Abstract. Suppose \( \mathcal{E} \to B \) is a non-isotrivial elliptic surface defined over a number field, for smooth projective curve \( B \). Let \( k \) denote the function field \( \mathbb{Q}(B) \) and \( E \) the associated elliptic curve over \( k \). In this article, we construct adelically metrized \( \mathbb{R} \)-divisors \( \overline{D}_X \) on the base curve \( B \) over a number field, for each \( X \in E(k) \otimes \mathbb{R} \). We prove non-degeneracy of the Arakelov-Zhang intersection numbers \( \overline{D}_X \cdot \overline{D}_Y \), as a biquadratic form on \( E(k) \otimes \mathbb{R} \). As a consequence, we have the following Bogomolov-type statement for the Néron-Tate height functions on the fibers \( E_t(\mathbb{Q}) \) of \( E \) over \( t \in B(\mathbb{Q}) \): given points \( P_1, \ldots, P_m \in E(k) \) with \( m \geq 2 \), there exist a non-repeating infinite sequence \( t_n \in B(\mathbb{Q}) \) and small-height perturbations \( P_{i,t_n}^{\prime} \in E_{t_n}(\mathbb{Q}) \) of specializations \( P_{i,t_n} \) so that the set \( \{P_{1,t_n}^{\prime}, \ldots, P_{m,t_n}^{\prime}\} \) satisfies at least two independent linear relations for all \( n \), if and only if the points \( P_1, \ldots, P_m \) are linearly dependent in \( E(k) \). This gives a new proof of results of Masser and Zannier [MZ1, MZ2] and of Barroero and Capuano [BC] and extends our earlier results [DM]. We also prove a general equidistribution theorem for adelically metrized \( \mathbb{R} \)-divisors on curves (over a number field), extending the equidistribution theorems of Chambert-Loir, Thuillier, and Yuan [CL1, Th, Yu], and using results of Moriwaki [Mo2].

1. Introduction

Suppose \( \mathcal{E} \to B \) is an elliptic surface defined over a number field \( K \). That is, \( \mathcal{E} \) is a projective surface, \( B \) is a smooth projective curve, and there exists a section \( O : B \to \mathcal{E} \), all defined over \( K \), so that all but finitely many fibers \( E_t \), for \( t \in B(\overline{K}) \), are smooth elliptic curves with zero \( O_t \). We say that the elliptic surface \( \mathcal{E} \to B \) is isotrivial if all of the smooth fibers \( E_t \) are isomorphic over \( \overline{K} \). Let \( k \) denote the function field \( \mathbb{K}(B) \); we also view the surface \( \mathcal{E} \) as an elliptic curve \( E \) over the field \( k \).

In this article, we study the geometry and arithmetic of the set \( E(k) \) of rational points over the function field \( k \) when \( \mathcal{E} \to B \) is not isotrivial. To this end, we consider height functions associated to adelically metrized \( \mathbb{R} \)-divisors on the base curve \( B \) over a number field \( K \), and we prove an equidistribution theorem for these heights. We study the Arakelov-Zhang intersection number of metrized \( \mathbb{R} \)-divisors and prove that it induces a non-degenerate biquadratic form on \( E(k) \otimes \mathbb{R} \). We relate this theorem to existing results, and provide, for

Date: January 14, 2021.
example, a new proof of results of Masser and Zannier and of Barroero and Capuano on linear relations between specializations of independent sections.

1.1. Heights and the Arakelov-Zhang intersection of points in \( E(k) \). Assume that \( \mathcal{E} \to B \) is not isotrivial. Let \( \hat{h}_E \) denote the Néron-Tate canonical height on \( E(\overline{k}) \), associated to the choice of divisor \( O \) on \( E \); let \( \hat{h}_{E_t} \) denote the corresponding canonical height on the smooth fibers \( E_t(\overline{K}) \) for (all but finitely many) \( t \in B(\overline{K}) \). By non-isotriviality, a point \( P \in E(k) \) satisfies \( \hat{h}_E(P) = 0 \) if and only if it is torsion on \( E \). We denote the specializations of \( P \) by \( P_t \) in the fiber \( E_t \). Tate showed in [Ta] that the canonical height function

\[
h_P(t) := \hat{h}_{E_t}(P_t)
\]

is a Weil height on the base curve \( B(\overline{K}) \), up to a bounded error. More precisely, there exists a \( \mathbb{Q} \)-divisor \( D_P \) on \( B \) of degree equal to \( \hat{h}_E(P) \) so that \( h_P(t) = h_{D_P}(t) + O(1) \), where \( h_{D_P} \) is a Weil height on \( B(\overline{K}) \) associated to \( D_P \). In [DM], we showed that we can also understand the small values of the function (1.1), from the point of view of equidistribution. Assume that \( \hat{h}_E(P) > 0 \) (so that the function \( h_P \) is nontrivial) and that, as a section, \( P : B \to \mathcal{E} \) is defined over the number field \( K \). Building on work of Silverman [Si2, Si4, Si5], we showed that \( h_P \) is the height induced by an ample line bundle on \( B \) (with divisor \( D_P \)) equipped with a continuous, adelic metric of non-negative curvature defined over \( K \), denoted by \( \overline{D}_P \) and satisfying

\[
\overline{D}_P \cdot \overline{D}_P = 0
\]

for the Arakelov-Zhang intersection number introduced in [Zh4]. In particular, we can then apply the equidistribution theorems of [CL1, Th, Yu] to deduce that the Gal(\( \overline{K}/K \))-orbits of points \( t_n \in B(\overline{K}) \) with height \( h_P(t_n) \to 0 \) are uniformly distributed on \( B(\mathbb{C}) \) with respect to the curvature distribution \( \omega_P \) for \( \overline{D}_P \) at an archimedean place of \( K \). A similar equidistribution occurs at each place \( v \) of \( K \) to a measure \( \omega_{P,v} \) on the Berkovich analytification \( B^an_v \) [DM, Corollary 1.2].

As a consequence of our main result in [DM], and combined with the results of Masser and Zannier [MZ1, MZ2], we have

\[
\overline{D}_P \cdot \overline{D}_Q \geq 0 \quad \text{for all} \quad P,Q \in E(k), \quad \text{and}
\]

\[
\overline{D}_P \cdot \overline{D}_Q = 0 \quad \iff \quad \exists \alpha > 0 \text{ such that } h_P(t) = \alpha h_Q(t) \text{ for all } t \in B(\overline{K})
\]

\[
\iff \exists (n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\} \text{ such that } nP = mQ
\]

In particular, by the non-degeneracy of the Néron-Tate bilinear form \( \langle \cdot, \cdot \rangle_E \) on \( E(k) \), we have

\[
\overline{D}_P \cdot \overline{D}_Q = 0 \quad \iff \quad \hat{h}_E(P)\hat{h}_E(Q) = \langle P, Q \rangle^2_E
\]

for all \( P,Q \in E(k) \).
The main result of this article is the proof of a stronger version of (1.3):

**Theorem 1.1.** Let $\mathcal{E} \to B$ be a non-isotrivial elliptic surface defined over a number field $K$. Let $E$ be the corresponding elliptic curve over the field $k = \overline{K}(B)$. There exists a constant $c > 0$ so that

$$c \left( \hat{h}_E(P) \hat{h}_E(Q) - \langle P, Q \rangle^2_E \right) \leq D_P \cdot D_Q \leq c^{-1} \left( \hat{h}_E(P) \hat{h}_E(Q) - \langle P, Q \rangle^2_E \right)$$

for all $P, Q \in E(k)$, where $\langle \cdot, \cdot \rangle_E$ is the Néron-Tate bilinear form on $E(k)$.

**Remark 1.2.** The upper bound on $D_P \cdot D_Q$ in Theorem 1.1 is relatively straightforward. The difficulty lies in the lower bound; in Section 7, we observe that this is equivalent to proving that $D_X \cdot D_Y > 0$ for all independent $X, Y \in E(k) \otimes \mathbb{R}$.

### 1.2. Motivation and context.

Theorem 1.1 was inspired by the statements and proofs of the Bogomolov Conjecture [Ul, Zh3, SUZ], extending Raynaud’s theorem that settled the Manin-Mumford Conjecture [Ra], and the “Mordell-Lang plus Bogomolov” results of Poonen [Po] and Zhang [Zh1], in the spirit of the conjectures of Pink and Zilber. (See [Za] for background and additional references.)

Theorem 1.1 can be seen as a Bogomolov-type bound. The intersection number $D_P \cdot D_Q$ is related to the small values of the heights $\hat{h}_E(P) + \hat{h}_E(Q)$ in the fibers $E_t(K)$. Indeed, as a consequence of Zhang’s Inequality [Zh4, Theorem 1.10] applied to the sum $D_P + D_Q$, and the fact that $h_P(t) \geq 0$ at all points $t \in B(K)$ for every $P \in E(k)$ [DM, Proposition 4.3], we have

$$\frac{1}{2} \text{ess.min.} (h_P + h_Q) \leq \frac{D_P \cdot D_Q}{\hat{h}_E(P) + \hat{h}_E(Q)} \leq \text{ess.min.} (h_P + h_Q)$$

for every pair of non-torsion $P, Q \in E(k)$. The essential minimum is defined by $\text{ess.min.}(f) = \sup_F \inf_{B \setminus F} f$ over all finite sets $F$ in $B(K)$. Bogomolov-type bounds have found many applications in problems of unlikely intersections. In Section 7, we explain that Theorem 1.1 is equivalent to the following:

**Theorem 1.3.** Let $\mathcal{E} \to B$ be a non-isotrivial elliptic surface defined over a number field, and let $\pi : \mathcal{E}^m \to B$ be its $m$-th fibered power with $m \geq 2$. Let $\mathcal{E}^{m,\{2\}}$ denote the union of flat subgroup schemes of $\mathcal{E}^m$ of codimension at least 2, and consider the tubular neighborhood

$$T(\mathcal{E}^{m,\{2\}}, \epsilon) = \left\{ P \in \mathcal{E}^m : \exists P' \in \mathcal{E}^{m,\{2\}}(\overline{K}) \text{ with } \pi(P) = \pi(P') \text{ and } \hat{h}_{E_{\pi(P)}}(P - P') \leq \epsilon \right\}$$

Then, for any irreducible curve $C$ in $\mathcal{E}^m$, defined over a number field and dominating $B$, there exists $\epsilon > 0$ such that $C \cap T(\mathcal{E}^{m,\{2\}}, \epsilon)$ is contained in a finite union of flat subgroup schemes of positive codimension.
See, e.g., [BC, Lemma 2.2] for definitions and a classification of flat subgroup schemes.

**Remark 1.4.** The conclusion of Theorem 1.3 with $\epsilon = 0$ is a result of Barroero and Capuano [BC, Theorem 2.1]: using techniques involving o-minimality and transcendence theory, similar to those of [MZ1, MZ2] (which treated the intersections of curves $C$ with $T(E^{m,(m)}, 0)$, the torsion subgroups), they show that $C \cap T(E^{m,(2)}, 0)$ is contained in a finite union of flat subgroup schemes of positive codimension. Thus Theorem 1.3 may be seen as a Bogomolov-type extension of [BC, Theorem 2.1], while providing a new proof of results in [BC, MZ1, MZ2].

Our main result in [DM] treated the intersections of $C$ with the smaller tube $T(E^{m,(m)}, \epsilon)$.

In [Zh2], Zhang proposed the investigation of a function on the base curve $B$ that detects drops in rank of the specializations of a subgroup of $E(k)$: given a finitely-generated subgroup $\Lambda$ of $E(k)$ of rank $m \geq 1$, if the torsion-free part of $\Lambda$ is generated by $S_1, \ldots, S_m$, let

$$h_{\Lambda}(t) := \text{det}(\langle S_i, S_j \rangle_t)_{i,j} \geq 0$$

on $B(K)$, where defined, where $\langle \cdot, \cdot \rangle_t$ is the Néron-Tate bilinear form on the specialization $\Lambda_t$ in the fiber $E_t$.

We propose the following result as the analog of [Zh2, §4 Conjecture] for elliptic surfaces; Zhang’s conjecture was formulated for geometrically simple families of abelian varieties $A \to B$ of relative dimension $> 1$, and it does not hold as stated for elliptic surfaces [Zh2, §4 Remark 3].

**Theorem 1.5.** Let $E \to B$ be a non-isotrivial elliptic surface defined over a number field $K$, and let $E$ be the corresponding elliptic curve over the field $k = \overline{K}(B)$. Let $\Lambda \subset E(k)$ be a subgroup of rank $m \geq 2$, with torsion-free part generated by $S_1, \ldots, S_m \in E(k)$. For each $i = 1, \ldots, m$, let $\Lambda_i \subset \Lambda$ be generated by $\{S_1, \ldots, S_m\} \setminus \{S_i\}$. There is a constant $\epsilon = \epsilon(\Lambda) > 0$ so that

$$\{t \in B(K) : h_{\Lambda_1}(t) + \cdots + h_{\Lambda_m}(t) \leq \epsilon\}$$

is finite.

In Section 7, we observe that Theorem 1.5 is equivalent to Theorems 1.1 and 1.3.

**Remark 1.6.** Note that, for rank 1 groups $\Lambda$, the value $h_{\Lambda}(t)$ is the canonical height of the generating point $S_t$ in $E_t$. In general, recall that the Néron-Tate height $\hat{h}_{E_t}$ on a smooth fiber defines a positive definite quadratic form in $E_t(\overline{Q}) \otimes \mathbb{R}$; see e.g. [Si6, Ch. VIII, Prop. 9.6]. Thus, $h_{\Lambda}$ will vanish at $t \in B(K)$ if and only if rank $\Lambda_t < \text{rank} \Lambda$. The sum $h_{\Lambda_1}(t) + \cdots + h_{\Lambda_m}(t)$ will be zero if and only if the points $S_{1,t}, \ldots, S_{m,t}$ satisfy (at least) two independent linear relations over $\mathbb{Z}$ in the fiber $E_t$.

**Remark 1.7.** The independence of the points $S_1, \ldots, S_m \subset \Lambda$ in Theorem 1.5 is necessary for the finiteness statement to hold. Indeed, suppose that $S_m$ is a linear combination of
$S_1, \ldots, S_{m-1}$, and suppose that $\{t_n\} \subset B(\overline{K})$ is any infinite non-repeating sequence for which $h_{S_m}(t_n) \to 0$ (for example, we can take $t_n$ where $S_{m,t_n}$ is torsion; see e.g. [DM, Proposition 6.2]). As we will see in Proposition 5.3, we then have $h_{\Lambda_1}(t_n) + \cdots + h_{\Lambda_m}(t_n) \to 0$.

