Other related results were obtained by Lelong (1945), Avanissian (1961), and Imomkulov (1990). Wiegerinck (1988) constructed a function on $\mathbf{C}^2$ which is subharmonic with respect to each complex variable but not subharmonic as a function of four real variables. He considered the sequence $a_j = (1/j)e^{1/(j+1)}$, $j \in \mathbf{N}$, and the sets

$$K_j = \{z \in \bar{D}(0, j) : \ 1/j \leq \ \arg z \leq 2\pi\} \cup \{0\} \qquad (j \in \mathbf{N}).$$

In view of Runge's theorem (in one complex variable), one can find a sequence of complex polynomials $P_j : \mathbf{C} \longrightarrow \mathbf{C}$ such that $P_j(a_j) = j + 1$ and $\|P_j\|_{K_j} < 1/2$. Define $v_j = \max\{|P_j| - 1, 0\}$ for $j \in \mathbf{N}$, and

$$v_j(z, w) = \sum_{j=1}^{\infty} v_j(z)v_j(w) \qquad ((z, w) \in \mathbf{C}^2).$$

The function v is well-defined, because for each $z \in \mathbf{C}$ only finitely many of the numbers $v_j(z)$ are different from zero. It is easy to check that v is subharmonic with respect to each (complex) variable separately when the other variable is fixed. On the other hand, v is not upper semicontinuous because $\lim_{j\to\infty} v(a_j, a_j) = \infty$. Consequently, v is not subharmonic.

Proof (of Theorem 2.8.10) First, note that if f is separately holomorphic and locally bounded, then f is continuous. Indeed, suppose that $\bar{P}(a, 2r) \subset \Omega$ and $|f| \leq M$ in the polydisc. Then, by Cauchy's integral formula in one variable, if $w = (w_1, \ldots, w_n), z = (z_1, \ldots, z_n) \in P(a, r)$, we have

$$\begin{aligned} &|f(z) - f(w)| \\ &\leq \sum_{j=1}^{n} \left| f(w_1, \ldots, w_j, z_{j+1}, \ldots, z_n) - f(w_1, \ldots, w_{j-1}, z_j, \ldots, z_n) \right| \\ &= \frac{1}{2\pi} \sum_{j=1}^{n} \left| \int_{\partial D(a_j, 2r)} \left(\frac{f(w_1, \ldots, w_{j-1}, \zeta, z_{j+1}, \ldots, z_n)}{\zeta - w_j} \right.\right. \\ &\qquad\qquad \left.\left. - \frac{f(w_1, \ldots, w_{j-1}, \zeta, z_{j+1}, \ldots, z_n)}{\zeta - z_j} \right) d\zeta \right| \\ &\leq \frac{M}{2\pi} \sum_{j=1}^{n} \frac{|w_j - z_j|}{r^2} 2\pi 2r \leq \frac{2Mn}{r} \|w - z\|. \end{aligned}$$

Secondly, if f is separately holomorphic and continuous, it satisfies the Cauchy integral formula (1.1.2); consequently, it is $\mathcal{C}^\infty$ and satisfies the Cauchy–Riemann equations.

Therefore, in order to prove the Hartogs fundamental theorem, it is enough to show that every separately holomorphic function is locally bounded. We shall proceed by induction. The one-dimensional case is trivial. Suppose that the theorem is true for functions of $(n-1)$ variables (with $n>1$). We shall show that then it must also be true for functions of n variables. It is convenient to employ the following notation here: if $z=(z_1,\dots,z_n)\in\mathbf{C}^n$, then $z'=(z_1,\dots,z_{n-1})$ and so $z=(z',z_n)$.

The proof will be divided into three steps.

Step 1 *If $D_1,\dots,D_n$ are open discs such that $\bar D_1\times\dots\times\bar D_n\subset\Omega$, then there is an open disc $\tilde D_n\subset D_n$ such that $|f|$ is bounded in $D_1\times\dots\times D_{n-1}\times\tilde D_n$.*

To see this, define for each $m\in\mathbf{N}$ the set

$$F_m=\left\{z_n\in\bar D_n:\ \forall z'\in\bar D_1\times\dots\times\bar D_{n-1},\ |f(z',z_n)|\le m\right\}.$$

Clearly, F_m is closed for each m. In view of the inductive hypothesis, $\bigcup_{m\ge1}F_m=\bar D_n$. By Baire's theorem (see, e.g. Royden (1963)), for some m, the interior of F_m is non-empty. Now it suffices to take a disc $\tilde D_n\subset F_m$.

Step 2 *Let $P(a,R)\subset\Omega$, and suppose that, for some $r>0$, $f\in\mathcal{O}\big(P(a',R)\times D(a_n,r)\big)$. Then $f\in\mathcal{O}\big(P(a,R)\big)$.*

Without loss of generality we may suppose that $a=0\in\mathbf{C}^n$ and $|f|$ is bounded by a constant $M>1$ in $P(0',R)\times D(0_n,r)$. Choose R_1, R_2 such that $0<R_1<R_2<R$. We know that

$$f(z)=\sum_{j=0}^{\infty}c_j(z')z_n^j\qquad\big(z\in P(0,R)\big),\tag{2.8.1}$$

where

$$c_j(z')=\left(\frac{\partial}{\partial z_n}\right)^j\frac{f(z',0_n)}{j!}.$$

Moreover, $c_j\in\mathcal{O}\big(P(0',R)\big)$ and, by Cauchy's estimates,

$$|c_j(z')|\le\frac{M}{r^j}.$$

Define

$$u_j(z')=\frac{1}{j}\log|c_j(z')|\qquad\big(z'\in P(0',R)\big).$$

Then $u_j\in\mathcal{SH}\big(P(0',R)\big)$ and

$$u_j\le\log\frac{M}{r}.$$

Also, as the series (2.8.1) is convergent, $|c_j(z')|R_2^j$ converges to zero as j tends to ∞, and so

$$\limsup_{j\to\infty}u_j(z')\le-\log R_2\qquad\big(z'\in P(0',R)\big).$$

By the Hartogs lemma (Theorem 2.6.4), $u_j\le-\log R_1$ in $P(0',R_1)$, provided that j is sufficiently large. In other words,

$$|c_j(z')|R_1^j\le1$$

for large j and $z'\in P(0',R_1)$. Therefore we have shown that the series $\sum_{j=1}^{\infty}c_j(z')z_n^j$ converges absolutely and uniformly on compact sets in the polydisc $P(0,R)$; the sum of the series is then holomorphic by the Weierstrass theorem (Theorem 2.8.4).

Step 3 *f is holomorphic in Ω.*

Given $\zeta\in\Omega$, we can take $R>0$ such that $P(\zeta,2R)\subset\Omega$. In view of Step 1, there is an open disc $D(a_n,r)\subset D(\zeta_n,R)$ such that $f\in\mathcal{O}\big(P(\zeta',R)\times D(a_n,r)\big)$. By Step 2, $f\in\mathcal{O}\Big(P\big((\zeta',a_n),R\big)\Big)$. As $\zeta\in P\big((\zeta',a_n),R\big)$, it means that f is holomorphic in a neighbourhood of ζ. ∎

Finally, we shall give another useful application of the Hartogs lemma.

Let P be a *complex polynomial* in $\mathbf{C}^n$, i.e. a polynomial in n-complex variables that has complex coefficients. We say that P is *homogeneous of degree* $m\in\mathbf{Z}_+$ if $P(\zeta z)=\zeta^mP(z)$ for all $\zeta\in\mathbf{C}$ and $z\in\mathbf{C}^n$.

Suppose that f is a holomorphic function defined on an open polydisc P centered at 0. Then f can be expanded into a power series

$$f(z)=\sum_{\alpha\in\mathbf{Z}_+^n}a_\alpha z^\alpha\qquad(z\in P)$$

which is absolutely and uniformly convergent on compact subsets of P. If we put

$$f_j(z)=\sum_{|\alpha|=j}a_\alpha z^\alpha\qquad(z\in\mathbf{C}^n,\ j\in\mathbf{Z}_+),$$

then f_j is a homogeneous polynomial of degree j, for $j>0$, and the series

$$f(z)=\sum_{j=0}^{\infty}f_j(z)\qquad(z\in P)\tag{2.8.2}$$

is uniformly convergent on compact subsets of P. We say that the series in (2.8.2) is the *homogeneous expansion* of f (in P).

Recall that a set $S\subset\mathbf{C}^n$ is said to be *balanced* if for each $z\in S$ and each $\zeta\in\bar D(0,1)$, $\zeta z\in S$.

Theorem 2.8.11 *Let Ω be a balanced neighbourhood of the origin in $\mathbf{C}^n$. If $f\in\mathcal{O}(\Omega)$, then f has a locally uniformly convergent homogeneous expansion in Ω.*

Proof Choose $r > 0$ so that $\bar{P}(0,r) \subset \Omega$. According to what has been said, the homogeneous expansion $f = \sum_j f_j$ exists and is uniformly convergent on $\bar{P}(0,r)$.

