Math 274Z Problem Session

Problem Sheet 2

23 Feb 2023

Problem 1. Linearization and Basins of Attraction

The following two problems explain the local behavior of a rational function f near attracting periodic points. It suffices to study the case of fixed points, since a periodic point of f with period n is a fixed point of f^n.

Let z_0 be a fixed point of f with multiplier λ.

1. Prove that z_0 is an attracting fixed point (i.e., $|\lambda| < 1$) if and only if z_0 is *topologically attracting* in the following sense: there exists a neighborhood U of z_0 such that the sequence $\{f, f^2, \ldots\}$ converges uniformly on U to the constant function z_0.

2. Prove that z_0 is a repelling fixed point (i.e., $|\lambda| > 1$) if and only if z_0 is *topologically repelling* in the following sense: there exists a neighborhood U of z_0 such that for all $z \in U \setminus \{z_0\}$, there exists $n \geq 1$ such that $f^n(z) \notin U$.

From the Taylor series expansion

$$f(z) = z_0 + \lambda(z - z_0) + O((z - z_0)^2),$$

one expects that iterating f near z_0 "looks like" repeated multiplication by λ near 0. More precisely, we have the following:

Theorem 0.1 (Kœnigs linearization)**.** *Suppose $|\lambda| \neq 0, 1$. Then there exists a holomorphic change of coordinate $w = \phi(z)$ on a neighborhood U of z_0, such that $\phi(z_0) = 0$ and*

$$\phi(f(z)) = \lambda\phi(z)$$

for all $z \in U$. Moreover, ϕ is unique up to multiplication by a nonzero constant.

3. Prove the uniqueness statement in the Kœnigs linearization theorem. (Hint: classify all power series which fix 0 and commute with $z \mapsto \lambda z$.)

4. Prove the Kœnigs linearization theorem for $0 < |\lambda| < 1$. (Hint: if $z_0 = 0$, consider $\lim_{n\to\infty} \lambda^{-n} f^n(z)$.)

5. Prove the Kœnigs linearization theorem for $|\lambda| > 1$. (Hint: consider f^{-1}, but note that this is not a rational function.)

Now suppose that z_0 is attracting. The *basin of attraction* of z_0 is defined as

$$A(z_0) = \{z \mid \lim_{n\to\infty} f^n(z) = z_0\}.$$

The *immediate basin of attraction* of z_0 is the connected component of $A(z_0)$ containing z_0.

6. Prove that $A(z_0)$ is nonempty, open, and contained in the Fatou set of f.

7. Prove that $\partial A(z_0) = J(f)$.

8. Prove that the immediate basin of attraction of z_0 is also the component of the Fatou set of f containing z_0.

It turns out that the local linearization extends to a global linearization across the whole basin of attraction:

9. Prove that the Kœnigs coordinate ϕ extends to a holomorphic function on $A(z_0)$ such that

$$\phi(f(z)) = \lambda\phi(z)$$

for all $z \in A(z_0)$.

10. Prove that there is a maximal radius $0 < R < \infty$ such that ϕ^{-1} has an analytic continuation from a neighborhood of 0 to the disc $B(0, R)$. Deduce that $A(z_0)$ contains a critical point of f.

Problem 2. Superattracting Points and Böttcher Coordinates

Next we look at the case of $\lambda = 0$. In this case, we call z_0 a *superattracting* fixed point.

Note that we have the Taylor series expansion

$$f(z) = z_0 + a_m(z - z_0)^m + O((z - z_0)^{m+1}),$$

for some $m \geq 2$, $a_m \neq 0$. The integer m is called the *local degree* of f at z_0.

One might guess that iterating f near z_0 "looks like" repeated applications of the m-th power map near 0. More precisely, we have the following:

Theorem 0.2 (Böttcher)**.** *Suppose $\lambda = 0$. Then there exists a holomorphic change of coordinate $w = \phi(z)$ on a neighborhood U of z_0, such that $\phi(z_0) = 0$ and*

$$\phi(f(z)) = \phi(z)^m$$

for all $z \in U$. Moreover, ϕ is unique up to multiplication by a $(m-1)$-th root of unity.

1. Prove the uniqueness statement in Böttcher's theorem. (Hint: classify all power series which fix 0 and commute with $z \mapsto z^m$.)

2. Prove the existence statement in Böttcher's theorem. (Hint: if $z_0 = 0$ and $a_m = 1$, consider $\lim_{n\to\infty}(f^n(z))^{1/m^n}$, where the m^n-th root is chosen with power series $z + \cdots$.)

Problem 3. Subharmonic Functions

Let X be a topological space. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous if $u^{-1}((-\infty, \alpha))$ is open for all $\alpha \in \mathbb{R}$.

1. Let u be an upper semicontinuous function on a compact set K. Show that u attains a maximum on K.

2. Let u be an upper semicontinuous function on a metric space (X, d). Define
$$\phi_n(x) = \sup_{y \in X}(u(y) - nd(x, y)).$$
Show that each $\phi_n : X \to \mathbb{R}$ is continuous and $\phi_1 \geq \phi_2 \geq \cdots \geq u$ with $\lim_{n\to\infty} \phi_n = u$.

Let U be an open subset of $\mathbb{C}$. A function $u : U \to \mathbb{R} \cup \{-\infty\}$ is said to be subharmonic if it i s upper semicontinuous and for any $w \in U$, there exists $\rho > 0$ such that
$$u(w) \leq \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{it})dt$$
for any $0 \leq r < \rho$.

3. Let f be a holomorphic function on an open set U. Show that $\log|f|$ is subharmonic.

4. Let $u \in C^2(U)$ be continuously twice differentiable. Show that u is subharmonic if and only if $\triangle u \geq 0$.

We now introduce the notion of smoothing. Let $u : U \to \mathbb{R}\cup_{-\infty}$ be a locally integrable function and let $\phi : \mathbb{C} \to \mathbb{R}$ be a continuous function with support in $D(0, r)$. The convolution $u * \phi$ is defined as
$$u * \phi(z) = \int_{\mathbb{C}} u(z - w)\phi(w)dA(w).$$

5. Show that for any subharmonic function $u : U \to \mathbb{C}$, there exists a sequence of smooth subharmonic functions (u_n) such that $u_1 \geq u_2 \cdots \geq u$ and $\lim_{n\to\infty} u_n(z) = u$ for all $z \in U$. (Remark: the functions u_n might be defined on a smaller open subset $U_n \subseteq U$.

6. Let u be a subharmonic function on U. Show that $\triangle u \geq 0$ in the sense of distributions.

Let E be a subset of $\mathbb{C}$. If $E \subseteq \{u(z) = -\infty\}$ for some non-constant subharmonic function u, we say that E is a small set.

7. Let u be a subharmonic function on U. Show that if $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \in \partial U$, then $u \leq 0$ on U.

8. Let U be an open subset of $\mathbb{C}$, let E be a closed small set and let u be a subharmonic function on $U \setminus E$. Suppose that u is locally bounded on U. Prove that u has an unique subharmonic extension to the whole of U. (Hint: extend u by $u(w) = \limsup_{z\to w, z\in U\setminus E} u(z)$. To show it is subharmonic, consider $u + \epsilon v$ where $E = \{v(z) = -\infty\}$.)

9. Deduce a generalization of Riemann's removable singularity theorem: let $E \subseteq \mathbb{C}$ be a countable closed subset (not necessarily discrete) and let f be a holomorphic function on $\mathbb{C} \setminus E$ that is locally bounded on $\mathbb{C}$. Show that f extends to an entire function.