If the elliptic surface $E \to B$ is isotrivial, the conclusion of Theorem 1.3 with $\epsilon = 0$ was established by Viada [Vi1] and Galateau [Ga]. Moreover, in this isotrivial setting, Viada proved the analogue of Theorem 1.3 (for positive effective $\epsilon$) in [Vi2, Theorem 1.4], [Vi1, Theorem 1.2], providing in particular new proofs of instances of earlier results by Poonen [Po] and Zhang [Zh1] and extending the work in [RV]. It is worth pointing out that the aforementioned results invoked a different Bogomolov-type bound than the one in Theorem 1.1, established by Galateau [Ga]. The isotrivial case for $\epsilon = 0$ is completed for curves in arbitrary abelian varieties in [HP] using, amongst others, techniques from o-minimality. In the setting of the multiplicative group $\mathbb{G}_m^n$, Habegger established results of this flavor in arbitrary dimension [Ha1], generalizing a result of Bombieri-Masser-Zannier [BMZ].

Finally, we remark that analogues of Theorem 1.1, Theorem 1.3 and Theorem 1.5 can be formulated for products of non-isogenous elliptic surfaces, as we did in [DM, Theorem 1.4]. Our methods here would yield these results and, in particular, Barroero-Capuano’s [BC2, Theorem 1.1]. We omit them in this article to simplify our exposition.

1.3. Metrized $\mathbb{R}$-divisors on curves. For each $t \in B(\overline{K})$ with $E_t$ smooth, the canonical height $\hat{h}_{E_t}$ induces a positive definite quadratic form on $E_t(\overline{K}) \otimes \mathbb{R}$. The height functions $h_P$ on $B(\overline{K})$, defined by (1.1) for $P \in E(k)$, therefore make sense for elements of the finite-dimensional vector space $E(k) \otimes \mathbb{R}$. In Theorem 4.6, we prove that every nonzero element $X \in E(k) \otimes \mathbb{R}$ gives rise to a continuous, adelic, semi-positive metrization $\overline{D}_X$ of an ample $\mathbb{R}$-divisor on the base curve $B$, defined over a number field $K$, with height function $h_X(t) = \hat{h}_{E_t}(X_t)$ for $t \in B(\overline{K})$ when $E_t$ is smooth, satisfying $\overline{D}_X \cdot \overline{D}_X = 0$.

Consequently, we are able to employ results of Moriwaki [Mo2] in our proof of Theorem 1.1. Specifically, we use his arithmetic Hodge index theorem for adelically metrized $\mathbb{R}$-divisors on curves defined over a number field [Mo2, Corollary 7.1.2] to understand when $\overline{D}_X \cdot \overline{D}_Y = 0$ for $X, Y \in E(k) \otimes \mathbb{R}$.

Although not needed for Theorem 1.1, we provide another application of Moriwaki’s arithmetic Hodge index theorem [Mo2, Theorem 7.1.1], combined with the continuity of the arithmetic volume function [Mo2, Theorem 5.3.1] and a generalization of Zhang’s fundamental inequality [CLT, Lemme 5.1]. We prove that metrizations of $\mathbb{R}$-divisors on curves with these nice properties (continuous, adelic, semipositive) obey an equidistribution law. This extends the equidistribution theorems of Chambert-Loir, Thuillier, and Yuan [CL1, Th, Yu] and follows a proof strategy originating in [SUZ]:
Theorem 1.8. Let $B$ be a smooth projective curve defined over a number field $K$. Fix an ample $\mathbb{R}$-divisor $D$ on $B$, equipped with an adelic, semipositive and normalized metrization $\overline{D}$ over $K$. Let $M$ be any admissible adelic metrization on an $\mathbb{R}$-divisor $M$ over $K$. For any (non-repeating) sequence $x_n \in B(K)$ with $h_{\overline{D}}(x_n) \to 0$, we have

$$h_M(x_n) \to \frac{D \cdot M}{\deg D}.$$ 

Corollary 1.9. Fix any ample $\mathbb{R}$-divisor $D$ on $B$, equipped with an adelic, semipositive and normalized metrization $\overline{D}$ over $K$. For each place $v$ of $K$ and for any (non-repeating) infinite sequence $x_n \in B(K)$ with $h_{\overline{D}}(x_n) \to 0$, the discrete probability measures

$$\mu_n = \frac{1}{|\Gal(K/K) \cdot x_n|} \sum_{y \in \Gal(K/K) \cdot x_n} \delta_y$$

converge weakly in $B^\an v$ to the probability measure $\mu_\overline{D},v = \frac{1}{\deg \overline{D}} \omega_{\overline{D},v}$.

Here, $B^\an v$ denotes the Berkovich analytification of the curve $B$ over the complete and algebraically closed field $\mathbb{C}_v$.

Returning to the setting of $E \to B$ and its sections, Corollary 1.9 applies to sequences in the base curve $B$ where the specializations of points in $E(k)$ satisfy non-trivial linear relations. For example, generalizing [DM, Corollary 1.2], we show:

Theorem 1.10. Let $E \to B$ be a non-isotrivial elliptic surface defined over a number field $K$, and let $E$ be the corresponding elliptic curve over the field $k = K(B)$. Suppose that $P_1, \ldots, P_m$ is a collection of $m \geq 1$ linearly independent points in $E(k)$, also defined over $K$ as sections of $E \to B$. Suppose that $\{t_n\} \subset B(K)$ is an infinite non-repeating sequence where

$$a_{1,n}P_{1,t_n} + a_{2,n}P_{2,t_n} + \cdots + a_{m,n}P_{m,t_n} = O_{t_n}$$

for $a_{i,n} \in \mathbb{Z}$, with $[a_{i,n} : \cdots : a_{m,n}] \to [x_1 : \cdots : x_m]$ in $\mathbb{RP}^{m-1}$ as $n \to \infty$. Set

$$X = x_1P_1 + \cdots + x_nP_n \in E(k) \otimes \mathbb{R}.$$ 

Then

$$h_X(t_n) \to 0$$

for the height function associated to the metrized $\mathbb{R}$-divisor $\overline{D}_X$. Moreover, for each place $v$ of $K$, the $\Gal(K/K)$-orbits of $t_n$ in $B(K)$ are uniformly distributed on $B^\an v$ with respect to the curvature distribution

$$\omega_{X,v} = \sum_i \left( x_i^2 - \sum_{j \neq i} x_i x_j \right) \omega_{P_i,v} + \sum_{i < j} x_i x_j \omega_{P_i + P_j,v}.$$
Remark 1.11. For nonzero $X \in E(k) \otimes \mathbb{R}$, the height $h_X$ will have only finitely many zeros unless a positive real multiple $cX$ is represented by an element of $E(k)$; see Proposition 5.5. On the other hand, there is always an infinite sequence $\{t_n\}$ for which (1.6) is satisfied; see Proposition 5.1.

1.4. Measures at an archimedean place. In [DM], the curvature measures $\omega_P$ for the metrized divisor $\overline{D}_P$ associated to $P \in E(k)$, at an archimedean place, are computed as the pullback by $P$ of a certain $(1, 1)$-form on $E(\mathbb{C})$, via a dynamical construction. In [CDMZ], it is shown that $\omega_P = db_1 \wedge db_2$ in the Betti coordinates $(b_1, b_2)$ of $P$. We explain in Section 8 that elements $X \in E(k) \otimes \mathbb{R}$ are also represented by holomorphic curves in the surface $E$, and the Betti coordinates of $X$ are real linear combinations of the Betti coordinates of points $P_i \in E(k)$. We use this to prove that the measure $\omega_X$, at a single archimedean place of the number field $K$, is enough to uniquely determine the pair of points $X$ and $-X$:

**Theorem 1.12.** Fix $X$ and $Y$ in $E(k) \otimes \mathbb{R}$ and an archimedean place of the number field $K$. Let $\omega_X$ and $\omega_Y$ denote the curvature distributions on $B(\mathbb{C})$ at this place for the adelically metrized $\mathbb{R}$-divisors $\overline{D}_X$ and $\overline{D}_Y$. Then

$$\omega_X = \omega_Y \iff X = \pm Y.$$

We are grateful to Lars Kühne for helping us with the proof of Theorem 1.12; we use the holomorphic-antiholomorphic trick of André, Corvaja, and Zannier [ACZ, §5] and a transcendence result of Bertrand [Be, Théorème 5]. A special case of Theorem 1.12 was proved by a different method in [DWY, Proposition 1.9].

1.5. Example. Let $E_t$ be the Legendre elliptic curve defined by

$$y^2 = x(x - 1)(x - t)$$

for $t \in \mathbb{Q} \setminus \{0, 1\}$. By filling in the family over $t = 0, 1, \infty$, we obtain an elliptic surface $E \to B$ with $B = \mathbb{P}^1$ defined over $\mathbb{Q}$. Here $k = \mathbb{Q}(t)$. It is easy to see that rank $E(k) = 0$. However, by choosing any collection of $m$ distinct points $x_1, x_2, \ldots, x_m \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$, we can construct an elliptic surface $E' \to B'$ with rank $E'(k') \geq m$ where $k' = \mathbb{Q}(B')$. Indeed, we let $P_{x_i}$ be a point with constant $x$-coordinate equal to $x_i$. As the points $x_i$ are distinct, the structure of the field extensions $k_i/k$, determined by each $P_{x_i}$, implies that the points must be independent. We pass to a branched cover $B' \to B$ so that each $P_{x_i}$ defines a section over $B'$ and set $k' = \mathbb{Q}(B')$. These examples were first considered in [MZ1] and the associated measures $\omega_{P_{x_i}}$ on $B'(\mathbb{C})$ (or rather, their projections to $B = \mathbb{P}^1$) were computed in [DWY].

1.6. Outline of the article. In Section 3, we introduce metrizations on $\mathbb{R}$-divisors on curves defined over a number field and define their intersection numbers. We then prove the equidistribution results, Theorem 1.8 and Corollary 1.9. In Section 4, we prove that each
nonzero element $X \in E(k) \otimes \mathbb{R}$ induces a continuous, adelic, semipositive metrization $D_X$ on an ample $\mathbb{R}$-divisor on the base curve $B$. In Section 5, we study the sequences of small points for the height function $h_X$ on $B(\mathbb{Q})$ associated to $D_X$. In Section 6 we lay out the basic properties of the intersection number $(X,Y) \mapsto D_X \cdot D_Y$ as a biquadratic form on the vector space $E(k) \otimes \mathbb{R}$. In Section 7, we analyze the significance of $D_X \cdot D_Y = 0$ for nonzero $X,Y \in E(k) \otimes \mathbb{R}$, and we explain how to relate Theorems 1.1, 1.3, and 1.5. Section 8 contains a proof of Theorem 1.12, and we complete the proofs of Theorems 1.1, 1.3, and 1.5 in Section 9.

1.7. Acknowledgements. We spoke with many people about this work, and we are grateful to all of them for helpful discussions. Special thanks go to Fabrizio Barroero, Daniel Bertrand, Laura Capuano, Gabriel Dill, Philipp Habegger, Lars Kühne, Harry Schmidt, Xinyi Yuan, and Umberto Zannier.

2. Notation

Throughout this article, $K$ denotes a number field. We let $M_K$ denote its set of places, with absolute values $|\cdot|_v$ satisfying the product formula:

\begin{equation}
\prod_{v \in M_K} |x|_v^{[K_v:Q_v]} = 1
\end{equation}

for all nonzero $x$ in $K$. Here $K_v$ denotes the completion of $K$ with respect to $|\cdot|_v$. We set

\begin{equation}
r_v := \frac{[K_v:Q_v]}{[K:Q]}.
\end{equation}

For each place $v \in M_K$, we let $\mathbb{C}_v$ denote the completion of an algebraic closure of $K_v$.

We let $B$ denote a smooth projective curve defined over a number field $K$. For each $v \in M_K$, we let $B_v^{an}$ denote the Berkovich analytification of $B$ over the field $\mathbb{C}_v$.

We let $\text{Div}_Z(B)$ denote the group of divisors on $B$.

Throughout, $k$ denotes the function field $\overline{K}(B)$. Its places are in one-to-one correspondence with the elements $t \in B(\overline{K})$, with absolute values given by $|f|_t = \exp(-\text{ord}_t(f))$ for each nonzero $f \in \overline{K}(B)$.

3. $\mathbb{R}$-divisors and equidistribution

In this section, we introduce metrizations on $\mathbb{R}$-divisors on curves, following Moriwaki [Mo2]. We show that an equidistribution theorem holds for sequences of small height on a curve $B$ defined over a number field, for adelic semipositive metrizations $D$ associated to an
The proofs of Theorem 1.8 and Corollary 1.9 follow a known strategy for equidistribution. We mimic the presentation of Chambert-Loir and Thuillier in [CLT]; they themselves appeal to results of Yuan [Yu] and Zhang [Zh4], building on the ideas that originally appeared in [SUZ]. The key ingredient for passing from $\mathbb{Q}$-divisors to $\mathbb{R}$-divisors is the continuity of the arithmetic volume function on the space of metrized of $\mathbb{R}$-divisors, as proved by Moriwaki [Mo2]. We provide the details for completeness.

3.1. Metrizations of $\mathbb{R}$-divisors on curves. Let $B$ be a smooth projective curve defined over a number field $K$. Let $D = \sum a_i D_i$ be an ample $\mathbb{R}$-divisor on $B$, with $a_i \in \mathbb{R}$ and $D_i \in \text{Div}_Z(B)$ with support in $B(K)$, invariant under the action of $\text{Gal}(\overline{K}/K)$. By rewriting the sum if necessary, we may assume that each $D_i$ is associated to an ample line bundle $L_i$ that extends over the Berkovich analytification $B^\text{an}$ for each place $v$ of $K$.

A continuous, adelic metrization for $D$ is a collection of continuous functions $g_v : B^\text{an}_v \setminus \text{supp} D \to \mathbb{R}$ for $v \in M_K$, such that

1. For each $v$, the locally-defined function $\psi_v := g_v + \sum a_i \log |f_i|_v$ extends continuously to the support of $D$, where $f_i$ is a local defining equation for $D_i$ defined over $K$;
2. there exists a model $(B, D)$ of $(B, D)$ over the ring of integers $O_K$ so that $g_v$ is the associated model function for all but finitely many $v$, or equivalently, the function $\psi_v \equiv 0$ at all but finitely many places $v$ for the associated $\{f_i\}$ near each element of $\text{supp} D$.

See [Mo2, §0.2] and [CL2, §1.3.2] for the definition of model functions.

The metrization is semipositive if each $g_v$ is subharmonic on $B^\text{an}_v \setminus \text{supp} D$. An $\mathbb{R}$-divisor $D$ on $B$ and collection of continuous functions $g_v : B^\text{an}_v \setminus \text{supp} D \to \mathbb{R}$, for $v \in M_K$, is said to be admissible if $D = D_1 - D_2$ and $g_v = g_{v,1} - g_{v,2}$ for two adelic, semipositive metrizations on $\mathbb{R}$-divisors $\overline{D}_i = (D_i, \{g_i,v\})$. We write $\overline{D} = \overline{D}_1 - \overline{D}_2$ for the data $(D, \{g_v\})$. An associated height function is given by $h_\overline{D} = h_{\overline{D}_1} - h_{\overline{D}_2}$.