Let $K \subset \Omega$ be compact, and let $t > 1$ be a number such that $t^2 K \subset \Omega$. Consider the subharmonic functions $u_j = |f_j|^{1/j}$ for $j \in \mathbf{N}$. If $M > 0$ is such that $\|f_j\|_{\bar{P}(0,r)} \leq M$ for all j, then — due to homogeneity — we have the following estimates

$$u_j(z) \leq M^{1/j} \frac{|z|}{r} \qquad (j \in \mathbf{N},\, z \in \mathbf{C}^n).$$

Hence the family $\{u_j\}$ is locally uniformly bounded from above. Furthermore, if $z \in \mathbf{C}^n$, the function $\lambda \longmapsto f(\lambda z)$ is holomorphic in a neighbourhood of the closed unit disc, and thus the series

$$f(\lambda z) = \sum_{j=0}^{\infty} f_j(\lambda z) = \sum_{j=0}^{\infty} \lambda^j f_j(z)$$

is absolutely convergent. By Cauchy's convergence criterion (for numerical series),

$$\limsup_{j\to\infty} u_j(z) \leq 1 \qquad (z \in \Omega).$$

By the Hartogs lemma there exists an integer j_0 such that for $j \geq j_0$

$$u_j(z) \leq t \qquad (z \in t^2 K).$$

Hence, if $z \in K$ and $j \geq j_0$, then

$$|f_j(z)| = t^{-2j} |f_j(t^2 z)| \leq t^{-j}.$$

Consequently, the series $\sum_j f_j$ is uniformly convergent on K. ∎

2.9 PLURISUBHARMONIC FUNCTIONS

Let Ω be an open subset of $\mathbf{C}^n$, and let $u : \Omega \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of Ω. The function u is said to be *plurisubharmonic* if for each $a \in \Omega$ and $b \in \mathbf{C}^n$, the function $\lambda \longmapsto u(a + \lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbf{C} :\ a + \lambda b \in \Omega\}$. In this case, we write $u \in \mathcal{PSH}(\Omega)$. If $u \in \mathcal{C}^2(\Omega)$, $a \in \Omega$, and $b \in \mathbf{C}^n$, then

$$4\langle \mathcal{L}u(a)b, b\rangle = \Delta_\lambda\big(u(a + \lambda b)\big)\big|_{\lambda=0}. \tag{2.9.1}$$

Consequently, the above definition is consistent with that given in the first chapter.

The following characterization of plurisubharmonic functions is often useful.

Theorem 2.9.1 *Let $u :\ \Omega \longrightarrow [-\infty, \infty)$ be upper semicontinuous and not identically $-\infty$ on any connected component of $\Omega \subset \mathbf{C}^n$. Then $u \in \mathcal{PSH}(\Omega)$ if and only if for each $a \in \Omega$ and $b \in \mathbf{C}^n$ such that*

$$\{a + \lambda b :\ \lambda \in \mathbf{C}, |\lambda| \leq 1\} \subset \Omega,$$

we have

$$u(a) \leq l\,(u; a, b), \tag{2.9.2}$$

where

$$l\,(u; a, b) = \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{it} b)\,dt. \tag{2.9.3}$$

Moreover, plurisubharmonicity is a local property.

Proof The first part follows directly from the definition of plurisubharmonic functions and Theorem 2.4.1, since

$$l\,(u; a, b) = \mathbf{L}(v; 0, 1),$$

where $v(\lambda) = u(a + \lambda b)$. The second part is obvious, as subharmonicity is a local property. ∎

A number of important properties of plurisubharmonic functions can be derived from the next result. Similarly as in the case of subharmonic functions, we shall call it the *main approximation theorem for plurisubharmonic functions.* The notation we use here is the same as in Section 2.5.

Theorem 2.9.2 *Let Ω be an open subset of $\mathbf{C}^n$, and let $u \in \mathcal{PSH}(\Omega)$. If $\varepsilon > 0$ is such that $\Omega_\varepsilon \neq \emptyset$, then $u * \chi_\varepsilon \in \mathcal{C}^\infty \cap \mathcal{PSH}(\Omega_\varepsilon)$. Moreover, $u * \chi_\varepsilon$ monotonically decreases with decreasing ε, and $\lim_{\varepsilon\to 0} u * \chi_\varepsilon(z) = u(z)$ for each $z \in \Omega$.*

The proof will proceed along the same lines as the proof of the main approximation theorem for subharmonic functions. But first, we need a suitable replacement for commutativity of integral averages and convolutions (2.5.6).

Lemma 2.9.3 *Let $\Omega \subset \mathbf{C}^n$ be an open set, and let $u \in L^1_{\mathrm{loc}}(\Omega)$. Suppose that $a \in \Omega$, $b \in \mathbf{C}^n$, and $\{a + \lambda b :\ \lambda \in \mathbf{C}, |\lambda| \leq 1\} \subset \Omega$. Then*

$$\big(l\,(u;\ \cdot\,, b) * \chi_\varepsilon\big)(a) = l\,(u * \chi_\varepsilon; a, b). \tag{2.9.4}$$

Proof The left-hand side of (2.9.4) is equal to

$$\int_{\mathbf{C}^n}\left(\frac{1}{2\pi}\int_0^{2\pi} u(a+e^{it}b-w)dt\right)\chi_\varepsilon(w)d\lambda(w)\,.$$

By Fubini's theorem, this is equal to the right-hand side of (2.9.4). ∎

Now we can prove the theorem.

Proof By Proposition 2.5.2 (i), $u*\chi_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$. Theorem 2.9.1, in conjunction with the above lemma, implies that $u*\chi_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon)$. Using the same argument as in the proof of (2.5.7), with respect to each variable separately, we can prove (by induction on j) the following estimate:

$$(u*\chi_{\varepsilon_1})(z) \geq \int_{\mathbf{C}^{n-1}} I(w_1,\dots,w_{j-1},w_{j+1},\dots,w_n)d\lambda(w_1,\dots,w_{j-1},w_{j+1},\dots,w_n),$$

where

$$I(w_1,\dots,w_{j-1},w_{j+1},\dots,w_n) = \int_{\mathbf{C}} u(z_1+\varepsilon_2 w_1,\dots,z_j+\varepsilon_2 w_j, z_{j+1}+\varepsilon_1 w_{j+1},\dots,z_n+\varepsilon_1 w_n)\chi(w)d\lambda(w_j)\,,$$

$0\leq\varepsilon_2<\varepsilon_1$, and $z=(z_1,\dots,z_n)\in\Omega_{\varepsilon_1}$. Hence $(u*\chi_{\varepsilon_1})(z)\geq(u*\chi_{\varepsilon_2})(z)\geq u(z)$. The rest of the proof is the same as in Theorem 2.5.5. ∎

A natural question arises in connection with the theorem. Suppose that Ω is an open set in $\mathbf{C}^n$ and $u\in\mathcal{PSH}(\Omega)$. Could we find a decreasing sequence $\{u_j\}_{j\in\mathbf{N}}\subset\mathcal{PSH}\cap\mathcal{C}^\infty(\Omega)$ which is pointwise convergent to u? (Note that in the theorem the domain Ω_ε of u_ε is smaller than Ω if $\Omega\neq\mathbf{C}^n$.) In general, the answer is 'no', as shown by the following example due to Fornaess (Bedford 1982); (see also Cegrell 1978*a*, Example 2, p.321).

Example 2.9.4: Consider the connected open set $\Omega\subset\mathbf{C}^2$ given by the formula

$$\Omega=\left[\left(D(0,2)\setminus\partial D(0,1)\right)\times D(0,1)\right]\cup\bigcup_{j=2}^{\infty}\partial D(0,1)\times D(1/j,e^{-e^j})\,.$$

Note that $D(1/j,e^{-e^j})\cap D(1/k,e^{-e^k})=\emptyset$ if $k\neq j$. (It is easy to check by induction that $2j(j+1)\leq 2^{2^j}$ for $j\in\mathbf{N}$; therefore $1/j-1/(j+1)\geq 2/2^{2^j}>e^{-e^{j+1}}+e^{-e^j}$, which means that the jth and $(j+1)$th discs are disjoint.) Clearly, Ω contains the discs

$$D_j=D(0,2)\times\left\{\frac{1}{j}\right\}\qquad(j=2,3,\dots)$$

and the set

$$D_0=\left(D(0,2)\setminus\partial D(0,1)\right)\times\{0\}\,.$$

Note that D_j 'converge' in Ω to D_0 as j tends to ∞. If $u\in\mathcal{PSH}\cap\mathcal{C}(\Omega)$, then, by the maximum principle (in the disc $D(0,3/2)\times\{1/j\}\subset D_j$),

$$u\left(0,\frac{1}{j}\right)\leq\sup_{|z|=3/2}u\left(z,\frac{1}{j}\right);$$

thus, by continuity,

$$u(0,0)\leq\sup_{|z|=3/2}u(z,0)\,. \tag{2.9.5}$$

Consequently, (2.9.5) holds also for all $u\in\mathcal{PSH}(\Omega)$ that can be expressed as pointwise limits of decreasing sequences in $\mathcal{PSH}\cap\mathcal{C}(\Omega)$. Therefore it would be sufficient to find a function $u\in\mathcal{PSH}(\Omega)$ that does not satisfy (2.9.5).

Define

$$\psi(w)=\sum_{j=2}^{\infty}\frac{2^{-j}}{\log j}\log\left|w-\frac{1}{j}\right|\qquad(w\in\mathbf{C})\,.$$

Then $\psi\in\mathcal{SH}(\mathbf{C})$ and $\psi(0)=-1/2$. Furthermore, $\psi|D(1/j,e^{-e^j})\leq-1$ for $j\geq2$. (Indeed, if $w\in D(1/j,e^{-e^j})$, then $(2^{-j}\log|w-1/j|)/\log j\leq-1$, and $(2^{-k}\log|w-1/k|)/\log k<0$ for $k\neq j$.) Hence the function

$$u(z,w)=\begin{cases}\max\{\psi(w),-1\} & ((z,w)\in\Omega\text{ and }|z|<1)\\ -1 & ((z,w)\in\Omega\text{ and }|z|\geq1)\end{cases}$$

is plurisubharmonic in Ω. Moreover, $u(0,0)=-1/2$ and $u(z,0)=-1$ if $|z|=3/2$, and so u does not satisfy (2.9.5). ∎

It was shown by Richberg (1968) that if Ω is a bounded domain in $\mathbf{C}^n$ and $u\in\mathcal{C}(\Omega)$, then the answer to our question is 'yes'. Fornaess (1983) proved that this is no longer true if the assumption of continuity of u is dropped. On the other hand, according to Fornaess and Narasimhan (1980), the answer is 'yes' if Ω is pseudoconvex. (See next section for definition of pseudoconvexity.)

