Math 274Z Problem Session

Problem Sheet 1

16 Feb 2023

1 Montel's Theorem

We will assume the following theorem.

Theorem 1.1. *(Riemann's Uniformization) Any simply connected Riemann surface is isomorphic to either $\mathbb{P}^1$, $\mathbb{C}$ or the open disc $\mathbb{D}$.*

1. Let S be a Riemann surface (not necessarily compact) and $\pi : X \to S$ a topological covering space. Show that one can put a complex structure such that $\pi : X \to S$ is locally biholomorphic, and that all automorphisms of X as a covering space are holomorphic.

Thus any Riemann surface S has a universal cover that is also a Riemann surface. In particular, it is isomorphic to S'/Γ where Γ is an automorphism group of S'.

2. Show that if S has $\mathbb{P}^1$ as an universal cover, then $S \simeq \mathbb{P}^1$.
3. Show that if S has $\mathbb{C}$ as an universal cover, then $S \simeq \mathbb{C}, \mathbb{C}^*$ or a complex torus $\mathbb{C}/\Lambda$.
4. Consider the Poincare metric
$$ds = \frac{2|dz|}{1-|z|^2} \text{ for } z = x + iy \in \mathbb{D}$$
on the open disc. Let $\rho(x, y)$ be the distance corresponding to the Poincare metric. Prove that
$$\rho(f(x), f(y)) \leq \rho(x, y)$$
for any holomorphic map $f : \mathbb{D} \to \mathbb{D}$.

If S has $\mathbb{D}$ as an universal cover, we say that S is hyperbolic. It naturally inherits a Poincare metric from $\mathbb{D}$, which we will denote by ρ_S.

5. Let S, T be two hyperbolic surfaces and $f : S \to T$ a holomorphic map. Then
$$\rho_T(f(x), f(x')) \leq \rho_S(x, x').$$

From now on, we assume that S is hyperbolic. Let $\{f_n\}$ be a sequence of maps from S to an hyperbolic open $U \subset \mathbb{P}^1$.

6. Let $\{z_j\} \subset S$ be a countable dense subset. Show that one can choose a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\lim_{n\to\infty} g_n(z_j) \in \overline{U}$ exists for all j.

7. Suppose that each of this limit points lie within U itself. Show that $\{g_n\}$ converges uniformly on any compact set K. (Hint: For a given compact set K and $\epsilon > 0$, choose finitely many z_j's such that $\rho_S(z, z_j) < \epsilon$ for all $z \in K$.)

8. Assume that one of the limit points a is on the boundary of U. Prove that $g_m(z)$ converges uniformly to a. Thus conclude the following theorem of Montel:

   Theorem 1.2. *(Montel's Theorem) Let S be a hyperbolic Riemann surface. If a collection $\mathcal{F}$ of holomorphic maps from S to $\mathbb{P}^1$ has three distinct points that never occur as values, then $\mathcal{F}$ is normal.*

2 Julia Sets of Rational Maps

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function defined over $\mathbb{C}$ with degree $d \geq 2$. The Fatou set consists of points $z \in \mathbb{P}^1$ who has an open neighbourhood U such that $\{f, f^2, \dots, \}$ is a normal family when restricted to U. The Julia set $J(f)$ is the complement of the Fatou set.

1. Prove that $z \in J(f)$ if and only if $f(z) \in J(f)$.

2. Show that $J(f^n) = J(f)$.

3. Let z_1 be any point on the Julia set. Show that for any neighbourhood U of z_1, the union of the forward images $f^n(U)$ omits at most two points of $\mathbb{P}^1$.

4. For $z_1 \in J(f)$, show that the set of iterated pre-images

   $$\{z \mid f^n(z) = z_1 \text{ for some } n \geq 0\}$$

   is everywhere dense in $J(f)$.