We denote this data by $\overline{D} = (D, \{g_v\})$. Moriwaki calls $\overline{D}$ an adelic arithmetic $\mathbb{R}$-divisor of $(C^0 \cap \text{PSH})$-type on $B$ [Mo2]. It extends Zhang’s notion of an adelic, semipositive metric on a line bundle to $\mathbb{R}$-divisors [Zh4]. Indeed, for $D$ an ample divisor on $B$ associated to a line bundle $L$, equipped with an adelic metric $\{\|\cdot\|_v\}_v$, and $s$ a meromorphic section of $L$ with $(s) = D$, we put $g_v = - \log \|s\|_v$ at each place $v$ of the number field $K$.

We let $\omega_{\overline{D},v}$ denote the curvature distribution on $B^\text{an}_v$ associated to the metrization $\overline{D}$; by definition, this is a positive measure of total mass $\deg D$, equal to the Laplacian of $g_v$ away from $\text{supp} D$. See, for example, [BR] for more information about the distribution-valued...
Laplacian on Berkovich curves. The associated probability measure is denoted by

$$\mu_{D,v} := \frac{1}{\deg D} \omega_{D,v}.$$  

There is an associated height function on $B(\mathbb{K})$ defined by

$$(3.1)\quad h_D(x) := \sum_{v \in M_K} \frac{r_v}{|\text{Gal}(\overline{K}/K) \cdot x|} \sum_{x' \in \text{Gal}(\overline{K}/K) \cdot x} g_v(x'),$$

for $x \notin \text{supp } D$. Recall that $r_v$ was defined in (2.2). For any rational function $\phi$ on $B$ defined over $K$, and for any real $a \in \mathbb{R}$, note that

$$h_D(x) = \sum_{v \in M_K} \frac{r_v}{|\text{Gal}(\overline{K}/K) \cdot x|} \sum_{x' \in \text{Gal}(\overline{K}/K) \cdot x} (g_v - a \log |\phi|_v)(x')$$

away from $(\text{supp } D) \cup (\text{supp } \phi)$, from the product formula (2.1). This allows definition (3.1) to extend to the points $x \in \text{supp } D$, by choosing any $\phi$ so that $x \in \text{supp } \phi$ and $a$ so that $g_v - a \log |\phi|_v$ extends continuously at $x$ for every $v$. For an $\mathbb{R}$-divisor $D' = \sum_i b_i [x_i]$ with support in $B(\overline{K})$, we will write

$$h_D(D') := \sum_i b_i h_D(x_i).$$

3.2. Intersection. For divisors $D_1, D_2 \in \text{Div}_{\mathbb{Z}}(B)$ associated to line bundles $L_1$ and $L_2$, respectively, equipped with continuous, adelic metrics $\overline{D}_1$ and $\overline{D}_2$, the arithmetic intersection number is defined in [Zh4] (see also [CL2]) as

$$(3.2)\quad \overline{D}_1 \cdot \overline{D}_2 := h_{\overline{D}_1}(s_2) + \sum_{v \in M_K} r_v \int_{B_v^{\mathbb{R}}} (-\log \|s_2\|_{\overline{D}_{2,v}}) \, d\omega_{\overline{D}_{1,v}}$$

$$= h_{\overline{D}_1}(s_2) + h_{\overline{D}_2}(s_1) + \sum_{v \in M_K} r_v \int_{B_v^{\mathbb{R}}} (-\log \|s_2\|_{\overline{D}_{2,v}}) \, d\omega_{\overline{D}_{1,v}} - \delta(s_1)$$

$$= h_{\overline{D}_1}(s_2) + h_{\overline{D}_2}(s_1) + \sum_{v \in M_K} r_v \int_{B_v^{\mathbb{R}}} (-\log \|s_2\|_{\overline{D}_{2,v}}) \Delta(-\log \|s_1\|_{\overline{D}_{1,v}})$$

$$= \overline{D}_2 \cdot \overline{D}_1,$$

where $s_i$ is a meromorphic section of $L_i$ defined over $K$, for $i = 1, 2$, with divisors $(s_1)$ and $(s_2)$ of disjoint support. For the continuous, adelic metrizations of $\mathbb{R}$-divisors, we extend by $\mathbb{R}$-linearity, so that

$$(3.3)\quad \overline{D}_1 \cdot \overline{D}_2 = h_{\overline{D}_1}(D_2) + \sum_{v \in M_K} r_v \int_{B_v^{\mathbb{R}}} g_{\overline{D}_{2,v}} \, d\omega_{\overline{D}_{1,v}} = \overline{D}_2 \cdot \overline{D}_1.$$
Remark 3.1. The intersection number (3.3) coincides with $\widehat{\deg}(D_1 \cdot D_2)$ of [Mo2]. Indeed, [Mo2, Theorem 4.1.3] states that each $D$ can be uniformly approximated by metrizations associated to models, and it is known that the intersection numbers coincide for these model metrics [Mo1, Proposition 2.1.1].

We say $D$ is normalized if its self-intersection number satisfies

$$D \cdot D = 0.$$ 

Note that any continuous, adelic metrization can be normalized by adding a constant to $g_v$ at some place.

For each $a \in \mathbb{R}$ and $D = (D, \{g_v\})$, we write $aD$ for the pair $(aD, \{ag_v\})$. Normalized metrized divisors $D_1$ and $D_2$ on $B$ are isomorphic if $D_1 - D_2$ is principal, meaning that there are rational functions $\phi_1, \ldots, \phi_m \in K(B)$ and real numbers $a_1, \ldots, a_m$, so that

$$D_1 - D_2 = \sum_{i=1}^m a_i \left((\phi_i), \{-\log |\phi_i|_v\}_{v \in M_K}\right).$$

Note that by the product formula the height function $h_D$ depends only on the isomorphism class of $D$.

We formulate Moriwaki’s arithmetic Hodge-index theorem as follows:

Theorem 3.2. [Mo2, Corollary 7.1.2] Suppose $D_1$ and $D_2$ are normalized continuous semi-positive adelic metrizations on ample $\mathbb{R}$-divisors with $\deg D_1 = \deg D_2$. Then $D_1 \cdot D_2 \geq 0$, and $D_1 \cdot D_2 = 0$ if and only if $D_1$ and $D_2$ are isomorphic.

Proof. Set $D = D_1 - D_2$, so that the underlying divisor $D$ has degree 0, and

$$D \cdot D = -2D_1 \cdot D_2.$$

From [Mo2, Corollary 7.1.2], we have that $D \cdot D \leq 0$ with equality if and only if $D$ is principal, up to the addition of a constant to the metrization $g_v$ at some place. The normalization of $D_1$ and $D_2$ implies that this constant must be 0. \qed

3.3. Essential minima. Following [Zh4], the essential minimum of the height $h_D$ is defined as

$$(3.4) \quad e_1(D) := \sup_F \inf_{x \in B(\overline{K}) \setminus F} h_D(x),$$

over all finite subsets $F$ of $B(\overline{K})$, and we put

$$e_2(D) := \inf_{x \in B(\overline{K})} h_D(x).$$
**Theorem 3.3.** [Zh4, Theorem 1.10] For any adelic, semipositive metrization $D$ of an ample $\mathbb{R}$-divisor $D$, we have

$$e_1(D) \geq \frac{D \cdot D}{2 \deg D} \geq \frac{1}{2} \left(e_1(D) + e_2(D)\right)$$

**Proof.** Zhang proved the result for ample line bundles equipped with adelic, semipositive metrics [Zh4, Theorem 1.10]. It holds also for metrizations of $\mathbb{R}$-divisors because the height function associated to an $\mathbb{R}$-divisor is a uniform limit of heights associated to $\mathbb{Q}$-divisors, and the intersection number is a bilinear form on metrized divisors. □

Using the upper bound on $D \cdot D$ in Theorem 3.3, we can extend Theorem 3.2 to:

**Theorem 3.4.** Suppose $D_1$ and $D_2$ are normalized semipositive adelic metrizations on ample $\mathbb{R}$-divisors of the same degree, and suppose the essential minimum of at least one of the $D_i$ is 0. Then the following are equivalent:

1. $D_1 \cdot D_2 = 0$
2. $D_1$ and $D_2$ are isomorphic
3. $h_{D_1} = h_{D_2}$ on $B(\overline{K})$
4. $h_{D_1} = h_{D_2}$ at all but finitely many points of $B(\overline{K})$
5. there exists an infinite non-repeating sequence $t_n$ in $B(\overline{K})$ for which

$$\lim_{n \to \infty} \left( h_{D_1}(t_n) + h_{D_2}(t_n) \right) = 0.$$

**Proof.** We have (1) $\iff$ (2) from Theorem 3.2. The definition of the height function, in view of the product formula, implies that (2) $\implies$ (3), and we clearly have (3) $\implies$ (4). The essential minimum being 0 for $D_1$ or for $D_2$ gives (4) $\implies$ (5). Finally, assume (5). Theorem 3.3 implies that $e_1(D_i) \geq 0$, for $i = 1, 2$, because $D_i$ is normalized. Therefore, we also have $e_1(D_1 + D_2) \geq 0$ for the essential minimum of the sum $h_{D_1} + h_{D_2}$. The existence of the sequence $\{t_n\}$ thus implies that $e_1(D_1 + D_2) = 0$. As $D_1 \cdot D_2 \geq 0$ from Theorem 3.2 and $D_i \cdot D_i = 0$ for $i = 1, 2$ by assumption, we apply Zhang’s inequality (Theorem 3.3) to $D_1 + D_2$ to obtain

$$0 = e_1(D_1 + D_2) \geq \frac{2 D_1 \cdot D_2}{\deg D_1 + \deg D_2} \geq 0,$$

which allows us to deduce condition (1). □

We will use the equivalences of Theorem 3.4 repeatedly in our proofs of Theorems 1.1, 1.3, and 1.5.
3.4. Arithmetic volume. The rest of this section is devoted to the proof of the equidistribution results, Theorem 1.8 and Corollary 1.9.

For any \( \mathbb{R} \)-divisor \( D \) on \( B \) defined over \( K \), we set

\[
H^0(X, D) = \{ \phi \in K(B) : (\phi) + D \geq 0 \}.
\]

For ample \( D \in \text{Div}_Z(B) \), the volume of \( D \) is

\[
\text{vol}(D) = \lim_{k \to \infty} \frac{1}{k} \dim H^0(B, kD) = \deg D
\]

from Riemann-Roch. For a \( \mathbb{Q} \)-divisor \( D \), the volume can be defined by taking the limit along sequences where \( kD \in \text{Div}(B) \) and again we have \( \text{vol}(D) = \deg D \) when \( D \) is ample.

The arithmetic volume of an adelically-metrized \( \mathbb{R} \)-divisor \( \overline{D} \) is defined as follows. We first fix a family of norms on \( H^0(B, D) \) by

\[
\|\phi\|_{\sup, v} = \sup_{x \in B_v^\text{an} \setminus \text{supp } D} |\phi(x)|_v e^{-g_v(x)},
\]

for each place \( v \) of \( K \). Set

\[
\chi(\overline{D}) = -\log \mu((H^0(B, D) \otimes A_K)/H^0(B, D)) / \mu(\prod_v U_v)
\]

where \( A_K \) is the ring of adeles, \( \mu \) is a Haar measure on \( H^0(B, D) \otimes A_K \), and \( U_v \) is the unit ball in \( H^0(B, D) \otimes \mathbb{C}_v \) in the induced norm. Then

\[
\hat{\text{vol}}_\chi(\overline{D}) := \limsup_{k \to \infty} \chi(k\overline{D}) / k^2 / 2.
\]

In [Mo2, Theorem 5.2.1], Moriwaki proves that \( \hat{\text{vol}}_\chi \) defines a continuous function on a space of continuous, adelic metrizations on \( \mathbb{R} \)-divisors. As a consequence, he shows that for semipositive metrizations, we have \( \hat{\text{vol}}_\chi(\overline{D}) = \overline{D} \cdot \overline{D} \) [Mo2, Theorem 5.3.2]. Therefore, Zhang’s inequality (Theorem 3.3) implies that

\[
e_1(\overline{D}) \geq \hat{\text{vol}}_\chi(\overline{D}) / (2 \deg D).
\]

for all continuous, semipositive, adelic metrizations of \( \mathbb{R} \)-divisors on \( B \).

Remark 3.5. This volume function \( \hat{\text{vol}}_\chi \) is defined differently than the one studied by Moriwaki in [Mo2], but they coincide. See, for example, Appendix C.2 of [BG] and the discussion on page 615, for the comparison of an adelic volume to a Euclidean volume.

Proposition 3.6. For all admissible adelic metrizations on an ample \( \mathbb{R} \)-divisor \( D \) on a curve, we have

\[
e_1(\overline{D}) \geq \frac{\hat{\text{vol}}_\chi(\overline{D})}{2 \deg D}.
\]
Proof. From (3.5), the inequality holds for semipositive metrics. For admissible metrics, we write $D = D_1 - D_2$ for semipositive $D_i$ and approximate each $D_i$ with semipositive adelic metrics on $\mathbb{Q}$-divisors $D_{i,n}$ as $n \to \infty$. Because $D$ is ample, we can assume that $D_{1,n} - D_{2,n}$ is ample for all $n$. In that setting, we apply [CLT, Lemme 5.1], recalling that $\text{vol}(D_{1,n} - D_{2,n}) = \deg(D_{1,n} - D_{2,n})$. The result then follows by uniform convergence of the resulting height functions, so that $e_1$ is continuous, and by continuity of the volume function $\hat{\text{vol}}_\chi$ [Mo2, Theorem 5.2.1].

3.5. Proof of Equidistribution.

Proof of Theorem 1.8. Fix an ample $\mathbb{R}$-divisor $D$, equipped with an adelic, semipositive and normalized metrization $\overline{D}$. Let $x_n \in B(\overline{K})$ be any (non-repeating) sequence $x_n \in B(\overline{K})$ with $h_{\overline{D}}(x_n) \to 0$.