Further discussion of this and other related problems can be found in Fornaess (1982), Fornaess and Sibony (1986), and Fornaess and Stensønes (1987). It should be mentioned that though plurisubharmonic functions are generally not differentiable, a notion of a tangent space to a plurisubharmonic function can be introduced (Kiselman 1988).

Now we shall look at some consequences of the main approximation theorem.

In view of that theorem and formula (1.4.4), plurisubharmonicity is preserved by holomorphic substitutions.

Corollary 2.9.5 *Let Ω and Ω' be open sets in $\mathbf{C}^n$ and $\mathbf{C}^k$, respectively. If $u \in \mathcal{PSH}(\Omega)$ and $f: \Omega' \longrightarrow \Omega$ is a holomorphic mapping, then $u \circ f$ is plurisubharmonic in Ω'.* ∎

Corollary 2.9.6 *If Ω is an open subset of $\mathbf{C}^n$, then $\mathcal{PH}(\Omega) \subset \mathcal{PSH}(\Omega) \subset \mathcal{SH}(\Omega) \subset L^1_{\text{loc}}(\Omega)$.*

Proof The first inclusion is obvious. If u is of class $\mathcal{C}^2$, then, in view of (2.9.1), $u \in \mathcal{SH}(\Omega)$. Thus the middle inclusion follows from the main approximation theorem. The last one has already been shown (Corollary 2.4.7). ∎

If $n = 1$, $\mathcal{PSH} = \mathcal{SH}$, but in view of Example 2.2.12, $\mathcal{PSH} \neq \mathcal{SH}$ for $n > 1$.

Corollary 2.9.7 *Let Ω be an open subset of $\mathbf{C}^n$, and let $u: \Omega \longrightarrow \mathbf{R}$ be a function. Then $u \in \mathcal{PH}(\Omega)$ if and only if u and $-u$ are plurisubharmonic in Ω.*

Proof If $-u$ and u are plurisubharmonic, then, by Corollaries 2.9.6 and 2.4.3, $u \in \mathcal{C}^2(\Omega)$. Therefore $\langle \mathcal{L}u(a)b, b\rangle = 0$ for all eligible a, b, and so $u \in \mathcal{PH}(\Omega)$. The opposite implication is trivial. ∎

The fact that plurisubharmonic functions are subharmonic enables us to state several other properties (see also Section 2.4).

Corollary 2.9.8 *If $u, v \in \mathcal{PSH}(\Omega)$ and $u = v$ almost everywhere in Ω, then $u \equiv v$.* ∎

Corollary 2.9.9 *Plurisubharmonic functions satisfy the maximum principle in bounded domains, i.e. if Ω is a bounded connected open subset of $\mathbf{C}^n$ and $u \in \mathcal{PSH}(\Omega)$, then either u is constant or, for each $z \in \Omega$,*

$$u(z) < \sup_{w \in \partial\Omega} \left\{ \limsup_{\substack{y \to w \\ y \in \Omega}} u(y) \right\}.$$ ∎

Sometimes, the boundedness requirement for Ω can be relaxed (see Gauthier *et al.* 1988).

A set $E \subset \mathbf{C}^n$ is said to be *pluripolar* if for each point $a \in E$ there is a neighbourhood V of a and a function $u \in \mathcal{PSH}(V)$ such that $E \cap V \subset \{z \in V : u(z) = -\infty\}$.

Corollary 2.9.10 *Pluripolar sets have Lebesgue measure zero.* ∎

Plurisubharmonicity can also be characterized in terms of distributional derivatives.

Theorem 2.9.11 *If $\Omega \subset \mathbf{C}^n$ is open and $u \in \mathcal{PSH}(\Omega)$, then, for each $b = (b_1, \ldots, b_n) \in \mathbf{C}^n$,*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} b_j \bar{b}_k \geq 0$$

in Ω, in the sense of of distributions, i.e.

$$\int_\Omega u(z) \langle \mathcal{L}\varphi(z)b, b\rangle d\lambda(z) \geq 0$$

for any non-negative test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Conversely, if $v \in L^1_{\text{loc}}(\Omega)$ is such that for each $b = (b_1, \ldots, b_n) \in \mathbf{C}^n$

$$\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \bar{z}_k} b_j \bar{b}_k \geq 0 \tag{2.9.6}$$

in Ω, in the sense of distributions, then the function $u = \lim_{\varepsilon \to 0}(v * \chi_\varepsilon)$ is well-defined, plurisubharmonic in Ω, and equal to v almost everywhere in Ω.

Proof Let $u \in \mathcal{PSH}(\Omega)$, and let $u_\varepsilon = u * \chi_\varepsilon$ for $\varepsilon > 0$. Take a non-negative test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and a vector $b = (b_1, \ldots, b_n) \in \mathbf{C}^n$. Lebesgue's dominated convergence theorem, combined with integration by parts and the main approximation theorem, shows that

$$\begin{aligned} \int_\Omega u(z)\langle \mathcal{L}\varphi(z)b, b\rangle d\lambda(z) &= \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(z)\langle \mathcal{L}\varphi(z)b, b\rangle d\lambda(z) \\ &= \lim_{\varepsilon \to 0} \int_\Omega \langle \mathcal{L}u_\varepsilon(z)b, b\rangle \varphi(z) d\lambda(z) \geq 0, \end{aligned}$$

which proves the first part of the theorem.

Suppose that $v \in L^1_{\text{loc}}(\Omega)$, and that (2.9.6) is satisfied. Let $v_\varepsilon = v * \chi_\varepsilon$ for $\varepsilon > 0$. Then, in particular, $\Delta v \geq 0$ in Ω, in the sense of distributions. By Theorem 2.5.8, there is a (unique) subharmonic function u on Ω that coincides with v almost everywhere, and $u = \lim_{\varepsilon\to 0} v_\varepsilon$. Fubini's theorem and (2.9.6) imply that

$$\int_\Omega \langle \mathcal{L}v_\varepsilon(z)b, b\rangle \varphi(z) d\lambda(z) \geq 0$$

for all $b \in \mathbf{C}^n$, $\varphi \in \mathcal{C}_0^\infty(\Omega_\varepsilon)$, $\varphi \geq 0$. Therefore $\langle \mathcal{L}v_\varepsilon(z)b, b\rangle \geq 0$ for all $z \in \Omega_\varepsilon$, $b \in \mathbf{C}^n$, and thus $v_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon)$. As $v_{\varepsilon_1} < v_{\varepsilon_2}$ if $\varepsilon_1 < \varepsilon_2$, the limit function u is plurisubharmonic. ∎

We already know that the compositions of plurisubharmonic and holomorphic functions are plurisubharmonic. The next result can be regarded as a converse statement.

Theorem 2.9.12 *A function $u : \Omega \longrightarrow [-\infty, \infty)$ defined on an open set $\Omega \subset \mathbf{C}^n$ is plurisubharmonic in Ω if and only if $u \circ T$ is subharmonic in $T^{-1}(\Omega)$ for every $\mathbf{C}$-linear isomorphism $T : \mathbf{C}^n \longrightarrow \mathbf{C}^n$.*

Proof According to Corollary 2.9.5, the above condition is necessary. To prove that it is sufficient, take $b \in \mathbf{C}^n \setminus \{0\}$ and $v_2, \ldots, v_n \in \mathbf{C}^n$ such that $\{b, v_2, \ldots, v_n\}$ is an orthogonal basis for $\mathbf{C}^n$. For $\varepsilon > 0$, define

$$T_\varepsilon(z_1, \ldots, z_n) = z_1 b + \varepsilon \sum_{j=2}^n z_j v_j\,,$$

where $(z_1, \ldots, z_n) \in \mathbf{C}^n$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis for $\mathbf{C}^n$. Take a non-negative test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$. By (1.4.4), Proposition 1.3.2, and change of variables, we have

$$\begin{aligned}
&\int_\Omega u(z)\langle \mathcal{L}\varphi(z)b, b\rangle d\lambda(z) + \varepsilon^2 \sum_{j=2}^n \int_\Omega u(z)\langle \mathcal{L}\varphi(z)v_j, v_j\rangle d\lambda(z) \\
&= \sum_{j=1}^n \int_\Omega u(z)\langle \mathcal{L}\varphi(z)T_\varepsilon(e_j), T_\varepsilon(e_j)\rangle d\lambda(z) \\
&= |\det T_\varepsilon|^2 \int_{T_\varepsilon^{-1}(\Omega)} \left((u\circ T_\varepsilon)(w) \sum_{j=1}^n \langle \mathcal{L}(\varphi\circ T_\varepsilon)(w)e_j, e_j\rangle \right) d\lambda(w) \\
&= \frac{1}{4}|\det T_\varepsilon|^2 \int_{T_\varepsilon^{-1}(\Omega)} (u\circ T_\varepsilon)(w)\Delta(\varphi\circ T_\varepsilon)(w) d\lambda(w) \geq 0\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this means that

$$\int_\Omega u(z)\langle \mathcal{L}\varphi(z)b, b\rangle d\lambda(z) \geq 0\,.$$

Hence, by the previous theorem, u is equal to a plurisubharmonic function v almost everywhere in Ω. By Corollary 2.5.6, $u \equiv v \in \mathcal{PSH}(\Omega)$. ∎

Corollary 2.9.13 *Let Ω be an open subset of $\mathbf{C}^n$. A function $u \in \mathcal{C}^2(\Omega)$ is pluriharmonic in Ω if and only if $u \circ T$ is harmonic in $T^{-1}(\Omega)$ for every $\mathbf{C}$-linear isomorphism $T : \mathbf{C}^n \longrightarrow \mathbf{C}^n$.* ∎

The above theorem and corollary, combined with Corollary 2.9.5, justify our previous statement that plurisubharmonic (respectively, pluriharmonic) functions are those among subharmonic (respectively, harmonic) functions that are invariant with respect to biholomorphic substitutions.