For a periodic point $f^n(z) = z$ of minimal period n, the derivative $(f^n)'(z) = f'(z)f'(f(z)) \cdots f'(f^{n-1}(z))$ is called the multiplier of the periodic orbit, denoted by λ. If $|\lambda| < 1, |\lambda| = 1, |\lambda| > 1$, these are called attracting, indifferent and repelling cycles respectively.

5. Let $\{z_1, \dots, z_n\}$ be an attracting cycle. The basin of attraction Ω is the open set of all points $z \in \mathbb{P}^1$ for which $f^n(z), f^{2n}(z), \dots$ converge to some point of the cycle. Show that Ω is contained in the Fatou set, but the boundary $\partial\Omega$ is in the Julia set.

6. Show that repelling cycles lie in the Julia set.

Now let's further assume that $f(z)$ is a polynomial. We let $K_f = \{z \mid f^n(z) \not\to \infty\}$. This is called the filled Julia set.

7. Show that $\mathbb{P}^1 \setminus K_f$ is a connected set containing ∞.

8. Show that the topological boundary ∂K_f is exactly the Julia set $J(f)$. (Hint: Use Montel's theorem.)

3 Brolin's Theorem

1. Let T be a distribution on $\mathbb{R}$ that is positive, i.e. $T(f) \geq 0$ for all $f \geq 0$. Show that T is of order zero.

2. Let (u_n) be a sequence of uniformly bounded subharmonic functions on $U \subseteq \mathbb{R}^2$. Show that if u_n converges uniformly to u, then $\triangle u_n$ converges weakly to $\triangle u$ too.

Now let's fix a polynomial $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$.

3. Let $G_n(z) = \frac{1}{d^n} \log^+ |f^n(z)|$. Show that G_n converges uniformly to some function G_f.

4. Show that on $\mathbb{C} \setminus K(f)$, the functions G_n are all harmonic. Conclude that $\lim_{n\to\infty} G_n = \triangle G_f$ and that $\triangle G_f$ is supported on the Julia set $J(f)$.

5. Now let $a \in \mathbb{C}$ and consider the measure δ_a. Convolving with a disc of radius ϵ gives us a measure $(\delta_a)_\epsilon$ with a smooth potential $V_\epsilon(z)$. Show there exists a constant $C > 0$ such that

$$|V_\epsilon(z) - \log^+ |z|| \leq C.$$

6. Show that

$$\lim_{n\to\infty} \frac{1}{d^n} V_\epsilon(f^n(z)) = G_f(z)$$

and hence conclude that $(f^n)^*(\delta_{a,\epsilon}) = \mu_f$ where $\mu_f = \triangle G_f$.

Now assume the following lemma:

Lemma 3.1. *For any $\delta > 0$, there exists n_0 and k_0 so that at least $(1-\delta)d^n$ elements of $f^{-n}(z_0)$ lie in a connected component of $f^{-n}(U)$ having diameter $\leq \delta$, for all $k \geq k_0$ and $n \geq n_0$ and any sufficiently small neighbourhood U of z_0.*

7. Using the lemma, prove that

$$\frac{1}{d^n} \sum_{f^n(z)=a} \delta_z \to \mu_f.$$

Let's try to prove the lemma under the assumption that $\{f^n(z) = a\}$ are all distinct and we can find a neighbourhood $a \in U$ such that $f^{-n}(U)$ contain no critical points. We will need the following distortion theorem.

Theorem 3.2. *(Koebe's distortion theorem) Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent function such that $f(0) = 0$ and $f'(0) = 1$. Then*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

8. Show that each z satisfying $f^n(z) = a$, is contained in an unique connected component of $f^{-n}(U)$, denoted by U_z, and $f^{-n} : U \to U_z$ is a biholomorphism.

9. Now apply the Koebe's distortion theorem to conclude that for a suitable choice of U, each U_z is almost a round disc, i.e. its diameter is bounded by a fixed constant times the area.

10. Conclude Lemma 3.1. (Hint: The total area is bounded by some constant.)