Assume first that $\overline{M}$ is an adelic, semi-positive metrization on an ample $\mathbb{R}$-divisor $M$, not necessarily normalized. For each positive integer $m$, by Zhang’s inequality (Theorem 3.3) applied to $(mD) + \overline{M}$, we have

$$\liminf_{n \to \infty} (mh_{\overline{D}}(x_n) + h_{\overline{M}}(x_n)) \geq \frac{m^2 \overline{D} \cdot \overline{D} + 2m \overline{D} \cdot \overline{M} + \overline{M} \cdot \overline{M}}{2(m \deg \overline{D} + \deg \overline{M})} = \frac{2m \overline{D} \cdot \overline{M} + \overline{M} \cdot \overline{M}}{2(m \deg \overline{D} + \deg \overline{M})}.$$  

As the sequence $x_n$ is small for $\overline{D}$, this gives

$$\liminf_{n \to \infty} h_{\overline{M}}(x_n) \geq \frac{2m \overline{D} \cdot \overline{M} + \overline{M} \cdot \overline{M}}{2(m \deg \overline{D} + \deg \overline{M})}$$

for all $m$. Letting $m$ go to $\infty$, we obtain

$$\liminf_{n \to \infty} h_{\overline{M}}(x_n) \geq \frac{\overline{D} \cdot \overline{M}}{\deg \overline{D}}. \quad (3.6)$$

For the reverse inequality, we choose $m$ large enough so that $\deg(mD - M) > 0$. We can therefore apply Proposition 3.6 to obtain

$$\liminf_{n \to \infty} (mh_{\overline{D}}(x_n) - h_{\overline{M}}(x_n)) \geq \frac{\hat{\text{vol}}_\chi (mD - M)}{2 \deg(mD - M)}$$

so that

$$-\limsup_{n \to \infty} h_{\overline{M}}(x_n) \geq \frac{\hat{\text{vol}}_\chi (mD - M)}{2 \deg(mD - M)}.$$
On the other hand, again assuming that $\deg(mD - M) > 0$, Moriwaki's [Mo2, Theorem 7.1.1] implies that
\[
\widehat{\text{vol}}_x (mD - M) \geq (mD - M) \cdot (mD - M) \\
= m^2(D \cdot D) - 2m(D \cdot M) + M \cdot M \\
= -2m(D \cdot M) + M \cdot M,
\]
with the last equality because $D$ is normalized. (Compare [Yu, Theorem 2.2] and [CLT, Proposition 5.3].)

Therefore,
\[
\limsup_{n \to \infty} h_{\overline{M}}(x_n) \leq \frac{2m(D \cdot M) - M \cdot M}{2 \deg(mD - M)}
\]
for all sufficiently large $m$. As $\deg(mD - M) \approx m \deg D$ as $m \to \infty$, we obtain the desired upper bound of

(3.7)

\[
\limsup_{n \to \infty} h_{\overline{M}}(x_n) \leq \frac{D \cdot M}{\deg D}.
\]

Putting the two inequalities (3.6) and (3.7) together, we have
\[
\lim_{n \to \infty} h_{\overline{M}}(x_n) = \frac{D \cdot M}{\deg D}.
\]

Now suppose that $\overline{M}$ is admissible. By definition, we can write $\overline{M} = M_1 - M_2$ for semipositive $M_i$. By adding and subtracting any choice of ample divisor with semipositive metric, we can assume that each $M_i$ is ample, and we apply the result above to each $M_i$. We have $h_{\overline{M}} = h_{M_1} - h_{M_2}$ and $D \cdot \overline{M} = D \cdot M_1 - D \cdot M_2$. The theorem is a consequence of this linearity. \hfill \Box

Proof of Corollary 1.9. Fix a place $v \in M_K$, and let $\phi$ be a smooth real-valued function on $B_v^{an}$. By density as in [Mo2, Theorem 3.3.3] it is enough to consider these functions. We denote by $\overline{\mathcal{O}}_\phi$ the trivial divisor on $B$ equipped with the metrization given by $g_v = \phi$ and $g_w = 0$ for all $w \neq v$ in $M_K$. This metrization is clearly admissible.

Let $\mu_n$ denote the probability measure in $B_v^{an}$ supported uniformly on the Galois conjugates of $x_n$. Note that
\[
h_{\mathcal{O}_\phi}(x_n) = r_v \int_{B_v^{an}} \phi d\mu_n,
\]
by the definition of the height function. From (3.3), we have
\[
D \cdot \mathcal{O}_\phi(\phi) = r_v \int_{B_v^{an}} \phi d\omega_{\mathcal{O}_\phi,v}.
\]
Applying Theorem 1.8 to $M = \overline{O_v(\phi)}$, we get that
\[
\lim_{n \to \infty} h_{\overline{O_v(\phi)}}(x_n) = \frac{1}{\deg D} \left[ \frac{K_v : Q_v}{K : Q} \right] \int_{B^an_v} \phi d\omega_{D,v} = \frac{[K_v : Q_v]}{[K : Q]} \int_{B^an_v} \phi d\mu_{D,v},
\]
demonstrating weak convergence of $\mu_n$ to $\mu_{D,v}$ in $B^a_v$. □

4. A metrized $\mathbb{R}$-divisor for each element of $E(k) \otimes \mathbb{R}$

Throughout this section, we let $E \to B$ be a non-isotrivial elliptic surface defined over a number field $K$, and let $E$ be the corresponding elliptic curve over the field $k = K(B)$. We denote the zero by $O \in E(k)$. As $E(k)$ is finitely generated, we enlarge $K$ if needed so that all sections of $E \to B$ are defined over $K$. Recall that points $P_1, \ldots, P_m$ are independent in $E(k)$ if there is no nontrivial relation of the form
\[ a_1P_1 + \cdots + a_mP_m = O \]
in $E(k)$ with $a_i \in \mathbb{Z}$.

In this section, we show that each nonzero element $X \in E(k) \otimes \mathbb{R}$ naturally gives rise to an adelic, semipositive continuous metrization $D_X$ associated to an ample $\mathbb{R}$-divisor $D_X$; see Theorem 4.6. For $P \in E(k)$, these metrizations on $\mathbb{R}$-divisors coincide with the adelically metrized line bundles on $B$ that we studied in [DM].

4.1. Néron-Tate heights. Let $F$ be a number field or a function field in characteristic 0. Let $E/F$ be an elliptic curve with origin $O$, expressed in Weierstrass form as
\[ E = \{ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \} \]
with discriminant $\Delta$. Denote by
\[ \hat{h}_E : E(\overline{F}) \to [0, \infty) \]
a Néron-Tate canonical height function; it can be defined by
\[ \hat{h}_E(P) = \frac{1}{2} \lim_{n \to \infty} \frac{h(x(nP))}{n^2} \]
where $h$ is the naive Weil height on $\mathbb{P}^1$ and $x : E \to \mathbb{P}^1$ is the degree 2 projection to the $x$-coordinate.

For each $v \in M_F$, recall that $\mathcal{F}_v$ denotes the completion of $\mathcal{F}$ with respect to $| \cdot |_v$ and $\mathbb{C}_v$ denote the completion of the algebraic closure of $\mathcal{F}_v$. The canonical height has a decomposition into local heights, as
\[ \hat{h}_E(P) = \frac{1}{|\text{Gal}(\overline{F}/F) \cdot P|} \sum_{Q \in \text{Gal}(\overline{F}/F) \cdot P} \sum_{v \in M_F} r_v \hat{\lambda}_{E,v}(Q) \]
for all $P \in E(\mathbb{F}) \setminus \{O\}$, with $r_v$ defined by (2.2) in the number field case, and $r_v = 1$ for function fields. The local heights $\hat{\lambda}_{E,v}$ are characterized by the three properties [Si3, Chapter 6, Theorem 1.1]:

1. $\hat{\lambda}_{E,v}$ is continuous on $E(\mathbb{C}_v) \setminus \{O\}$ and bounded on the complement of any $v$-adic neighborhood of $O$;
2. the limit of $\hat{\lambda}_{E,v}(P) - \frac{1}{2} \log |x(P)|_v$ exists as $P \to O$ in $E(\mathbb{C}_v)$; and
3. for all $P = (x, y) \in E(\mathbb{C}_v)$ with $2P \neq O$,
   
   \[
   \hat{\lambda}_{E,v}(2P) = 4\hat{\lambda}_{E,v}(P) - \log |2y + a_1x + a_3|_v + \frac{1}{4} \log |\Delta|_v.
   \]

Property (3) may be replaced with the quasi-parallelogram law

\[
\hat{\lambda}_{E,v}(P + Q) + \hat{\lambda}_{E,v}(P - Q) = 2\hat{\lambda}_{E,v}(P) + 2\hat{\lambda}_{E,v}(Q) - \log |x(P) - x(Q)|_v + \frac{1}{6} \log |\Delta|_v
\]

under the assumption that none of $P$, $Q$, $P + Q$, or $P - Q$ is equal to $O$. Note that $\hat{\lambda}_{E,v}$ is independent of the choice of Weierstrass equation for $E$ over $\mathbb{F}$. It is useful to recall also the triplication formula; if $3P \neq O$, then

\[
\hat{\lambda}_{E,v}(3P) = 9\hat{\lambda}_{E,v}(P) - \log |(3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8)(P)|_v - \frac{2}{3} \log |\Delta|_v,
\]

where $b_i$ are the usual Weierstrass quantities; see e.g. [Si3, pg. 463].

4.2. **Metrized divisors for elements of** $E(k)$. Fix non-torsion $P \in E(k)$. Define

\[
D_P := \sum_{\gamma \in \mathcal{B}(K)} \hat{\lambda}_{E,\text{ord}_\gamma}(P) [\gamma].
\]

We remark that $\hat{\lambda}_{E,\text{ord}_\gamma}(P) \in \mathbb{Q}$ [La, Chapter 11, Theorem 5.1], so $D_P$ is a $\mathbb{Q}$-divisor on $B$. As $P$ is defined over $K$, the divisor is $\text{Gal}(\overline{K}/K)$-invariant.

In [DM, Theorem 1.1] we established that $D_P$ can be equipped with an adelic, semipositive, continuous and normalized metrization

\[
\overline{D}_P := (D_P, \{\lambda_{P,v}\}_{v \in \mathcal{M}_K})
\]

over the number field $K$, where $\lambda_{P,v}$ denotes the extension of $t \mapsto \hat{\lambda}_{E_t,v}(P_t)$ to $B_v^{an}$. It follows that the associated height functions satisfy

\[
h_P(t) := h_{\overline{D}_P}(t) = \hat{h}_{E_t}(P_t)
\]

for all $t \in B(\overline{K})$ for which $E_t$ is smooth. Both minima $e_1(\overline{D}_P)$ and $e_2(\overline{D}_P)$ (defined in §3.3) are equal to 0 [DM, Proposition 4.3].
For \( O \in E(k) \), we set
\[
\overline{D}_O := (0,0),
\]
the trivial divisor with all functions \( g_v = 0 \). For torsion points \( T \neq O \in E(k) \), the metrized divisor \( \overline{D}_T \) can also be defined by (4.4), with \( \lambda_{T, v}(t) := \hat{\lambda}_{E_t, v}(T_t) \) for all \( t \in B(K) \) with \( E_t \) smooth and \( T_t \neq O \). The following proposition is key for the passage from \( E(k) \) to \( E(k) \otimes \mathbb{R} \).

**Proposition 4.1.** The metrized divisor \( \overline{D}_P \) is well defined for \( P \in E(k)/E(k)_{\text{tors}} \), up to isomorphism. Moreover, for each \( m \geq 1 \) and any set of independent points \( P_1, \ldots, P_m \in E(k) \) and integers \( a_1, \ldots, a_m \), the following metrized divisors are isomorphic:
\[
(4.5) \quad \overline{D}_{a_1 P_1 + \cdots + a_m P_m} \simeq \sum_{i=1}^m (a_i^2 - a_i \sum_{j \neq i} a_j) \overline{D}_{P_i} + \sum_{1 \leq i < j \leq m} a_i a_j \overline{D}_{P_i + P_j}
\]

**Remark 4.2.** The proposition implies, in particular, that the functions
\[
t \mapsto \sum_{i=1}^m \left( a_i^2 - a_i \sum_{j \neq i} a_j \right) \lambda_{P_i, v}(t) + \sum_{1 \leq i < j \leq m} a_i a_j \lambda_{P_i + P_j, v}(t)
\]
are subharmonic on \( B^a_v \) (away from the points \( t \) where \( \lambda_{P_i, v}(t) \) or \( \lambda_{P_i + P_j, v}(t) \) is equal to \( \infty \)), for all choices of \( a_i \in \mathbb{Z} \), and at every place \( v \) of \( K \).

We begin with a lemma (c.f. [Si3, Exercise 6.4]):

**Lemma 4.3.** Fix \( P \in E(k) \). For each nonzero \( m \in \mathbb{Z} \) with \( |m| \geq 2 \) such that \( mP \neq O \), there exist \( h \in K(B) \) and constant \( c \in \mathbb{Q} \) so that
\[
\hat{\lambda}_{E_t, v}(mP_t) = m^2 \hat{\lambda}_{E_t, v}(P_t) + c \log |h(t)|_v
\]
at every place \( v \) and for each \( t \in B(K) \) such that \( E_t \) is smooth. If \( P \) is torsion of order \( m \geq 2 \), we have
\[
\hat{\lambda}_{E_t, v}(P_t) = c \log |h(t)|_v,
\]
for some \( c \in \mathbb{Q} \) and \( h \in K(B) \).

**Proof.** Upon replacing \( P \) by \(-P\), it suffices to prove the statement for \( m \geq 2 \). The duplication formula (4.1) provides the desired result for \( m = 2 \), assuming that \( 2P \neq O \). Now fix any \( m \geq 3 \) and \( P \in E(k) \), and assume that \( mP \neq O \), \( (m-1)P \neq O \) and \( (m-2)P \neq O \). Then the quasi-parallelogram law (4.2) implies
\[
(4.6) \quad \hat{\lambda}_{E_t, v}(mP_t) = 2 \hat{\lambda}_{E_t, v}((m-1)P_t) + 2 \hat{\lambda}_{E_t, v}(P_t) - \hat{\lambda}_{E_t, v}((m-2)P_t)
\]
\[
- \log |x((m-1)P_t) - x(P_t)|_v + \frac{1}{6} \log |\Delta_t|_v,
\]
for each \( t \in B(C_v) \) such that \( E_t \) is smooth and \( mP_t \neq O_t \), \( (m-1)P_t \neq O_t \) and \( (m-2)P_t \neq O_t \) and therefore for all \( t \in B(C_v) \) by the continuity of the local height \( t \mapsto \hat{\lambda}_{E_t, v}(P_t) \in \mathbb{R} \cup \{ \pm \infty \} \).
The desired relation, for all non-torsion points and for all $m \geq 3$, then follows from (4.6) by an easy induction.