Most of the results from Sections 2.6 and 2.7 carry over to the plurisubharmonic case. We list these properties here without proofs; all of them can be derived easily from the subharmonic case, and either the definition of plurisubharmonic functions or the invariance criterion described in Theorem 2.9.12.

Theorem 2.9.14 *Let Ω be an open subset of $\mathbf{C}^n$.*

(i) *The family $\mathcal{PSH}(\Omega)$ is a convex cone, i.e. if α, β are non-negative numbers and $u, v \in \mathcal{PSH}(\Omega)$, then $\alpha u + \beta v \in \mathcal{PSH}(\Omega)$.*
(ii) *If Ω is connected and $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{PSH}(\Omega)$ is a decreasing sequence, then $u = \lim_{j\to\infty} u_j \in \mathcal{PSH}(\Omega)$ or $u \equiv -\infty$.*
(iii) *If $u : \Omega \longrightarrow \mathbf{R}$, and if $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{PSH}(\Omega)$ converges to u uniformly on compact subsets of Ω, then $u \in \mathcal{PSH}(\Omega)$.*
(iv) *Let $\{u_\alpha\}_{\alpha\in A} \subset \mathcal{PSH}(\Omega)$ be such that its upper envelope $u = \sup_{\alpha\in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization u^* is plurisubharmonic in Ω.* ∎

Corollary 2.9.15 *Let Ω be an open set in $\mathbf{C}^n$, and let ω be a non-empty proper open subset of Ω. If $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PSH}(\omega)$, and $\limsup_{x\to y} v(x) \leq u(y)$ for each $y \in \partial\omega \cap \Omega$, then the formula*

$$w = \begin{cases} \max\{u, v\} & \text{in } \omega \\ u & \text{in } \Omega \setminus \omega \end{cases}$$

defines a plurisubharmonic function in Ω. ∎

Let Ω be an open subset of $\mathbf{C}^n$.

We say that a holomorphic mapping $f : \Omega \longrightarrow \mathbf{C}^m$ is *non-degenerate* in Ω if in each connected component of Ω one can find a point z such that the (complex) rank of $\partial_z f$ is m.

The next proposition shows that regularizations of families of plurisubharmonic functions behave well under holomorphic changes of variables (Klimek 1982*a* or 1982*b*).

Proposition 2.9.16 *Let $f : \Omega \longrightarrow \mathbf{C}^m$ be a non-degenerate holomorphic mapping on an open set $\Omega \subset \mathbf{C}^n$, and let Ω' be an open neighbourhood of $f(\Omega)$ in $\mathbf{C}^m$. Let $\{u_\alpha\}_{\alpha\in A} \subset \mathcal{PSH}(\Omega')$ be such that its upper envelope $u = \sup_{\alpha\in A} u_\alpha$ is locally upper bounded. Then*

$$(u \circ f)^* = (u^* \circ f).$$

Proof Denote by A the zero set of the Jacobian of f, i.e.

$$A = \{z \in \Omega : \det \partial_z f = 0\}.$$

Since $z \longmapsto \det \partial_z f$ is a holomorphic function, A is pluripolar; hence, by Corollary 2.9.10, A is of Lebesgue measure zero. As the restriction of the mapping f to $\Omega \setminus A$ is open (by the inverse mapping theorem) and continuous, we have:

$$\begin{aligned}(u \circ f)^*(a) &= \lim_{\varepsilon\to 0} \sup \{u(f(z)) : z \in B(a,\varepsilon)\} \\ &= \lim_{\varepsilon\to 0} \sup \{u(w) : w \in f(B(a,\varepsilon))\} \\ &= (u^* \circ f)(a),\end{aligned}$$

for any $a \in \Omega\setminus A$. Therefore $(u\circ f)^* = (u^*\circ f)$ almost everywhere in Ω. Also, $(u \circ f)^*$, $(u^* \circ f) \in \mathcal{PSH}(\Omega)$. Thus, by Corollary 2.9.8, $(u \circ f)^* = (u^* \circ f)$ in Ω. ∎

Proposition 2.9.17 *Let the sequence $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{PSH}(\Omega)$ be locally uniformly bounded above. Define $u(z) = \limsup_{j\to\infty} u_j(z)$ for $z \in \Omega$. Then the upper semicontinuous regularization u^* is plurisubharmonic in Ω.* ∎

Theorem 2.9.18 *Let $\{u_j\}_{j\in\mathbf{N}} \subset \mathcal{PSH}(\Omega)$ be locally uniformly bounded above in $\Omega \subset \mathbf{C}^n$. Suppose that*

$$\limsup_{j\to\infty} u_j(z) \le M$$

for each $z \in \Omega$ and some constant M. Then, for each $\varepsilon > 0$ and each compact set $K \subset \Omega$, there exists a natural number j_0 such that, for $j \ge j_0$,

$$\sup_{z\in K} u_j(z) \le M + \varepsilon.$$ ∎

Theorem 2.9.19 *Let Ω be an open subset of $\mathbf{C}^n$.*
(i) *Let u, v be pluriharmonic in Ω and $v > 0$. If $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ is convex, then $v\phi(u/v)$ is plurisubharmonic in Ω.*

(ii) *Let $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PH}(\Omega)$, and $v > 0$ in Ω. If $\phi : \mathbf{R} \longrightarrow \mathbf{R}$ is convex and increasing, then $v\phi(u/v)$ is plurisubharmonic in Ω. ($\phi(-\infty)$ is interpreted as $\lim_{x\to-\infty} \phi(x)$.)*
(iii) *Let $u, -v \in \mathcal{PSH}(\Omega)$, $u \ge 0$ in Ω, and $v > 0$ in Ω. If $\phi : [0,\infty) \longrightarrow [0,\infty)$ is convex and $\phi(0) = 0$, then $v\phi(u/v) \in \mathcal{PSH}(\Omega)$.* ∎

Corollary 2.9.20 *If $u \in \mathcal{PSH}(\Omega)$, then also $e^u \in \mathcal{PSH}(\Omega)$. If $u \in \mathcal{PSH}(\Omega)$ and $u \ge 0$, then $u^\alpha \in \mathcal{PSH}(\Omega)$ for any number $\alpha \ge 1$.* ∎

Corollary 2.9.21 *If u_1, u_2 are non-negative functions in $\Omega \subset \mathbf{C}^n$ and $\log u_1, \log u_2 \in \mathcal{PSH}(\Omega)$, then $u_1u_2 \in \mathcal{PSH}(\Omega)$ and $\log(u_1 + u_2) \in \mathcal{PSH}(\Omega)$.* ∎

Theorem 2.9.22 *Let Ω be an open subset of $\mathbf{C}^n$, and let F be a closed subset of Ω of the form $F = \{z \in \Omega : v(z) = -\infty\}$, where $v \in \mathcal{PSH}(\Omega)$. If $u \in \mathcal{PSH}(\Omega \setminus F)$ is bounded above, then the function $\tilde{u}$ defined by the formula*

$$\tilde{u}(z) = \begin{cases} u(z) & (z \in \Omega \setminus F) \\ \limsup\limits_{\substack{y\to z \\ y\notin F}} u(y) & (z \in F) \end{cases}$$

is plurisubharmonic in Ω. If u is pluriharmonic and bounded in $\Omega\setminus F$, then $\tilde{u}$ is pluriharmonic in Ω. If Ω is connected, then so is $\Omega \setminus F$. ∎

Proposition 2.9.23 *If $u \in \mathcal{PSH}(\mathbf{C}^n)$ and u is bounded above, then u is constant.* ∎

The removable singularity theorem allows us to find the exact form of the Taylor expansion of order 2 of some plurisubharmonic functions (Klimek 1989). The proof was derived from the proof of a version of the Schwarz lemma obtained by Sibony (1981).

Proposition 2.9.24 *Let u be a non-negative function of class C^2 defined on a neighbourhood V of $0 \in \mathbf{C}^n$ such that $\log u \in \mathcal{PSH}(V)$ and $u(0) = 0$. Then*

$$u(z) = \langle \mathcal{L}u(0)z, z\rangle + o(\|z\|^2). \tag{2.9.7}$$

Proof Suppose first that $n = 1$. Since u attains a minimum at 0, its first partial derivatives vanish there. In view of Taylor's formula for u at 0, the function $v(z) = (u(z)/|z|^2)^*$ is subharmonic in V and

$$\lim_{t\to 0^+} \frac{u(t\alpha, t\beta)}{t^2} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(0)\alpha^2 + \frac{\partial^2 u}{\partial x\partial y}(0)\alpha\beta + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(0)\beta^2,$$

for any $\alpha + i\beta$ from the unit circle. According to Theorem 2.7.2, $[0,1]$ is not thin at 0, and so the limit of the left-hand side of the above equality is $v(0)$. By substituting $\alpha + i\beta$ equal to 1, i, and $(1+i)/\sqrt{2}$, respectively, we conclude that

$$\frac{\partial^2 u}{\partial x^2}(0) = \frac{\partial^2 u}{\partial y^2}(0) \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y}(0) = 0 .$$

Hence the Taylor expansion looks exactly as in the proposition.

The higher-dimensional case follows from the one-dimensional one applied to $u \circ F_\xi$, where $\xi \in \mathbf{C}^n \setminus \{0\}$ and $F_\xi : \mathbf{C} \longrightarrow \mathbf{C}^n$ is given by the formula $F_\xi(\lambda) = \lambda\xi$. ∎

The following example complements the statement of the proposition.

Example 2.9.25 Let $u : \mathbf{C}^n \longrightarrow [0,\infty)$ be plurisubharmonic and smooth in a neighbourhood of the origin. Suppose also that

$$u(\lambda z) = |\lambda|^2 u(z) \qquad (2.9.8)$$

for $\lambda \in \mathbf{C}$, $z \in \mathbf{C}^n$. Then

$$u(z) = \langle \mathcal{L}u(0)z, z\rangle \qquad (z \in \mathbf{C}^n) .$$

Indeed, in order to prove this formula, it is enough to apply the operator $\partial^2/\partial\lambda\partial\bar{\lambda}|_{\lambda=0}$ to both sides of (2.9.8). Note also that since $\mathcal{L}u(0)$ is positive definite, after a (complex) change of coordinates in $\mathbf{C}^n$, u can be written as

$$u(z_1, \dots, z_n) = |z_1|^2 + \dots + |z_p|^2 \qquad \big((z_1, \dots, z_n) \in \mathbf{C}^n\big) \qquad (2.9.9)$$

for some $p \in \{0, 1, \dots, n\}$. In particular, if $u \not\equiv 0$, $\log u \in \mathcal{PSH}(\mathbf{C}^n)$. Indeed, if $a \in \mathbf{C}^n$ is such that $u(a) \neq 0$ and $w \in \mathbf{C}^n$, then

$$\begin{aligned}\langle \mathcal{L} \log u(a)w, w\rangle \\ &= \frac{1}{u^2}\left(\sum_{j=1}^{p}|a_j|^2 \sum_{k=1}^{p}|w_k|^2 - \sum_{j=1}^{p}\bar{a}_j w_j \sum_{k=1}^{p} a_k \bar{w}_k\right), \text{ by differentiation,} \\ &\geq 0, \quad \text{by the Cauchy–Schwarz inequality} .\end{aligned}$$

Now the required property follows from Theorem 2.9.22. ∎

It seems appropriate to mention here another property of plurisubharmonic functions. Before stating it, however, we have to describe some basic facts concerning proper holomorphic mappings. We shall not give proofs

here; that would take us too far from our principal subject. Elegant proofs can be found in (Rudin 1980).

Let Ω, Ω' be open connected sets in $\mathbf{C}^n$, and let $f : \Omega \longrightarrow \Omega'$ be a proper mapping. It is easy to check that f is closed, i.e. it maps closed sets onto closed sets.

If, in addition, f is holomorphic, then:

(i) f is open and, in particular, $f(\Omega) = \Omega'$ (because f is also closed, see Section 2.8);

(ii) if $A = \{z \in \Omega : \det \partial_z f = 0\}$, then for each $a \in \Omega'$ there is an open ball B, centred at a and contained in Ω', and a function $g \in \mathcal{O}(B)$ such that $g \not\equiv 0$ and $f(A) \cap B = g^{-1}(0)$;

(iii) the fibres of f, that is, the sets $f^{-1}(w)$ where $w \in \Omega'$, are finite.

It should be noted that (ii) is a special case of Remmert's proper mapping theorem (e.g. Gunning and Rossi (1965), Rudin (1980)). In fact, we only need to know that $f(A)$ is closed and pluripolar.

Proposition 2.9.26 *Let $f : \Omega \longrightarrow \Omega'$ be a proper holomorphic surjection between two open sets in $\mathbf{C}^n$. If $u \in \mathcal{PSH}(\Omega)$, then the formula*

$$v(z) = \max\{u(w) : w \in f^{-1}(z)\} \qquad (z \in \Omega')$$

defines a plurisubharmonic function.

Proof (Klimek 1982*a*) Without loss of generality we may suppose that Ω' is connected.

If G is a relatively compact open subset of Ω', then the open set $f^{-1}(G)$ is relatively compact in Ω, because f is proper. Therefore, in view of the main approximation theorem, it is enough to show that the proposition is true for continuous plurisubharmonic functions.

Suppose that $u \in \mathcal{C} \cap \mathcal{PSH}(\Omega)$. If a and b are real numbers such that $a < b$, then

$$v^{-1}\big((a,b)\big) = f\Big(u^{-1}\big((a,\infty)\big)\Big) \setminus f\Big(u^{-1}\big([b,\infty)\big)\Big) .$$

Consequently, v is continuous in Ω'. By Theorem 1.3.1, the proper surjection

$$f|f^{-1}(\Omega' \setminus f(A)) : f^{-1}(\Omega' \setminus f(A)) \longrightarrow \Omega' \setminus f(A)$$

is locally biholomorphic. Therefore there is a unique number $k \in \mathbf{N}$ such that for each $z \in \Omega' \setminus f(A)$ there exist a neighbourhood $V \subset \Omega' \setminus f(A)$ of z, and mutually disjoint neighbourhoods

$$U_1, \dots, U_k \quad \text{of} \quad w_1, \dots, w_k, \quad \text{where} \quad \{w_1, \dots, w_k\} = f^{-1}(z) ,$$

such that

(i) $(f|U_j) : U_j \longrightarrow V$ is a biholomorphic mapping,
(ii) $f^{-1}(V) = U_1 \cup \ldots \cup U_k$.
Accordingly, $v \in \mathcal{PSH}(\Omega' \setminus f(A))$. Since v is continuous and $f(A)$ is pluripolar, plurisubharmonicity of v follows from Theorem 2.9.22. ∎

2.10 PSEUDOCONVEXITY

One of the central themes of the theory of functions of several complex variables is the study of domains of holomorphy or, equivalently, domains of existence of holomorphic functions. A proper exposition of these notions and related topics would take us too far away from the intended subject of this book. In this section we state only basic definitions and properties of domains of holomorphy, concentrating instead on the equivalent notion of pseudoconvexity. Krantz (1982) presents an excellent discussion of domains of holomorphy (see also Hörmander 1973).

The general notion of strict (or strong) pseudoconvexity does not play a prominent role in this book; it is described briefly in the exercises at the end of this chapter. In various considerations where normally strictly pseudoconvex domains are used, we shall use hyperconvex domains (defined in this section) or just Euclidean balls.

Let Ω be an open subset of $\mathbf{C}^n$, and let K be a compact set contained in Ω. We shall be interested in four types of natural 'envelopes' of K.

The *convex hull* of K, denoted by conv K, is the smallest convex set in $\mathbf{C}^n$ containing K. Clearly,

$$\operatorname{conv} K = \{z \in \mathbf{C}^n : \ \varphi(z) \le \sup \varphi(K) \text{ for all } \varphi \in \mathcal{L}(\mathbf{C}^n, \mathbf{R})\},$$

where $\mathcal{L}(\mathbf{C}^n, \mathbf{R})$ is the family of all $\mathbf{R}$-linear functionals $\varphi : \ \mathbf{C}^n \longrightarrow \mathbf{R}$. Obviously, conv K is compact.

The *polynomially convex hull* of K is the set

$$\hat{K} = \{z \in \mathbf{C}^n : \ |p(z)| \le \|p\|_K \text{ for all } p \in \mathcal{P}_n\},$$

where $\mathcal{P}_n$ denotes the family of all polynomials of n complex variables.

The *holomorphically convex hull* of K (in Ω) is defined as follows:

$$\hat{K}_{\mathcal{O}(\Omega)} = \{z \in \Omega : \ |f(z)| \le \|f\|_K \text{ for all } f \in \mathcal{O}(\Omega)\}.$$

Note that $\hat{K}_{\mathcal{O}(\Omega)}$ is closed in Ω.

The *plurisubharmonically convex hull* of K (in Ω) is the set

$$\hat{K}_{\mathcal{PSH}(\Omega)} = \{z \in \Omega : \ u(z) \le \sup u(K) \text{ for all } u \in \mathcal{PSH}(\Omega)\}.$$

Note that

$$\hat{K}_{\mathcal{PSH}(\Omega)} \subset \hat{K}_{\mathcal{O}(\Omega)} \subset \hat{K} \subset \operatorname{conv} K.$$

The first inclusion follows from the fact that $|f| \in \mathcal{PSH}(\Omega)$ if $f \in \mathcal{O}(\Omega)$. The second inclusion is obvious. The last one becomes clear when one notices that $\hat{K} = \hat{K}_{\mathcal{O}(\mathbf{C}^n)}$ and if $\varphi \in \mathcal{L}(\mathbf{C}^n, \mathbf{R})$, then

$$\Phi(z) = \varphi(z) - i\varphi(iz) \qquad (z \in \mathbf{C}^n)$$

is $\mathbf{C}$-linear and $|\exp \Phi| = \exp \varphi$.