Now suppose that $2P = O$ with $P \neq O$. Then we have $3P = P \neq O$, and the triplication formula (4.3) implies that
\[
\hat{\lambda}_{E_t,v}(3P) = 9\hat{\lambda}_{E_t,v}(P) - c \log |h(t)|_v = \hat{\lambda}_{E_t,v}(P),
\]
for a constant $c \in \mathbb{Q}$ and $h \in K(B)$ and for all but finitely many $t$. The equation then holds for all $t \in B(\mathbb{C}_v)$ by the continuity of the local heights, and it implies that $\hat{\lambda}_{E_t,v}(P) = \frac{c}{3} \log |h(t)|_v$.

For a torsion point $P$ of order 3, we have $2P = -P \neq O$, so we may apply the duplication formula (4.1) to see that
\[
\hat{\lambda}_{E_t,v}(2P) = 4\hat{\lambda}_{E_t,v}(P) - c \log |h(t)|_v = \hat{\lambda}_{E_t,v}(-P) = \hat{\lambda}_{E_t,v}(P),
\]
for a constant $c \in \mathbb{Q}$ and $h \in K(B)$. It follows that $\hat{\lambda}_{E_t,v}(P) = \frac{c}{3} \log |h(t)|_v$.

Finally, suppose that $P$ is torsion of order $n \geq 4$, and note that $(n - 1)P = -P \neq O$, $(n - 2)P \neq O$ and $(n - 3)P \neq O$. We infer from (4.6) with $3 \leq m \leq n - 1$ inductively that
\[
\hat{\lambda}_{E_{t,v}}((n - 1)P) = (n - 1)^2\hat{\lambda}_{E_{t,v}}(P) - c \log |h(t)|_v = \hat{\lambda}_{E_{t,v}}(-P) = \hat{\lambda}_{E_{t,v}}(P),
\]
for a rational function $h \in K(B)$ and $c \in \mathbb{Q}$, so that $\hat{\lambda}_{E_{t,v}}(P) = \frac{c}{n^2 - 2n} \log |h(t)|_v$. \qed

**Proof of Proposition 4.1.** Lemma 4.3 implies that
\[ D_P \simeq D_O \]
for every torsion point $P \in E(k)$. Furthermore, for any non-torsion point $P$, Lemma 4.3 also implies that
\[ \overline{D}_{nP} \simeq a^2 \overline{D}_P \]
for all $a \in \mathbb{Z}$, demonstrating (4.5) for $m = 1$. Therefore, if $P$ is non-torsion and $Q$ is torsion of order $n \geq 2$, we have
\[ \overline{D}_{P+Q} \simeq \frac{1}{n^2} \overline{D}_{n(P+Q)} = \frac{1}{n^2} \overline{D}_{nP} \simeq \overline{D}_P. \]
This proves that the metrized divisors depend only on the class in $E(k)/E(k)_{\text{tors}}$, up to isomorphism.

Now fix any $m \geq 2$, and any collection of independent points $P_1, \ldots, P_m \in E(k)$ and integers $a_1, \ldots, a_m$. Define a divisor on $B$ by
\[ D' = \sum_{i=1}^{m} \left( a_i^2 - a_i \sum_{j \neq 1} a_j \right) D_{P_i} + \sum_{1 \leq i < j \leq m} a_i a_j D_{P_i + P_j}, \]
and consider the metrization on $D'$ defined by

$$g_v(t) = \sum_{i=1}^{m} \left( a_i^2 - a_i \sum_{j \neq i} a_j \right) \lambda_{P_i,v}(t) + \sum_{1 \leq i < j \leq m} a_i a_j \lambda_{P_i + P_j,v}(t).$$

To prove the proposition, we will use the quasi-parallelogram law (4.2) to show that there exists a rational function $f \in K(B)$ so that

$$(4.7) \quad g_v(t) - \hat{\lambda}_{E_t,v}(a_1 P_{1,t} + \cdots + a_m P_{m,t}) = \log |f(t)|_v$$

at all places $v$ of $K$ and for all but finitely many $t \in B(\hat{C}_v)$.

**Lemma 4.4.** Let $P, Q, R \in E(k)$ be independent points defined over $K$. Then, there is a rational function $f_{P,Q,R} \in K(B)$ such that

$$\hat{\lambda}_{E_t,v}(P_t + Q_t + R_t) = \hat{\lambda}_{E_t,v}(P_t + R_t) + \hat{\lambda}_{E_t,v}(P_t + Q_t) + \hat{\lambda}_{E_t,v}(Q_t + R_t) - \hat{\lambda}_{E_t,v}(P_t) - \hat{\lambda}_{E_t,v}(Q_t) - \hat{\lambda}_{E_t,v}(R_t) - \log |f_{P,Q,R}(t)|_v.$$ 

for all $t \in B(\hat{K})$ such that $E_t$ is smooth and all $v \in M_K$.

**Proof.** The proof follows by applying the quasi-parallelogram law (4.2) for the pairs $\{P + R, Q\}$, $\{P, R - Q\}$, $\{P + Q, R\}$ and $\{R, Q\}$ and taking an alternating sum as in [Si6, Theorem 9.3].

**Lemma 4.5.** Fix independent $P, Q \in E(k)$. For each $(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, there is rational function $h_{a,b} \in K(B)$ such that

$$\hat{\lambda}_{E_t,v}(aP_t + bQ_t) = \begin{cases} (a^2 - ab)\hat{\lambda}_{E_t,v}(P_t) + ab\hat{\lambda}_{E_t,v}(P_t + Q_t) + (b^2 - ab)\hat{\lambda}_{E_t,v}(Q_t) - \log |h_{a,b}|_v, \\ \end{cases}$$

for all $t \in B(\hat{K})$ such that $E_t$ is smooth and all $v \in M_K$.

**Proof.** The proof follows from the quasi-parallelogram law by an easy induction. Lemma 4.3 provides the desired result if one of $a$ or $b$ is 0. Next we will show that for each $n \in \mathbb{Z}$ there is a rational function $g \in K(B)$ so that

$$(4.8) \quad \hat{\lambda}_{E_t,v}(nP_t + Q_t) = (n^2 - n)\hat{\lambda}_{E_t,v}(P_t) + n\hat{\lambda}_{E_t,v}(P_t + Q_t) + (1 - n)\hat{\lambda}_{E_t,v}(Q_t) - \log |g|_v.$$ 

Replacing $P$ by $-P$ we may assume that $n \geq 1$. For $n = 1$ the statement is clear. For $n \geq 1$, the quasi-parallelogram law (4.2) implies that

$$\hat{\lambda}_{E_t,v}((n + 1)P_t + Q_t) = \hat{\lambda}_{E_t,v}(nP_t + (P + Q)_t)$$

$$= 2\hat{\lambda}_{E_t,v}(nP_t) + 2\hat{\lambda}_{E_t,v}(P_t + Q_t) - \hat{\lambda}_{E_t,v}((n - 1)P_t - Q_t) - \log |x(nP_t) - x(P_t + Q_t)|_v + \frac{1}{6} \log |\Delta|_v,$$
and (4.8) follows inductively from Lemma 4.3. Using (4.8) we now have a rational function $h \in K(B)$ so that
\begin{equation}
\hat{\lambda}_{E_t,v}(aP_t + bQ_t) = (a^2 - a)\hat{\lambda}_{E_t,v}(P_t) + a\hat{\lambda}_{E_t,v}(P_t + bQ_t) + (1 - a)\hat{\lambda}_{E_t,v}(bQ_t) - \log |h|_v.
\end{equation}

The lemma then follows by another application of (4.3) and (4.8), exchanging the roles of $P$ and $Q$. \hfill \Box

Finally, a simple induction using Lemmas 4.4 and 4.5 implies that for any $m \geq 2$, and for any integers $a_1, \ldots, a_m$, the equality (4.7) holds, for some rational $f$. This completes the proof. \hfill \Box

4.3. Metrized divisors for elements of $E(k) \otimes \mathbb{R}$. Fix nonzero $X \in E(k) \otimes \mathbb{R}$. Choose independent points $P_1, \ldots, P_m \in E(k)$ that define a basis for $E(k) \otimes \mathbb{R}$, and write $X = x_1P_1 + \cdots + x_mP_m$ with $x_i \in \mathbb{R}$. With a slight abuse of notation, we identify the two isomorphic metrized divisors in Proposition 4.1 and define an adelically metrized $\mathbb{R}$-divisor on $B(K)$, over the number field $K$, by
\begin{equation}
\overline{D}_X := \sum_{i=1}^m \left( x_i^2 - x_i \sum_{j \neq i} x_j \right) \overline{D}_{P_i} + \sum_{1 \leq i < j \leq m} x_i x_j \overline{D}_{P_i + P_j}
\end{equation}
for the $\overline{D}_P$ defined by (4.4). It defines a height function
\begin{equation}
h_X(t) = \sum_{i=1}^m \left( x_i^2 - x_i \sum_{j \neq i} x_j \right) \hat{h}_{E_t}(P_{i,t}) + \sum_{1 \leq i < j \leq m} x_i x_j \hat{h}_{E_t}(P_{i,t} + P_{j,t})
\end{equation}
at all points $t \in B(K)$ for which $E_t$ is smooth.

**Theorem 4.6.** Fix nonzero $X \in E(k) \otimes \mathbb{R}$. The metrized divisor $\overline{D}_X$ of (4.10) is continuous, adelic, semipositive and normalized. The degree of the underlying $\mathbb{R}$-divisor $D_X$ is $\hat{h}_{E}(X) > 0$. Its associated height function satisfies
\begin{equation}
h_X(t) = \hat{h}_{E_t}(X_t)
\end{equation}
for all $t \in B(K)$ with smooth fiber $E_t$. Further, up to isomorphism, $\overline{D}_X$ is independent of the choice of basis for $E(k)$.

**Proof.** Fix $x_1, \ldots, x_m \in \mathbb{R}$ and choose sequences of rational numbers $a_{n,i}/a_{n,0} \to x_i$ for $i = 1, \ldots, m$. From Proposition 4.1 we know that the functions
\begin{equation}
\frac{1}{a_{n,0}^2} \left( \sum_{i=1}^m \left( a_{n,i}^2 - a_{n,i} \sum_{j \neq i} a_{n,j} \right) \lambda_{P_j,v}(t) + \sum_{1 \leq i < j \leq m} a_{n,i} a_{n,j} \lambda_{P_i + P_j,v}(t) \right)
\end{equation}
are continuous, subharmonic functions on $B_y^{an}$ (away from their logarithmic singularities), because they define a metrized divisor isomorphic to $a_{n,0}^{-2} \overline{D}_{a_{n,1}P_1 + \cdots + a_{n,m}P_m}$. The limit as $n \to \infty$ clearly exists as a continuous, semipositive, adelic metrization on an $\mathbb{R}$-divisor

$$D_X = \sum_{i=1}^{m} \left( x_i^2 - x_i \sum_{j \neq i} x_j \right) D_{P_i} + \sum_{1 \leq i < j \leq m} x_i x_j D_{P_i + P_j}.$$ 

To see that $\overline{D}_X$ is normalized, recall that by [DM, Theorem 1.1] we have

$$\overline{D}_{a_{n,1}P_1 + \cdots + a_{n,m}P_m} \cdot \overline{D}_{a_{n,1}P_1 + \cdots + a_{n,m}P_m} = 0,$$

for all $n \in \mathbb{N}$. In view of Proposition 4.1 we then have

$$\frac{1}{a_{n,0}^4} \left( \sum_{i=1}^{m} \left( a_{n,i}^2 - a_{n,i} \sum_{j \neq i} a_{n,j} \right) \overline{D}_{P_j} + \sum_{1 \leq i < j \leq m} a_{n,i} a_{n,j} \overline{D}_{P_i + P_j} \right)^2 = 0,$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get $\overline{D}_X \cdot \overline{D}_X = 0$.

Equation (4.12) follows from the properties of $\hat{h}_{E_t}$ as a quadratic form on each smooth fiber $E_t$. Specifically, we have

$$\hat{h}_{E_t}(P_t + Q_t) = \hat{h}_{E_t}(P_t) + 2\langle P_t, Q_t \rangle_{E_t} + \hat{h}_{E_t}(Q_t)$$

for the Néron-Tate bilinear form $\langle P_t, Q_t \rangle_{E_t}$ and for any pair of points $P, Q \in E(k)$ and $t \in B(\overline{K})$ with $E_t$ smooth. It follows that

$$\hat{h}_{E_t}(yP_t + zQ_t) = y^2 \hat{h}_{E_t}(P_t) + yz(\hat{h}_{E_t}(P_t + Q_t) - \hat{h}_{E_t}(P_t) - \hat{h}_{E_t}(Q_t)) + z^2 \hat{h}_{E_t}(Q_t)$$

$$= (y^2 - yz)\hat{h}_{E_t}(P_t) + yz \hat{h}_{E_t}(P_t + Q_t) + (z^2 - yz) \hat{h}_{E_t}(Q_t)$$

for all $y, z \in \mathbb{R}$. Therefore, by induction, we deduce that

$$\hat{h}_{E_t}(x_1P_{1,t} + \cdots + x_mP_{m,t}) = \sum_{i=1}^{m} x_i^2 \hat{h}_{E_t}(P_{i,t}) + 2 \sum_{i<j} x_i x_j \langle P_{i,t}, P_{j,t} \rangle_{E_t}$$

$$= \sum_{i=1}^{m} \left( x_i^2 - x_i \sum_{j \neq i} x_j \right) \hat{h}_{E_t}(P_{i,t}) + \sum_{1 \leq i < j \leq m} x_i x_j \hat{h}_{E_t}(P_{i,t} + P_{j,t})$$

for any collection $P_1, \ldots, P_m \in E(k)$ and real numbers $x_1, \ldots, x_m$, so that

$$h_X(t) = \hat{h}_{E_t}(X_t)$$

for all $t \in B(\overline{K})$ with $E_t$ smooth. That $\overline{D}_X$ does not depend on its presentation or the choice of basis follows easily from Proposition 4.1. \qed
5. Small sequences

As before, we let $E \to B$ be a non-isotrivial elliptic surface defined over a number field $K$, and let $E$ be the corresponding elliptic curve over the field $k = \overline{K}(B)$. In the previous section, we constructed metrized $\mathbb{R}$-divisors $\mathcal{D}_X$ and associated height functions $h_X$ for each element $X \in E(k) \otimes \mathbb{R}$. In this section, we look at the sets of “small” points for the height $h_X$. We conclude the section with a proof of Theorem 1.10.

5.1. Small sequences exist. For an adelic, continuous, semipositive, and normalized metrized $\mathbb{R}$-divisor $D$, an infinite sequence $\{t_n\} \subset B(k)$ is said to be small if

$$h_D(t_n) \to 0$$
as $n \to \infty$.