Observe that Ω is convex if and only if conv K is contained in Ω for each compact subset K of Ω. By analogy, Ω is said to be *polynomially convex* (respectively, *holomorphically convex*, *pseudoconvex*) if for each compact subset K of Ω, $\hat{K}$ (respectively, $\hat{K}_{\mathcal{O}(\Omega}$, $\hat{K}_{\mathcal{PSH}(\Omega)}$) is relatively compact in Ω. We shall discuss some properties of polynomially convex sets at a later stage. Here, we take a look at holomorphic convexity and pseudoconvexity.

It is obvious that if an open set is holomorphically convex, it is pseudoconvex. It is clear that in the one-dimensional case all open sets are holomorphically convex (and pseudoconvex). Indeed, let K be a compact set in $\Omega \subset \mathbf{C}$. If $\Omega = \mathbf{C}$, the situation is trivial. Suppose that $\Omega \neq \mathbf{C}$. By (2.10.1), $\hat{K}_{\mathcal{O}(\Omega)}$ is bounded; it is also closed in Ω by its definition. We claim that

$$\operatorname{dist}\left(\hat{K}_{\mathcal{O}(\Omega)}, \partial\Omega\right) > 0.$$

If this were not so, one would be able to find a sequence $\{a_n\}_{n\in\mathbf{N}} \subset \hat{K}_{\mathcal{O}(\Omega)}$ convergent to a point $a \in \partial\Omega$. Hence

$$\frac{1}{|a_n - a|} \le \sup\left\{\frac{1}{|z-a|} : \ z \in K\right\} < \infty \qquad (n \in \mathbf{N}).$$

As the left-hand side tends to ∞ when n tends to ∞, the hypothesis leads to a contradiction.

In higher dimensions the situation is much more complicated; usually it is difficult to prove whether a particular set is holomorphically convex or not. The following classical result will provide us with examples of open sets in $\mathbf{C}^n$ which are not holomorphically convex.

Theorem 2.10.1 (Hartogs) *Let Ω be an open neighbourhood of a closed polydisc K in $\mathbf{C}^n$, where $n \ge 2$. If $f \in \mathcal{O}(\Omega \setminus K)$, then there exists a holomorphic function $\tilde{f}$ on Ω, such that $\tilde{f}|\Omega \setminus K = f$.*

Proof As before, if $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, then $z' = (z_1, \ldots, z_{n-1})$.

Without loss of generality we may suppose that $K = \bar{P}(0,1)$. Choose $r > 1$ so that $\bar{P}(0,r) \subset \Omega$. It would be enough to find a function $\tilde{f} \in \mathcal{O}(P(0,r))$ that extends f. Note that if $z' \in \bar{P}(0', r)$ is fixed, the function $z_n \longmapsto f(z', z_n)$ is holomorphic in a neighbourhood of the annulus $\bar{D}(0_n, r) \setminus \bar{D}(0,1)$. Therefore the coefficients c_j of the Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} c_j(z') z_n^j \tag{2.10.1}$$

are holomorphic in $P(0',r)$; indeed,

$$c_j(z') = \frac{1}{2\pi i}\int\limits_{|\zeta|=r} \frac{f(z',\zeta)}{\zeta^{j+1}}d\zeta\,. \tag{2.10.2}$$

If $1 < |z'| < r$, the function $z_n \longmapsto f(z',z_n)$ is holomorphic in the disc $D(0_n,r)$. In consequence, $c_j(z') = 0$ for such z' and $j < 0$. By the identity principle (Theorem 2.8.1), $c_j \equiv 0$ for $j < 0$. Define

$$\tilde{f}(z) = \sum_{j=0}^{\infty} c_j(z')z_n^j \qquad (z \in P(0,r))\,.$$

The above series is absolutely convergent in view of (2.10.2) and Abel's lemma (in one variable z_n). In fact, it is convergent locally uniformly in $P(0,r)$. Indeed, by Cauchy's estimates, if $z \in P(0,r)$ and $j \geq 0$, then

$$\left|c_j(z')z_n^j\right| \leq M\left(\frac{|z_n|}{r}\right)^j,$$

where $M = \|f\|_{\bar{P}(0',r)\times\partial D(0_n,r)}$. The estimate implies the convergence. Thus, in view of the Weierstrass theorem, $\tilde{f}$ is holomorphic. Since $\tilde{f}$ coincides with f on a non-empty open subset of $P(0,r)\setminus\bar{P}(0,1)$, the identity principle guarantees that $\tilde{f}$ is the required extension of f. ∎

It should be mentioned that Hartogs' theorem remains true if K is an arbitrary compact set such that $\Omega\setminus K$ is connected (e.g. Hörmander 1973).

Corollary 2.10.2 *If Ω and K are as in the above theorem, $\Omega\setminus K$ is not holomorphically convex.*

Proof The statement follows immediately from Theorem 2.10.1 and the maximum principle. ∎

In order to gain a better idea of the relevance of holomorphic convexity in complex analysis, we need to introduce the notion of a domain of holomorphy.

An open set $\Omega \subset \mathbf{C}^n$ is called a *domain of holomorphy* if there are no open sets Ω_1, Ω_2 with the following properties:
(i) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$;
(ii) Ω_2 is connected and $\Omega_2 \setminus \Omega \neq \emptyset$;
(iii) for each $f \in \mathcal{O}(\Omega)$ there exists $\tilde{f} \in \mathcal{O}(\Omega_2)$ such that $\tilde{f}|\Omega_1 = f$.

The basic idea behind this definition is quite straightforward: Ω is a domain of holomorphy if there is no part of the boundary of Ω across which every holomorphic function on Ω can be holomorphically extended.

If one can find $f \in \mathcal{O}(\Omega)$ for which there are no open sets Ω_1, Ω_2 satisfying (i), (ii), and such that $f|\Omega_1 = \tilde{f}|\Omega_1$ for some $\tilde{f} \in \mathcal{O}(\Omega_2)$, then Ω is said to be the *domain of existence* of f.

Before stating the next result, we need some additional notation. By a *distance function* on $\mathbf{C}^n$ we shall mean a non-negative continuous function δ defined on $\mathbf{C}^n$, and such that:
(i) $\delta(z) = 0$ if and only if $z = 0$;
(ii) $\delta(\lambda z) = |\lambda|\delta(z)$ for all $z \in \mathbf{C}^n$, $\lambda \in \mathbf{C}$.

Let $\Omega \subset \mathbf{C}^n$ be an open set, and let δ be a distance function. We define the δ-distance from the boundary of Ω as follows:

$$\delta_\Omega(z) = \inf\{\delta(z-w):\ w \in \mathbf{C}^n\setminus\Omega\}\,,$$

where $\inf\emptyset = \infty$. It is easy to check that δ_Ω is continuous and

$$\delta_\Omega(z) = \sup\{r \geq 0:\ z + \zeta w \in \Omega,\ \zeta \in D(0,r),\ \delta(w) \leq 1\}\,.$$

The Cartan–Thullen theorem below gives a comprehensive characterization of domains of holomorphy.

Theorem 2.10.3 (Cartan–Thullen) *Let δ be a distance function on $\mathbf{C}^n$, and let Ω be an open subset of $\mathbf{C}^n$. The following conditions are equivalent:*
(i) Ω is a domain of holomorphy;
(ii) Ω is holomorphically convex;
(iii) for any compact set $K \subset \Omega$ and $f \in \mathcal{O}(\Omega)$, if $|f| \leq \delta_\Omega$ on K, then $|f| \leq \delta_\Omega$ on $\hat{K}_{\mathcal{O}(\Omega)}$;
(iv) Ω is the domain of existence of a function $f \in \mathcal{O}(\Omega)$.

Proof See, for example, Krantz (1982). ∎

In view of this theorem, every domain of holomorphy is pseudoconvex. A classical (and very difficult to prove) result due to Oka, Bremermann, and Norguet (e.g. Hörmander 1973) says that the converse is also true. Therefore, by studying pseudoconvexity, one can shed some light on the behaviour of holomorphic functions, even if the latter are not mentioned explicitly.

The basic characterization of pseudoconvexity is contained in the next theorem.

Theorem 2.10.4 *Let Ω be an open proper subset of $\mathbf{C}^n$, and let $\delta:\ \mathbf{C}^n \longrightarrow \mathbf{R}_+$ be a distance function. The following conditions are equivalent:*
(i) $-\log\delta_\Omega \in \mathcal{PSH}(\Omega)$;
(ii) there exists a continuous plurisubharmonic function $u:\ \Omega \longrightarrow \mathbf{R}$ such that for each $c \in \mathbf{R}$ the set $\{z \in \Omega:\ u(z) < c\}$ is relatively compact in Ω;

(iii) if K is a compact subset of Ω, then so is $\hat{K}_{\mathcal{PSH}(\Omega)}$;
(iv) Ω is pseudoconvex;
(v) for each $a \in \partial\Omega$ there is a neighbourhood W of a such that $W \cap \Omega$ is pseudoconvex.