**Proposition 5.1.** For every nonzero $X \in E(k) \otimes \mathbb{R}$, there exist small sequences for $D_X$, so that the essential minimum is $e_1(D_X) = 0$. More precisely, write $X = x_1P_1 + \cdots + x_mP_m$ for $x_i \in \mathbb{R}$ and independent $P_i \in E(k)$, and choose integers $a_{i,n}$ for $i = 1, \ldots, m$ and $n \in \mathbb{N}$ so that $[a_{1,n} : \cdots : a_{m,n}] \to [x_1 : \cdots : x_m]$ as $n \to \infty$ in the real projective space $\mathbb{RP}^{m-1}$. Then there exists an infinite non-repeating sequence of points $t_n \in B(\overline{K})$ at which

$$(a_{1,n}P_1 + \cdots + a_{m,n}P_m)_{t_n}$$
is torsion in the fiber $E_{t_n}(\overline{K})$. Moreover, for any such sequence $\{t_n\} \subset B(\overline{K})$ we have

$$h_X(t_n) \to 0$$
as $n \to \infty$.

To prove Proposition 5.1, we begin with a well-known statement that follows from Silverman’s specialization theorem [Si1].

**Lemma 5.2.** Fix any set of independent points $P_1, \ldots, P_m$ in $E(k)$, and let $h$ be any Weil height function on $B$ associated to a divisor of degree 1. The set of all $t$ for which there exist integers $a_1, \ldots, a_m$, not all zero, such that

$$a_1P_{1,t} + \cdots + a_mP_{m,t} = O_t$$
in $E_t$ has bounded $h$-height.

**Proof.** For each non-torsion point $Q \in E(k)$ we have [Si1]

$$\lim_{h(t) \to \infty} \frac{\hat{h}(Q_t)}{h(t)} = \hat{h}_E(Q) > 0.$$

Because $\det(\langle P_i, P_j \rangle_E)_{i,j} > 0$, it follows that the set

$$\mathcal{R}(P_1, \ldots, P_m) = \{t \in B(\overline{K}) : \det(\langle P_{i,t}, P_{j,t} \rangle_t) = 0\}$$
also has bounded height. This set $\mathcal{R}(P_1, \ldots, P_m)$ clearly contains the set of $t$ at which the points become linearly dependent.

Proof of Proposition 5.1. Write $X = x_1 P_1 + \cdots + x_m P_m$ for independent $P_1, \ldots, P_m \in E(k)$ and $x_1, \ldots, x_m \in \mathbb{R}$. Fix a sequence of positive integers $M_n \to \infty$ as $n \to \infty$. For $i = 1, \ldots, m$, choose any sequence of integers $a_{i,n}$ so that $a_{i,n}/M_n \to x_i$ as $n \to \infty$, and set

$$Q_n = a_{1,n} P_1 + \cdots + a_{m,n} P_m \in E(k),$$

so that $\frac{1}{M_n} Q_n \to X$ in $E(k) \otimes \mathbb{R}$.

Consider the set

$$\text{Tor}(Q_n) = \{t \in B(K) : Q_{n,t} \text{ is torsion in } E_t\}$$

For each $n$, the set Tor$(Q_n)$ is infinite; in fact, it is dense in $B(\mathbb{C})$ [DM, Proposition 6.2] [Za, §III.2 and Notes to Chapter III]. Moreover, from Lemma 5.2, this set has bounded height in the base curve $B$ with respect to any chosen Weil height $h$, and the height is bounded independent of $n$. Therefore, from [Si1, Theorem A], we can find $H > 0$ so that

$$h_{P_i}(t) \leq H \quad \text{and} \quad h_{P_i + P_j}(t) \leq H$$

for all $t \in \bigcup_n \text{Tor}(Q_n)$ and for all $i, j$.

From the formula for the height $h_X$ given in (4.11) and the formula for the height of $Q_n$ appearing in Proposition 4.1, we have the following. For any given $\varepsilon > 0$, there exists $N > 0$ so that

$$\left| h_X(t) - \frac{1}{M_n^2} h_{Q_n}(t) \right| = \left| \sum_{i=1}^m \left( x_i^2 - x_i \sum_{j \neq 1} x_j - \frac{a_{i,n}^2}{M_n^2} + \frac{a_{i,n}}{M_n} \sum_{j \neq i} \frac{a_{i,j,n}}{M_n} \right) h_{P_i}(t) \right. + \left. \left( \sum_{1 \leq i < j \leq m} x_i x_j - \frac{a_{i,n} a_{j,n}}{M_n^2} \right) h_{P_i + P_j}(t) \right| < \varepsilon$$

for all $n > N$ and for all $t$ where $h_{P_i}(t) \leq H$ and $h_{P_i + P_j}(t) \leq H$ for all $i, j$. In particular, the estimate holds for all $t \in \bigcup_{n \geq 1} \text{Tor}(Q_n)$.

For each $n$ and for every $t \in \text{Tor}(Q_n)$, we have $h_{Q_n}(t) = 0$. Choosing any sequence of distinct points $t_n \in B(K)$ so that $Q_{n,t_n}$ is torsion in $E_{t_n}$, we may conclude that

$$h_X(t_n) \to 0$$

as $n \to \infty$. 

5.2. Characterization of small sequences. Here, we observe that small sequences for real points $X \in E(k) \otimes \mathbb{R}$ always arise from a construction similar to that of Proposition 5.1, where relations between the generators are “almost” satisfied. We will use this next proposition in the proof of Theorem 7.1.
Proposition 5.3. Let $M$ be a torsion-free subgroup of $E(k)$ of rank $m$, generated by $S_1,\ldots,S_m$. Set $h_M(t) = \det(\langle S_{i,t}, S_{j,t} \rangle_t)$, for the Néron-Tate bilinear form $\langle \cdot, \cdot \rangle_t$ on the fiber $E_t(K)$. For a non-repeating infinite sequence $t_n \in B(\overline{K})$, the following are equivalent

1. $\liminf_{n \to \infty} h_M(t_n) = 0$;
2. there is a non-zero $X \in M \otimes \mathbb{R}$ such that $\liminf_{n \to \infty} h_X(t_n) = 0$.
3. there are sequences of points $s_{i,n} \in E_{t_n}(\overline{K})$, for $i = 1,\ldots,m$, satisfying

$$
\liminf_{n \to \infty} \left( \max_i \hat{h}_{E_{t_n}}(s_{i,n}) \right) = 0
$$

and so that the points

$$
S_{1,t_n} - s_{1,n}, \ldots, S_{m,t_n} - s_{m,n}
$$

satisfy a linear relation over $\mathbb{Z}$ in $E_{t_n}(\overline{K})$.

This proposition relies heavily on Silverman’s specialization results [Si1, Theorem A and Theorem B]. We point out that [Si1, Theorem B] holds for real points $X \in E(k) \otimes \mathbb{R}$ by the bilinearity of the Néron-Tate pairing. We begin with a lemma.

Lemma 5.4. Assume we are in the setting of Proposition 5.3. Assume further that there are sequences of points $s_{i,n} \in E_{t_n}(\overline{K})$, for $i = 1,\ldots,m$, satisfying

$$
\sup_n \left( \max_i \hat{h}_{E_{t_n}}(s_{i,n}) \right) < \infty
$$

for which the points

$$
S_{1,t_n} - s_{1,n}, \ldots, S_{m,t_n} - s_{m,n}
$$

satisfy a linear relation over $\mathbb{Z}$ in $E_{t_n}(\overline{K})$. Then the sequence $\{t_n\}$ will have bounded height in $B(\overline{K})$ with respect to any Weil height on $B$.

Proof. Fix any Weil height $h$ on $B(\overline{K})$ of degree 1. Consider the $m \times m$ matrix

$$
A_n := (\langle S_{i,t_n} - s_{i,n}, S_{j,t_n} - s_{j,n} \rangle_{t_n})_{i,j}
$$

where $\langle \cdot, \cdot \rangle_{t_n}$ is the Néron-Tate inner product on the fiber $E_{t_n}(\overline{K})$. Our assumption implies that

$$
\det A_n = 0
$$

for all $n$. Assume that $h(t_n) \to \infty$. Then by Silverman’s specialization theorem [Si1, Theorem B] we have

$$
\frac{\langle S_{i,t_n}, S_{j,t_n} \rangle_{t_n}}{h(t_n)} \to \langle S_i, S_j \rangle_E,
$$
as $n \to \infty$ for all $i, j = 1, \ldots, m$. On the other hand, the bounded height of the perturbations $s_{i,n}$ and the Cauchy-Schwarz inequality for $\langle \cdot, \cdot \rangle_{t_n}$ implies that

\[
\left| \frac{\langle s_{i,n}, s_{j,n} \rangle_{t_n}}{h(t_n)} \right| \leq \frac{\sqrt{\hat{h}_{E_{t_n}}(s_{i,n}) \hat{h}_{E_{t_n}}(s_{j,n})}}{h(t_n)} \to 0.
\]

Using Silverman’s specialization [Si1, Theorem A] we also have

\[
\left| \frac{\langle S_{i,t_n}, s_{j,n} \rangle_{t_n}}{h(t_n)} \right| \leq \frac{\sqrt{\hat{h}_{E_{t_n}}(S_{i,t_n}) \hat{h}_{E_{t_n}}(s_{j,n})}}{h(t_n)} \to 0.
\]

Combining these estimates, we obtain

\[
0 = \frac{\det A_n}{(h(t_n))^m} \to \det(\langle S_i, S_j \rangle_E)_{i,j} \neq 0,
\]

which is a contradiction. \hfill \Box

**Proof of Proposition 5.3.** Assume condition (2). Let $X \in M \otimes \mathbb{R}$ be nonzero and $\{t_n\}$ a sequence for which $\liminf_{n \to \infty} h_X(t_n) = 0$. Write $X = x_1 S_1 + \cdots + x_\ell S_\ell$, for $x_i \in \mathbb{R}$ not all equal to 0. After reordering the points $S_i$ we may assume that $x_1 \neq 0$. Notice that

\[
\det \begin{pmatrix}
\hat{h}_{E_{t_n}}(X_{t_n}) & \langle X_{t_n}, S_{2,t_n} \rangle & \cdots & \langle X_{t_n}, S_{\ell,t_n} \rangle \\
\langle S_{2,t_n}, X_{t_n} \rangle & \hat{h}_{E_{t_n}}(S_{2,t_n}) & \cdots & \langle S_{2,t_n}, S_{\ell,t_n} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle S_{\ell,t_n}, X_{t_n} \rangle & \langle S_{\ell,t_n}, S_{2,t_n} \rangle & \cdots & \hat{h}_{E_{t_n}}(S_{\ell,t_n})
\end{pmatrix} = x_1^2 h_M(t_n),
\]

which easily follows by subtracting from the first column the sum of $x_i$ times the $j$-th column over all $j = 2, \ldots, \ell$ and then subtracting from the first row the sum of $x_i$ times the $i$-th row over all $i = 2, \ldots, \ell$. Expanding the determinant along the first column we get

\[
x_1^2 h_M(t_n) = h_X(t_n) f_{1,n} + \sum_{j=2}^\ell \langle S_{j,t_n}, X_{t_n} \rangle f_{j,n},
\]

where for all $n \in \mathbb{N}$ the $f_{j,n}$ are polynomial functions of the quantities $\langle S_{j,t_n}, X_{t_n} \rangle$ and $\langle S_{j,t_n}, S_{k,t_n} \rangle$ for $j, k = 2, \ldots, \ell$. Passing to a subsequence of $\{t_n\}$ we have $\lim_{n \to \infty} h_X(t_{k_n}) = 0$. In particular, since $X$ is non-trivial, [Si1, Theorem B] yields that $\{h(t_{k_n})\}_{n \in \mathbb{N}}$ is a bounded sequence. Using then [Si1, Theorem A], the functoriality of heights and the Cauchy-Schwarz inequality we get

\[
\max\{|f_{1,k_n}|, \ldots, |f_{\ell,k_n}|\} \leq L,
\]

for some $L > 0$. Moreover, for all $j = 2, \ldots, \ell$ and all $n \in \mathbb{N}$ we have

\[
|\langle S_{j,t_{k_n}}, X_{t_{k_n}} \rangle|^2 \leq \hat{h}_{E_{t_{k_n}}}(S_{j,t_{k_n}}) \hat{h}_{E_{t_{k_n}}}(X_{t_{k_n}}) \leq L h_X(t_{k_n}) \to 0.
\]
Our assumption on $X$ together with (5.1), (5.2) and (5.3) yield
\[ \liminf_{n \to \infty} h_M(t_n) = 0, \]
proving that condition (1) holds.

Now assume (1). Let $A_t = (\langle S_{i,t}, S_{j,t} \rangle)_{i,j}$, so that $h_M(t) = \det A_t$, and consider the family of quadratic forms
\[ q_t(\vec{z}) := \hat{h}_{E_t}(z_1 S_{1,t} + \cdots + z_m S_{m,t}) \]
\[ = \sum_{k=1}^{m} z_k^2 \hat{h}_{E_t}(S_{k,t}) + 2 \sum_{i<j} z_i z_j \langle S_{i,t}, S_{j,t} \rangle = \vec{z} A_t \vec{z}^T, \]
for $\vec{z} = (z_1, \ldots, z_m) \in \mathbb{R}^m$, indexed by $t \in B(K)$ where $E_t$ is smooth. Since $q_t \geq 0$ for all $t$, we have that $A_t$ has non-negative eigenvalues. Our assumption is that
\[ \liminf_{n \to \infty} \det A_{t_n} = 0, \]
so, if $\lambda_n$ is the smallest eigenvalue of $A_{t_n}$, then
\[ \liminf_{n \to \infty} \lambda_n = 0. \]
Let $\vec{v}_n = (v_{1,n}, \ldots, v_{m,n}) \neq 0$ be an eigenvector of $A_{t_n}$ corresponding to $\lambda_n$. Then
\[ \vec{v}_n A_{t_n} \vec{v}_n^T = \lambda_n \| \vec{v}_n \|^2, \]
so that
\[ \liminf_{n \to \infty} q_{t_n}(\vec{v}_n/n) = \liminf_{n \to \infty} \lambda_n = 0. \]
Passing to a subsequence of the $\{t_n\}$, we have $\lim_{n \to \infty} h_M(t_n) = 0$, and passing to a further subsequence, we may set
\[ \vec{x} := \lim_{n \to \infty} \frac{\vec{v}_n}{\| \vec{v}_n \|} \in \mathbb{R}^m \setminus \{0\}. \]
By [Si1, Theorem B], the height of $\{t_n\}$ is bounded with respect to any choice of Weil height on $B$ (because $\det(\langle S_i, S_j \rangle) \neq 0$). In view of [Si1, Theorem A], we get that the sequences $\{\langle S_{i,t_n}, S_{j,t_n} \rangle\}_n$ for $i, j = 1, \ldots, \ell$ are bounded. Thus (5.4) yields
\[ \lim_{n \to \infty} q_{t_n}(\vec{x}) = 0. \]
In other words, for $X = x_1 S_1 + \cdots + x_m S_m$ we have $\lim_{n \to \infty} h_X(t_n) = 0$, providing condition (2).

Assuming (2), we now prove (3). Reordering the points and rescaling $X$ if necessary, we may assume that $x_1 = 1$. Passing to a subsequence, we have
\[ \hat{h}_{E_{t_n}}(S_{1,t_n} + x_2 S_{2,t_n} + \cdots + x_m S_{m,t_n}) \to 0, \]
as \( n \to \infty \). Let \( a_{2,n}, \ldots, a_{m,n} \) be infinite sequences of integers satisfying \( a_{i,n}/n \to x_i \) for each \( i = 2, \ldots, m \). As \( \hat{h}_{E}(X) \neq 0 \), we have by Silverman specialization [Si1, Theorem B] that the sequence \( \{t_n\} \) has bounded height in \( B \). Invoking [Si1, Theorem A] we get that all sequences \( \{\langle S_{i,t_n}, S_{j,t_n} \rangle_{t_n} \}_{n \in \mathbb{N}} \) are bounded. Using the fact that each \( \hat{h}_{E_{t_n}}(\cdot) \) defines a quadratic form on \( E_{t_n}(\overline{K}) \), line (5.5) yields

\[
\hat{h}_{E_{t_n}} \left( S_{1,t_n} + \frac{1}{n} \left( a_{2,n} S_{2,t_n} + \cdots + a_{\ell,n} S_{\ell,t_n} \right) \right) \to 0.
\]

Since \( \overline{K} \) is algebraically closed we may find \( s_n \in E_{t_n}(\overline{K}) \) so that

\[
ns_n = a_{2,n} S_{2,t_n} + \cdots + a_{\ell,n} S_{\ell,t_n}.
\]

Letting \( s_{1,n} := S_{1,t_n} + s_n \) and \( s_{i,n} := O_{t_n} \) for all \( i = 2, \ldots, m \), equation (5.6) yields

\[
\hat{h}_{E_{t_n}}(s_{i,n}) \to 0
\]

for each \( i = 1, \ldots, m \). Moreover by (5.7) we have that the set \( \{S_{1,t_n} - s_{1,n}, S_{2,t_n}, \ldots, S_{\ell,t_n}\} \) is linearly dependent in \( E_{t_n} \) for every \( n \).

Last, we assume condition (3) and prove (2). Pass to a subsequence so that

\[
\lim_{n \to \infty} \left( \max_i \hat{h}_{E_{t_n}}(s_{i,n}) \right) = 0.
\]

Choose sequences of integers \( a_{i,n} \) for \( i = 1, \ldots, m \), not all 0, so that

\[
a_{1,n}(S_{1,t_n} - s_{1,n}) + \cdots + a_{m,n}(S_{m,t_n} - s_{m,n}) = O_{t_n}
\]

for all \( n \). Now, letting \( M_n = \max_i a_{i,n} \), we can pass to a further subsequence so that

\[
\frac{a_{i,n}}{M_n} \to x_i \in \mathbb{R}
\]

as \( n \to \infty \) for each \( i \), with at least one \( x_i \) nonzero. This implies that

\[
\hat{h}_{E_{t_n}} \left( \frac{1}{M_n} (a_{1,n} S_{1,t_n} + \cdots + a_{m,n} S_{m,t_n}) \right) = \hat{h}_{E_{t_n}} \left( \frac{1}{M_n} (a_{1,n} s_{1,n} + \cdots + a_{m,n} s_{m,n}) \right) \to 0
\]

as \( n \to \infty \). Finally set

\[
X = x_1 S_1 + \cdots + x_m S_m.
\]

From Lemma 5.4, we know that the sequence \( \{t_n\} \) has bounded height and by [Si1, Theorem A] we get that the sequences \( \{\hat{h}_{E_{t_n}}(S_{i,t_n})\}_n \) are bounded. Therefore, from the definition of \( h_X \) in (4.11), line (5.8) implies that \( h_X(t_n) \to 0 \).
5.3. Proof of Theorem 1.10. From Theorem 4.6, we know that $\overline{D}_X$ is a continuous, adelic, semipositive, and normalized metrization on an ample $\mathbb{R}$-divisor. Thus, Corollary 1.9 applies to sequences with small height for $h_X$. From Proposition 5.1, we have $h_X(t_n) \to 0$ along any sequence $t_n$ for which $\sum_i r_{i,n} P_{i,t} = O_t$ with $r_{i,n} \in \mathbb{Q}$ satisfying $r_{i,n} \to x_i$. The formula for $\omega_{X,v}$ at each place follows from the definition of $\overline{D}_X$ in (4.10). This completes the proof. □

5.4. Height 0. As we shall see, it follows from Theorem 1.1 that although small sequences exist as in Proposition 5.1, we don’t always have sequences with height 0:

Proposition 5.5. Fix nonzero $X \in E(k) \otimes \mathbb{R}$. There exist infinitely many $t \in B(\overline{K})$ for which $h_X(t) = 0$ if and only if there exists a real $c > 0$ so that $cX$ is represented by an element of $E(k)$.

Proof. Suppose first that $cX$ is represented by an element $P \in E(k)$ for some real $c > 0$. Then $h_X(t) = \frac{1}{c^2} h_P(t)$ at all $t$, so that $h_X(t) = 0$ whenever $P_t$ is torsion in $E_t$. This holds at infinitely many points $t \in B(\overline{K})$. (See, e.g., [DM, Proposition 6.2].)

For the converse, write $X = x_1 P_1 + \cdots + x_m P_m$ for independent $P_1, \ldots, P_m \in E(k)$ and $x_i \in \mathbb{R}$, and assume that $h_X(t) = 0$ for infinitely many $t$. We can rewrite $X$ as

$$X = \alpha_1 Q_1 + \cdots + \alpha_s Q_s$$

for $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ a basis for the span of $\{x_1, \ldots, x_m\}$ over $\mathbb{Q}$ and $Q_1, \ldots, Q_s \in E(k)$. For $s = 1$, we see that are we back in the setting where a multiple of $X$ is represented by an element of $E(k)$, so we may assume $s > 1$. But, for each $t$ where $h_X(t) = 0$, we must have that $X_t = 0$ in $E_t(\overline{Q}) \otimes \mathbb{R}$. By the choices of the $\alpha_i$, this means that each of the specializations $Q_{i,t}$ must be 0 in $E_t(\overline{Q}) \otimes \mathbb{R}$. (Compare [Mo2, Lemma 1.1.1].) In other words, the points $Q_1, \ldots, Q_s$ are simultaneously torsion at infinitely many $t$. From Theorem 1.1, combined with (1.4), this implies that each pair $Q_i$ and $Q_j$ is linearly related. (Alternatively, here one could use the main results of [MZ1, MZ2].) Thus we infer that $X = cQ$ for some $c \in \mathbb{R}$ and $Q \in E(k)$. □

6. The Intersection Number as a Biquadratic Form on $E(k) \otimes \mathbb{R}$

Let $E \to B$ be a non-isotrivial elliptic surface defined over a number field $K$, and let $E$ be the corresponding elliptic curve over the field $k = \overline{K}(B)$. Recall that we can enlarge the number field $K$, if necessary, to ensure that each section $P : B \to E$ is defined over $K$. For each $P \in E(k)$, a metrized divisor $\overline{D}_P$ is defined by (4.4). We extended this definition to elements $X \in E(k) \otimes \mathbb{R}$ with the definition (4.10). In this section, we study the basic properties of the Arakelov-Zhang intersection number

$$(X,Y) \mapsto \overline{D}_X \cdot \overline{D}_Y$$
as a biquadratic form on the finite-dimensional vector space $E(k) \otimes \mathbb{R}$.

For $X,Y \in E(k) \otimes \mathbb{R}$, consider the metrized divisor

$$D_{\langle X,Y \rangle} := \frac{1}{2} (D_{X+Y} - D_X - D_Y),$$

of degree equal to the Néron-Tate inner product of $X$ and $Y$,

$$\langle X,Y \rangle_E = \frac{1}{2} \left( \hat{h}_E(X + Y) - \hat{h}_E(X) - \hat{h}_E(Y) \right).$$

Our goal in this section is to prove:

**Proposition 6.1.** Fix $X, Y, Z \in E(k) \otimes \mathbb{R}$. The following hold:

1. $D_X \cdot D_Y \geq 0$,
2. $D_X \cdot D_{Y+Z} + D_X \cdot D_{Y-Z} = 2 D_X \cdot D_Y + 2 D_X \cdot D_Z$,
3. $D_X \cdot D_{X+Y} = D_X \cdot D_Y$, and
4. for all $x \in \mathbb{R}$, we have $D_X \cdot D_{xY} = x^2 D_X \cdot D_Y$.

For each $X \in E(k) \otimes \mathbb{R}$ the map $Y \mapsto D_X \cdot D_{\langle Y,Z \rangle}$ defines a positive semidefinite quadratic form on $E(k) \otimes \mathbb{R}$, induced by the bilinear form $(Y,Z) \mapsto D_X \cdot D_{\langle Y,Z \rangle}$.

We begin with a lemma.

**Lemma 6.2.** We have $D_X \cdot D_Y \geq 0$ for all $X, Y \in E(k) \otimes \mathbb{R}$.

**Proof.** From Theorem 4.6, both $D_X$ and $D_Y$ are normalized, semipositive, continuous adelic metrized divisors on $B$ over $K$, so the lemma follows immediately from Theorem 3.2. Or we can see it as a consequence of Theorem 1.8, because the height functions satisfy $h_X, h_Y \geq 0$ at all points of $B(\overline{K})$. \[\square\]

The following lemma is a version of the Cauchy-Schwarz inequality.

**Lemma 6.3.** For each $X \in E(k) \otimes \mathbb{R}$, the intersection

$$(Y,Z) \mapsto D_X \cdot D_{\langle Y,Z \rangle}$$

is bilinear in $Y, Z \in E(k) \otimes \mathbb{R}$. Moreover,

$$(D_X \cdot D_{\langle Y,Z \rangle})^2 \leq (D_X \cdot D_Y)(D_X \cdot D_Z)$$

for all $X, Y, Z \in E(k) \otimes \mathbb{R}$. 

Proof. Fix $X, Y, Z, W \in E(k) \otimes \mathbb{R}$, and $a, b \in \mathbb{R}$. It is easy to see that $\overline{D}_X \cdot \overline{D}_{(Y,Z)} = \overline{D}_Z \cdot \overline{D}_{(Z,Y)}$. Then definitions (6.1) and (4.10) give

$$\overline{D}_X \cdot \overline{D}_{(Y,aZ+bW)} = \frac{1}{2} \left( \overline{D}_X \cdot \overline{D}_{Y+aZ+bW} - \overline{D}_X \cdot \overline{D}_Y - \overline{D}_X \cdot \overline{D}_{aZ+bW} \right)$$

$$= \frac{1}{2} \left( \overline{D}_X \cdot \left[ (1-a-b)\overline{D}_Y + (a^2-a-ab)\overline{D}_Z + (b^2-b-ab)\overline{D}_W \right. \right.$$

$$+ a\overline{D}_{Y+Z} + b\overline{D}_{Y+W} + ab\overline{D}_{Z+W} \left. - \overline{D}_X \cdot \overline{D}_Y \right.$$

$$\left. \left. - \overline{D}_X \cdot \left[ (a^2-ab)\overline{D}_Z + (b^2-ab)\overline{D}_W + ab\overline{D}_{Z+W} \right] \right)$$

$$= \frac{1}{2} \left( a\overline{D}_X \cdot [\overline{D}_{Y+Z} - \overline{D}_Y - \overline{D}_Z] + b\overline{D}_X \cdot [\overline{D}_{Y+W} - \overline{D}_Y - \overline{D}_W] \right)$$

$$= a \overline{D}_X \cdot \overline{D}_{(Y,Z)} + b \overline{D}_X \cdot \overline{D}_{(Y,W)},$$

demonstrating bilinearity.

By Lemma 6.2 we have $f(x) = \overline{D}_X \cdot \overline{D}_{Y+xZ} \geq 0$ for all $x \in \mathbb{R}$. From definition (4.10), we have

$$\overline{D}_{Y+xZ} = (1-x)\overline{D}_Y + x\overline{D}_{Y+Z} + (x^2-x)\overline{D}_Z.$$ **Definition (6.1)** then yields

$$f(x) = \overline{D}_X \cdot \overline{D}_{Y} + 2x\overline{D}_X \cdot \overline{D}_{(Y,Z)} + x^2\overline{D}_X \cdot \overline{D}_Z \geq 0$$

for all $x \in \mathbb{R}$. Thus the quadratic $f(x)$ has non-positive discriminant. The inequality follows. □

We are now ready to prove the proposition.

**Proof of Proposition 6.1.** Fix $X, Y, Z \in E(k) \otimes \mathbb{R}$. The symmetry in (1) follows immediately from the symmetry of the intersection number, shown explicitly in (3.2) and extending to (3.3) by linearity. The non-negativity is the content of Lemma 6.2.

For (2), note that by Proposition 4.1 we have $\overline{D}_{Y+Z} + \overline{D}_{Y-Z} \simeq 2\overline{D}_Y + 2\overline{D}_Z$, for any $X, Y, Z \in E(k)$. The same follows for $X, Y, Z \in E(k) \otimes \mathbb{R}$ by definition (4.10). The intersection number is invariant under isomorphism for metrized divisors, and so

$$\overline{D}_X \cdot (\overline{D}_{Y+Z} + \overline{D}_{Y-Z}) = \overline{D}_X \cdot (2\overline{D}_Y + 2\overline{D}_Z).$$

For (3), we use Lemma 6.3 to compute that

$$(\overline{D}_X \cdot \overline{D}_{(X,Y)})^2 \leq (\overline{D}_X \cdot \overline{D}_X) (\overline{D}_X \cdot \overline{D}_Y) = 0$$
because $D_X$ is normalized. Therefore,
\[
0 = D_X \cdot D_{(X,Y)} = \frac{1}{2} (D_X \cdot D_{X+Y} - D_X \cdot D_Y) = \frac{1}{2} (D_X \cdot D_{X+Y} - D_X \cdot D_Y),
\]
so that
\[
D_X \cdot D_{X+Y} = D_X \cdot D_Y.
\]

For (4), fix $x \in \mathbb{R}$. By the definition of $D_{xY}$ in (4.10), and the bilinearity of the intersection number, we have
\[
D_X \cdot D_{xY} = x^2 D_X \cdot D_Y.
\]

Properties (1) – (4) show that $Y \mapsto D_X \cdot D_Y$ defines a positive semidefinite quadratic form as claimed. Finally, we note that
\[
D_X \cdot D_{(Y,Y)} = \frac{1}{2} (D_X \cdot D_{Y+Y} - D_X \cdot D_Y - D_X \cdot D_Y) = \frac{1}{2} (D_X \cdot D_{2Y} - 2 D_X \cdot D_Y) = D_X \cdot D_Y,
\]
completing the proof of the proposition.