Proof To prove that (i) implies (ii), it is enough to define

$$u(z) = \max\{|z|, \ -\log \delta_\Omega(z)\}.$$

Note that (ii) implies pseudoconvexity of Ω. Therefore, in order to show (ii)$\Longrightarrow$(iii), it suffices to prove that if (ii) is satisfied, then, for any compact set $K \subset \Omega$,

$$\hat{K}_{\mathcal{PSH}(\Omega)} = \{z \in \Omega : \ v(z) \le \sup v(K), \ v \in \mathcal{PSH} \cap \mathcal{C}(\Omega)\}.$$

Clearly, we have the inclusion '$\subset$'. To show the opposite inclusion, we have to prove that for each $a \in \Omega \setminus \hat{K}_{\mathcal{PSH}(\Omega)}$ there exists a continuous plurisubharmonic function $v : \ \Omega \longrightarrow \mathbf{R}$ such that $v(a) > \sup v(K)$. Take a compact set $K \subset \Omega$ and a point $a \in \Omega \setminus \hat{K}_{\mathcal{PSH}(\Omega)}$. Let u be the function from (ii). By adding a constant to u, we may modify u so that u is negative on the set $K \cup \{a\}$. Choose a function $w \in \mathcal{PSH}(\Omega)$ such that $w(a) > 0$ and $w|K < 0$. By the main approximation theorem for plurisubharmonic functions, one can find a function w_1 such that:
(α) $w_1 \in \mathcal{PSH}(G) \cap \mathcal{C}(\bar{G})$, where $G = \{z \in \Omega : \ u(z) < 1\}$;
(β) $w_1(a) > 0$;
(γ) $w_1|K < 0$.
Set $C = \sup\{w_1(z) : \ z \in G\}$ and define

$$v(z) = \begin{cases} \max\{w_1(z), Cu(z)\} & (z \in G) \\ Cu(z) & (z \in \Omega \setminus G). \end{cases}$$

Then $v \in \mathcal{PSH} \cap \mathcal{C}(\Omega)$, $v(a) > 0$, and $v|K < 0$, as required.

The implications (iii)$\Longrightarrow$(iv) and (iv)$\Longrightarrow$(v) are obvious.

The proof will be complete if we can show (iv)$\Longrightarrow$(i) and (v)$\Longrightarrow$(ii). First, we prove (iv)$\Longrightarrow$(i). Let $a \in \Omega$ and $w \in \mathbf{C}^n \setminus \{0\}$. Choose $r > 0$ so that

$$D = \{a + \zeta w : \ \zeta \in \bar{D}(0,r)\} \subset \Omega.$$

Let h be a harmonic function defined on a neighbourhood of the closed disc $\bar{D}(0,r)$, such that

$$-\log \delta_\Omega(a + \zeta w) \le h(\zeta) \qquad (\zeta \in \partial D(0,r)).$$

We can find a holomorphic function f, defined on a neighbourhood of $\bar{D}(0,r)$, such that $\mathrm{Re}\, f|\bar{D}(0,r) = h|\bar{D}(0,r)$. Therefore

$$\delta_\Omega(a + \zeta w) \ge \left|e^{-f(\zeta)}\right| \qquad (\zeta \in \partial D(0,r)).$$

We want to prove that the same inequality holds for $\zeta \in D(0,r)$.

Take $b \in \mathbf{C}^n$ such that $\delta(b) \le 1$. For each $t \in [0,1]$ define the mapping:

$$\varphi_t \ : \bar{D}(0,r) \longrightarrow \mathbf{C}^n,$$
$$\varphi_t(\zeta) = a + \zeta w + tbe^{-f(\zeta)}.$$

Denote by D_t the range of φ_t, and set

$$T = T_b = \{t \in [0,1] : \ D_t \subset \Omega\}.$$

Since $0 \in T$, the set T is non-empty. Clearly, T is open.

The theorem would follow if we could show that T is closed. Indeed, then we could conclude that $T = [0,1]$, and so $D_1 \subset \Omega$. As b was arbitrarily chosen, this would mean that

$$a + \zeta w + be^{-f(\zeta)} \in \Omega$$

if $\delta(b) \le 1$ and $\zeta \in \bar{D}(0,r)$. Consequently,

$$\delta_\Omega(a + \zeta w) \ge \left|e^{-f(\zeta)}\right| \qquad (\zeta \in \bar{D}(0,r))$$

or, equivalently,

$$-\log \delta_\Omega(a + \zeta w) \le \mathrm{Re}\, f(\zeta) = h(\zeta) \qquad (\zeta \in \bar{D}(0,r)),$$

as required.

Let us fix $b \in \delta^{-1}([0,1])$. It remains to be proved that $T = T_b$ is closed. Define

$$K = \{a + \zeta w + tbe^{-f(\zeta)} : \ \zeta \in \partial D(0,r), \ t \in [0,1]\}.$$

Of course, K is a compact set in $\mathbf{C}^n$. Moreover, if $\zeta \in \partial D(0,r)$ and $t \in [0,1]$,

$$\delta_\Omega(a + \zeta w) \ge \left|e^{-f(\zeta)}\right| \ge \delta\left(tbe^{-f(\zeta)}\right).$$

Hence $K \subset \Omega$. If $u \in \mathcal{PSH}(\Omega)$ and $t \in T$, then the function

$$\zeta \longmapsto u\left(a + \zeta w + tbe^{-f(\zeta)}\right)$$

is subharmonic in a neighbourhood of $\bar{D}(0,r)$. By the maximum principle,

$$u\left(a + \zeta w + tbe^{-f(\zeta)}\right) \le \sup u(K) \qquad (\zeta \in \bar{D}(0,r)).$$

Consequently, if $t \in T$, we have

$$D_t \subset \hat{K}_{\mathcal{PSH}(\Omega)} \subset \Omega \tag{2.10.3}$$

by (iv). Since $\hat{K}_{\mathcal{PSH}(\Omega)}$ is relatively compact in Ω, (2.10.2) implies that T is closed.

Now we prove that (v) implies (ii).

Let $a \in \partial\Omega$. Choose a neighbourhood W of a such that $W \cap \Omega$ is pseudoconvex. Then, according to the equivalences we have already shown, $-\log \delta_{W\cap\Omega} \in \mathcal{PSH}(W \cap \Omega)$. Moreover, if U is a sufficiently small neighbourhood of a, $\delta_\Omega = \delta_{W\cap\Omega}$ in $U \cap \Omega$. The same can be repeated for any other point in $\partial\Omega$.

Consequently, we have proved that there exists a closed set F in $\mathbf{C}^n$, such that $F \subset \Omega$ and $-\log \delta_\Omega \in \mathcal{PSH}(\Omega \setminus F)$.

Let $\varphi: \mathbf{R} \longrightarrow \mathbf{R}$ be a convex increasing function such that

$$\lim_{t\to\infty} \varphi(t) = \infty \text{ and } \varphi(|z|) > -\log \delta_\Omega(z)$$

for each $z \in F$. Clearly, the function

$$u(z) = \max\{\varphi(|z|),\ -\log \delta_\Omega(z)\} \qquad (z \in \Omega)$$

is plurisubharmonic; furthermore, it satisfies (ii). ∎

In this book we shall often use a more particular type of pseudoconvexity. We shall say that an open bounded set $\Omega \subset \mathbf{C}^n$ is *hyperconvex* if it is connected and there is a continuous plurisubharmonic function $\varrho: \Omega \longrightarrow (-\infty, 0)$ such that the set

$$\{z \in \Omega:\ \varrho(z) < c\}$$

is a relatively compact subset of Ω, for each $c \in (-\infty, 0)$ (Stehlé 1975; Kerzman and Rosay 1981).

It follows from Theorem 2.10.4 (ii) that every pseudoconvex domain is the union of an increasing sequence of hyperconvex sets. Obviously, every hyperconvex set is pseudoconvex.

As we have seen, the concept of pseudoconvexity is non-trivial only in dimensions higher than one. It should be noted however that in one complex variable the function δ_Ω, where δ is the Euclidean distance, carries some information about the convexity (or concavity) of Ω.

Armitage and Kuran (1985) have shown that if Ω is a domain in $\mathbf{C}$ such that $-\delta_\Omega \in \mathcal{SH}(\Omega)$, then Ω is convex. They have also proved that in higher dimensions the subharmonicity of $-\delta_\Omega$ does not necessarily imply the convexity of Ω. Parker (1988) demonstrated that if Ω is a proper open subset of the complex plane, such that δ_Ω is subharmonic in a neighbourhood of $\partial\Omega$, then $\mathbf{C} \setminus \Omega$ is convex; moreover, the result fails in higher dimensions. For further results in this direction see Armitage and Kuran (1985) and Gardiner (1991*b*).

EXERCISES

1. Define $u(x) = -(s_m \max\{1, m-2\})^{-1} g(\|x\|)$ for $x \in \mathbf{R}^m$, where $g(r) = -\log r$ if $m = 2$, and $g(r) = r^{2-m}$ if $m > 2$. Prove that for any test function $\varphi \in C_0^\infty(\mathbf{R}^m)$,

$$\int_{\mathbf{R}^m} (\Delta\varphi) g\, d\lambda = \varphi(0)\,.$$

[Hint: Apply Green's theorem on the domain $B(0,R) \setminus \bar{B}(0,r)$, where $0 < r < R$.]

2. Let ϱ denote the reflection with respect to the unit sphere in $\mathbf{R}^m$, i.e.

$$\varrho = (\varrho_1, \ldots, \varrho_m):\ \mathbf{R}^m \setminus \{0\} \longrightarrow \mathbf{R}^m \setminus \{0\}\,,$$
$$\varrho(x) = \frac{x}{\|x\|^2}\,.$$

Notice that the mapping ϱ is bijective, $\varrho^{-1} = \varrho$, $\|\varrho(x)\| = 1/\|x\|$, and $\varrho(x) = \operatorname{grad}(\log\|x\|)$ for $x \neq 0$. Prove that the vectors $\operatorname{grad} \varrho_j(x)$, where $j = 1, \ldots, m$, form an orthogonal system and $\|\operatorname{grad} \varrho_j(x)\| = \|x\|^{-1}$.

If u is a real-valued function whose domain Ω is contained in $\mathbf{R}^m \setminus \{0\}$, we define the *Kelvin transform* u^* of u by the formula

$$u^*(x) = \|x\|^{2-m} u(\varrho(x)) \qquad (x \in \varrho(\Omega))\,.$$

Prove that if $u \in \mathcal{SH}(\Omega)$, then $u^* \in \mathcal{SH}(\varrho(\Omega))$.

3. Let D be a bounded domain in $\mathbf{R}^m$, and let $y \in D$. The function

$$G = G_{D,y}:\ \bar{D} \longrightarrow \mathbf{R} \cup \{\infty\}$$

is said to be the *classical Green function* of D with pole at y if the following conditions are satisfied:

(i) $G \in \mathcal{H}(D \setminus \{y\}) \cap \mathcal{C}(\bar{D} \setminus \{y\})$;
(ii) $G(x) = 0$ for all $x \in \partial D$;
(iii) $x \longmapsto G(x) - g(\|x - y\|)$ extends to a harmonic function on D, where g is as in Exercise 1 above.

Note that $-G$ is a negative subharmonic function on D. Prove that the Green function is unique if it exists.

Suppose that D has a smooth boundary, $y \in D$, and the classical Green function of D with pole at y exists. Prove that if u is harmonic in a neighbourhood of $\bar{D}$, then

$$u(y) = \frac{1}{s_m \max\{1, m-2\}} \int_{\partial D} \sum_{j=1}^m u \frac{\partial}{\partial x_j} \mathbf{G}(\cdot, y) \sigma_j \qquad (y \in B(0,1))\,,$$

where

$$\sigma_j = (-1)^{j+1} dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_m$$

and ∂D has the natural orientation induced from D.

4. Prove that the set $\mathbf{R}$ is not polar in $\mathbf{C}$. [Hint: Supposing that the result is false, construct a subharmonic function on $\mathbf{C}$ that is equal to $-\infty$ on the set $\{x + iy : y < 0\}$.]

5. (*Schwarz's reflection principle.*) Let u be a harmonic function on a domain D in $\mathbf{R}^m$. Suppose that $D \subset \{x : x_1 > 0\}$, E is an open subset of the hyperplane $x_1 = 0$, $E \subset \partial D \cap \{x : x_1 = 0\}$, and that u can be extended to a continuous function on $D \cup E$ by setting $u|_E \equiv 0$. Prove that u can be extended to a harmonic function on the set $D \cup E \cup D'$ by the formula

$$u(x_1, \ldots, x_m) = u(-x_1, \ldots, x_m) \qquad (x \in D'),$$

where D' denotes the reflection of D in the hyperplane $x_1 = 0$.

6. Let Ω be an open subset of $\mathbf{R}^m$, and let F be a closed subset of Ω. Suppose that $c \in [-\infty, \infty)$, and that $u : \Omega \longrightarrow [c, \infty)$ is upper semicontinuous. Prove that if $F \subset u^{-1}(c)$ and $u \in \mathcal{SH}(\Omega \setminus F)$, then $u \in \mathcal{SH}(\Omega)$.

7. Let Ω be an open subset of $\mathbf{R}^n$ with $n \geq 1$. A function $u : \Omega \longrightarrow \mathbf{R}$ is said to be *convex* if for each $a, b \in \Omega$ such that the line segment joining a and b is contained in Ω, and for each $t \in [0, 1]$ we have

$$u\big((1-t)a + tb\big) \leq (1-t)u(a) + tu(b).$$

If Ω is a convex open set, convexity of u means that the set

$$\{(x, t) : u(x) < t\} \subset \mathbf{R}^{n+1}$$

is convex. Note that convex functions are locally bounded. Use this fact to show that convex functions are continuous. Prove that the definition given here is consistent with that given in Chapter 1 for smooth functions. Observe that, by analogy to the definition of subharmonic functions, convex functions could be called 'subaffine'.

8. Prove that the main approximation theorem for subharmonic functions remains true when the word 'subharmonic' is replaced by the word 'convex' throughout the theorem. Conclude that convex functions (of at least two variables) are subharmonic; moreover, convex functions in $\mathbf{C}^n$ are plurisubharmonic. Notice that if $\Omega \subset \mathbf{R}^n$ and $u : \Omega \longrightarrow \mathbf{R}$, then u is convex if and only if the function

$$(z_1, \ldots, z_n) \longmapsto u(\operatorname{Re} z_1, \ldots, \operatorname{Re} z_n)$$

is plurisubharmonic in the set $\Omega + i\mathbf{R}^n$.

9. Prove a counterpart of Theorem 2.10.4 for convex functions, sets, and hulls.

10. Let $u : \mathbf{C}^n \longrightarrow \mathbf{R}$ be a pluriharmonic function which is homogeneous of degree $m \neq 0$ (i.e. $u(tz) = t^m u(z)$ for all $t > 0$ and $z \in \mathbf{C}^n$). Define

$$f(z) = \frac{2}{m} \sum_{j=1}^{n} z_j \frac{\partial u}{\partial z_j}(z) \qquad (z \in \mathbf{C}^n).$$

Prove that $f \in \mathcal{O}(\mathbf{C}^n)$ and u is the real part of f.

11. Let f be a complex function on an open connected set in $\mathbf{C}$.

(a) Prove that the following conditions are equivalent:

(i) for each $a \in \Omega$ and for every subharmonic function u defined on a neighbourhood of $f(a)$, the composition $u \circ f$ is subharmonic in a neighbourhood of a;

(ii) the function $u \circ f$ is subharmonic in Ω if $u(x + iy) = \pm x, \pm y, \pm xy$ and $\pm(x^2 - y^2)$;

(iii) either f or $\bar{f}$ is holomorphic.

(b) Show that if the compositions of plurisubharmonic functions of two variables with $z \longmapsto (z, f(z))$ are subharmonic, then f is holomorphic.

12. Let $n > 1$, and let D be an open set in $\mathbf{C}^{n-1}$. Let R be a positive lower semicontinuous function on D. The set

$$H = \{(z', z_n) \in D \times \mathbf{C} : |z_n| < R(z')\}$$

is called the *complete Hartogs domain* defined by D and R.

(a) Prove that H is pseudoconvex if and only if $-\log R \in \mathcal{PSH}(D)$.

(b) Show that if $f \in \mathcal{O}(\Omega)$, then there is a family $\{f_j\}_{j \in \mathbf{Z}_+} \subset \mathcal{O}(D)$ such that

$$f(z', z_n) = \sum_{j=0}^{\infty} f_j(z') z_n^j \qquad \big((z', z_n) \in H\big)$$

and the series is absolutely and uniformly convergent in H.

(c) (Bremermann 1956) Suppose that H is the domain of existence of $f \in \mathcal{O}(H)$. Define $u(z') = -\log R(z')$, $z' \in D$. Prove that in D

$$u = \left(\limsup_{j \to \infty} \frac{1}{j} \log |f_j|\right)^*, \qquad (\dagger)$$

where the f_j are as in (b).

Conclude that every plurisubharmonic function u is locally of the form $(\dagger)$ (for some functions f_j).

13. Let $h: \mathbf{C}^n \longrightarrow [0, \infty)$ be an upper semicontinuous function which is not identically 0 and is positive homogeneous, i.e. $h(tz) = |t|h(z)$ for all $t \in \mathbf{C}$ and $z \in \mathbf{C}^n$. Prove that the following conditions are equivalent:
(i) $\log h$ is plurisubharmonic;
(ii) h is plurisubharmonic;
(iii) the set $\{z \in \mathbf{C}^n : h(z) < 1\}$ is pseudoconvex.

14. A bounded domain $\Omega \subset \mathbf{R}^n$ is said to be *strongly convex* if it is of the form $\Omega = \{z \in \mathbf{C}^n : \varrho(z) < 0\}$, where ϱ is a $\mathcal{C}^2$ function on a neighbourhood of $\bar{\Omega}$ satisfying the following conditions: $d_z\varrho \neq 0$ for each $x \in \partial\Omega$, and $d_x^2 u(y, y) > 0$ for all $x \in \partial\Omega$ and y belonging to the tangent space to $\partial\Omega$ at x. Prove that if Ω is strongly convex, then it is convex and each of its boundary points is extreme. (Recall that a point $a \in A \subset \mathbf{R}^n$ is said to be an *extreme* point of A if it is an end point of every closed line segment containing it and contained in A.)

15. A bounded domain $\Omega \subset \mathbf{C}^n$ is said to be *strictly pseudoconvex* if it is of the form $\Omega = \{z \in \mathbf{C}^n : \varrho(z) < 0\}$, where ϱ is a $\mathcal{C}^2$ function on a neighbourhood of $\bar{\Omega}$ satisfying the following conditions: $d_z\varrho \neq 0$ for each $x \in \partial\Omega$, and ϱ is strictly plurisubharmonic in a neighbourhood of $\partial\Omega$. The function ϱ is said to be a *defining function* for Ω. Note that strict pseudoconvexity implies pseudoconvexity. Prove that for each point $p \in \partial\Omega$ there exists a neighbourhood V of p and a biholomorphic mapping $\varphi: V \longrightarrow \varphi(V)$ such that $\varphi(V \cap \Omega)$ is strongly convex. [Hint: Suppose that $p = 0$, and write the second order Taylor expansion of ϱ at 0 in terms of z and $\bar{z}$.]

Part II