7. Equivalent formulations of Theorem 1.1

Recall that $E \to B$ denotes a non-isotrivial elliptic surface defined over a number field $K$, and let $E$ be the corresponding elliptic curve over the field $k = \overline{K}(B)$. We extend $K$ so that all sections of $E \to B$ are defined over $K$. In this section we prove:

**Theorem 7.1.** Let $E \to B$ be a non-isotrivial elliptic surface defined over a number field $K$, and let $E$ be the corresponding elliptic curve over the field $k = \overline{K}(B)$. Let $\Lambda$ be any subgroup of $E(k)$. The following are equivalent:

1. the conclusion of Theorem 1.1 holds for all $P, Q \in \Lambda$;
2. the conclusion of Theorem 1.3 holds for all sections $C$ of $E^t$ defined by the graph $t \mapsto (Q_{1,t}, \ldots, Q_{\ell,t})$ for points $Q_1, \ldots, Q_\ell \in \Lambda$, for all $\ell \geq 2$;
3. the conclusion of Theorem 1.5 holds for this $\Lambda$;
4. the biquadratic form $(X,Y) \mapsto D_X \cdot D_Y$ on $\Lambda \otimes \mathbb{R}$ is non-degenerate, meaning that $D_X \cdot D_Y = 0$ if and only if $X$ and $Y$ are linearly dependent over $\mathbb{R}$;
5. for any pair $X, Y \in \Lambda \otimes \mathbb{R}$, if heights $h_X(t) = h_Y(t)$ for all $t \in B(\overline{K})$, then $X = \pm Y$;
6. for any pair $X, Y \in \Lambda \otimes \mathbb{R}$, if the Néron-Tate inner product satisfies $(X_t, Y_t)_{E_t} = 0$ for all $t \in B(\overline{K})$ with $E_t$ smooth, then either $X$ or $Y$ is 0;
(7) for any pair $X, Y \in \Lambda \otimes \mathbb{R}$, if there exists an infinite (non-repeating) sequence of points $t_n \in B(\overline{K})$ for which

$$
\lim_{n \to \infty} h_X(t_n) + h_Y(t_n) = 0,
$$

then $X$ and $Y$ are linearly dependent over $\mathbb{R}$.

Throughout this section, we rely on the work carried out in Sections 3 – 6. Specifically, for each $X \in E(k) \otimes \mathbb{R}$, we can express $X$ as a finite $\mathbb{R}$-linear combination of elements $P_1, \ldots, P_m \in E(k)$. We appeal to Theorem 4.6 to know that $D_X$ is a well-defined, semi-positive, normalized, continuous adelic metrization on $B$, defined over the number field $K$. Further, $(X,Y) \mapsto D_X \cdot D_Y$ is a well-defined semipositive biquadratic form on $E(k) \otimes \mathbb{R}$ by Proposition 6.1.

7.1. Intersection number 0. We begin by elaborating on the consequences of the existence of a pair $X, Y \in E(k) \otimes \mathbb{R}$ for which $D_X \cdot D_Y = 0$.

Recall that $\langle \cdot, \cdot \rangle_t$ denotes the Néron-Tate bilinear form on the fiber $E_t(\overline{K}) \otimes \mathbb{R}$, and $\langle \cdot, \cdot \rangle_E$ denotes the corresponding form on $E(k) \otimes \mathbb{R}$.

**Proposition 7.2.** Fix nonzero $X, Y \in E(k) \otimes \mathbb{R}$, and assume that $D_X \cdot D_Y = 0$. Then for all $t \in B(\overline{K})$ for which the fiber $E_t$ is smooth, we have

$$
h_X(t) = \frac{\hat{h}_E(X)}{\hat{h}_E(Y)} h_Y(t) \quad \text{and} \quad \langle X_t, Y_t \rangle_t = \left(\frac{X_t, Y_t}{\hat{h}_E(Y)}\right) h_Y(t).
$$

Moreover, we have $\overline{D}_X \cdot \overline{D}_Y = 0$ for all $X', Y' \in \text{Span}_\mathbb{R} \{\{X, Y\}\}$.

**Proof.** Assume that $\overline{D}_X \cdot \overline{D}_Y = 0$. From Theorem 4.6, each of $\overline{D}_X$ and $\overline{D}_Y$ is a continuous, normalized, semipositive adelic metrization on an $\mathbb{R}$-divisor on $B$. The degree of $D_X$ (respectively $D_Y$) is $\hat{h}_E(X)$ (respectively, $\hat{h}_E(Y)$). The relation between the heights $h_X$ and $h_Y$ follows immediately from Theorem 3.4.

Using now part (3) of Proposition 6.1 our assumption that $\overline{D}_X \cdot \overline{D}_Y = 0$ implies that $\overline{D}_X + Y \cdot \overline{D}_Y = \overline{D}_X + Y \cdot \overline{D}_X = 0$, and so

$$
\overline{D}_{xY} \cdot \overline{D}_{aX+bY} = ((x^2 - xy)\overline{D}_X + xy\overline{D}_X + (y^2 - xy)\overline{D}_Y)
\cdot ((a^2 - ab)\overline{D}_X + ab\overline{D}_Y + (b^2 - ab)\overline{D}_Y)
= 0
$$

for all $x, y, a, b, \in \mathbb{R}$. In particular, we have

$$
\hat{h}_{E_t}(X + Y) = \frac{\hat{h}_E(X + Y)}{\hat{h}_E(Y)} \hat{h}_{E_t}(Y).
$$
so that
\[ \hat{h}_{E_t}(X_t + Y_t) = \hat{h}_{E_t}(X_t) + \langle X_t, Q_t \rangle_t + \hat{h}_{E_t}(Y_t) \]
implies
\[ \langle X_t, Y_t \rangle_t = \frac{\langle X, Y \rangle_E}{\hat{h}_E(Y)} h_Y(t) \]
for all \( t \in B(\overline{K}) \) for which \( E_t \) is smooth. \( \square \)

The following proposition extends the observations of Proposition 5.3 to two independent relations.

**Proposition 7.3.** Let \( \Lambda \) be a subgroup of \( E(k) \) generated by independent, non-torsion elements \( P_1, \ldots, P_m \), with \( m \geq 2 \). The following are equivalent:

1. there exist an infinite, non-repeating sequence \( t_n \in B(\overline{K}) \) and points \( p_{i,n} \in E_{t_n}(\overline{K}) \) for \( i = 1, \ldots, m \), for which \( \hat{h}_{E_{t_n}}(p_{i,n}) \to 0 \) as \( n \to \infty \), and the points
\[ P_{1,t_n} - p_{1,n}, \ldots, P_{m,t_n} - p_{m,n} \]
satisfy two independent linear relations on \( E_{t_n} \);
2. there exist independent \( X, Y \in \Lambda \otimes \mathbb{R} \) for which
\[ D_X \cdot D_Y = 0. \]

**Proof.** Assume first that \( D_X \cdot D_Y = 0 \). Write 
\[ X = x_1 P_1 + \cdots + x_m P_m \quad \text{and} \quad Y = y_1 P_1 + \cdots + y_m P_m \]
for linearly independent coefficient vectors \( \vec{x}, \vec{y} \in \mathbb{R}^m \). From Proposition 7.2, we can replace \( X \) and \( Y \) by linear combinations of \( X \) and \( Y \) (and relabel the points \( P_i \) if needed) and so assume that \( x_1 = 1 = y_m \) and \( x_m = y_1 = 0 \). From Theorem 4.6, we know that \( D_X \) and \( D_Y \) are normalized, semipositive, continuous adelic metrizations. By Proposition 5.1, we know that \( e_1(D_X) = e_1(D_Y) = 0 \). Theorem 3.4 then implies that there is an infinite non-repeating sequence \( \{ t_n \} \subset B(\overline{K}) \) so that
\[ h_X(t_n) + h_Y(t_n) \to 0 \] (7.1)
as \( n \to \infty \). We now apply Proposition 5.3 to each of \( h_X \) and \( h_Y \) to show that small perturbations of the specializations \( P_{i,t_n} \) must satisfy two independent relations in the fibers \( E_{t_n}(\overline{K}) \). More precisely, we choose integers \( a_{i,n}, b_{i,n} \) for each \( n \geq 1 \) and each \( i = 2, \ldots, m-1 \), so that
\[ \frac{a_{i,n}}{n} \to x_i \quad \text{and} \quad \frac{b_{i,n}}{n} \to y_i \]
as \( n \to \infty \). As in the proof of Proposition 5.3 (2) \( \implies \) (3), we choose \( p_n \in E_{t_n}(\overline{K}) \) so that
\[ np_n = a_{2,n} P_2 + \cdots + a_{m-1,n} P_{m-1}. \]
Set $p_{1,n} = P_{1,t_n} + p_n \in E_{t_n}(\overline{K})$. Then
\[
\hat{h}_{E_{t_n}}(p_{1,n}) = \hat{h}_{E_{t_n}} \left( P_{1,t_n} + \frac{1}{n} (a_{2,n}P_2 + \cdots + a_{m-1,n}P_{m-1}) \right) \to 0,
\]
and $\{P_{1,t_n} - p_{1,t_n}, P_{2,t_n}, \ldots, P_{m-1,t_n}\}$ satisfy a linear relation. On the other hand, we can repeat the same argument with $Y$ and find point $q_n \in E_{t_n}(\overline{K})$ so that
\[
n q_n = b_{2,n}P_2 + \cdots + b_{m-1,n}P_{m-1}
\]
and set $p_{m,n} = P_{m,t_n} + q_n$. Then
\[
\hat{h}_{E_{t_n}}(p_{m,n}) = \hat{h}_{E_{t_n}} \left( \frac{1}{n} (b_{2,n}P_2 + \cdots + b_{m-1,n}P_{m-1}) + P_{m,t_n} \right) \to 0,
\]
and $\{P_{2,t_n}, \ldots, P_{m-1,t_n}, P_{m,t_n} - p_{m,n}\}$ satisfy a linear relation. It follows that the points
\[
\{P_{1,t_n} - p_{1,t_n}, P_{2,t_n}, \ldots, P_{m-1,t_n}, P_{m,t_n} - p_{m,n}\}
\]
satisfy two independent linear relations in $E_{t_n}(\overline{K})$ for all $n$.

For the converse direction, we assume there are an infinite, non-repeating sequence $t_n \in B(\overline{K})$ and points $p_{i,n} \in E_{t_n}(\overline{K})$ for $i = 1, \ldots, m$, for which $\hat{h}_{E_{t_n}}(p_{i,n}) \to 0$ as $n \to \infty$, and such that
\[
\{P_{1,t_n} - p_{1,t_n}, \ldots, P_{m,t_n} - p_{m,n}\}
\]
satisfy two independent linear relations on $E_{t_n}$. From Lemma 5.4, we know that the sequence $\{t_n\}$ must have bounded height. Choose integers $a_{i,n}, b_{i,n}$ for $n \geq 1$ and $i = 1, \ldots, m$ so that the independent relations are expressed as
\[
a_{1,n}(P_{1,t_n} - p_{1,n}) + \cdots + a_{m,n}(P_{m,t_n} - p_{m,n}) = O_{t_n}
\]
and
\[
b_{1,n}(P_{1,t_n} - p_{1,n}) + \cdots + b_{m,n}(P_{m,t_n} - p_{m,n}) = O_{t_n}
\]
Relabeling the points if necessary, we can rewrite the expressions as
\[
(P_{1,t_n} - p_{1,n}) + r_{2,n}(P_{2,t_n} - p_{2,n}) + \cdots + r_{m,n}(P_{m,t_n} - p_{m,n}) = O_{t_n}
\]
and
\[
r_{1,n}'(P_{1,t_n} - p_{1,n}) + \cdots + r_{m-1,n}'(P_{m-1,t_n} - p_{m-1,n}) + (P_{m,t_n} - p_{m,n}) = O_{t_n}
\]
for bounded sequences of rational numbers $r_{2,n}, \ldots, r_{m,n}$ and $r_{1,n}', \ldots, r_{m-1,n}'$. Passing to a subsequence we may assume that
\[
r_{i,n} \to x_i \in \mathbb{R} \quad \text{and} \quad r_{i,n}' \to y_i \in \mathbb{R}
\]
for each $i$. Then, recalling that $\{t_n\}$ has bounded height and that the perturbations $p_{i,n}$ have heights tending to 0 and using [Si1, Theorem A] to infer that $\{\hat{h}_{E_{t_n}}(P_{i,t_n})\}_n$ are bounded for each $i$, we conclude that

$$h_X(t_n) \to 0 \quad \text{and} \quad h_Y(t_n) \to 0$$

along this subsequence, for $X = P_1 + x_2 P_2 + \ldots + x_m P_m$ and $Y = y_1 P_1 + \ldots + y_{m-1} P_{m-1} + P_m$. From Theorem 3.4, we have that $\overline{D}_X \cdot \overline{D}_Y = 0$. \qed

7.2. **Proof of Theorem 7.1.** Throughout this proof, we fix a finitely-generated subgroup $\Lambda \subset E(k)$. Assume it is of rank $m \geq 1$ with torsion-free part generated by $P_1, \ldots, P_m \in E(k)$.

(1) $\iff$ (4) Recall that the Néron-Tate height $\hat{h}_E$ on $\Lambda$ extends to a positive definite quadratic form on $\Lambda \otimes \mathbb{R}$. It follows (by Cauchy-Schwarz) that the Néron-Tate regulator

$$R_E(X,Y) := \hat{h}_E(X)\hat{h}_E(Y) - \langle X, Y \rangle_E^2 \geq 0$$

extends to a biquadratic form on $\Lambda \otimes \mathbb{R}$ satisfying $R_E(X,Y) = 0$ if and only if $X$ and $Y$ are linearly dependent over $\mathbb{R}$. As

$$F(X,Y) := \overline{D}_X \cdot \overline{D}_Y$$

is also biquadratic on $\Lambda \otimes \mathbb{R}$ and satisfies $F(X,Y) \geq 0$ (see Proposition 6.1), the upper bound on $\overline{D}_X \cdot \overline{D}_Y$ in Theorem 1.1 follows. Condition (1) is then equivalent to the statement that $F(X,Y) = 0$ if and only if $X$ and $Y$ are linearly dependent over $\mathbb{R}$.

In details, if we assume (1), and if the pair $X = \sum_{i=1}^m x_i P_i$ and $Y = \sum_{i=1}^m y_i P_i$ with $P_i \in \Lambda$ satisfy $\overline{D}_X \cdot \overline{D}_Y = 0$, then we can approximate by rational combinations $P_n = \frac{1}{n} \sum_{i=1}^m a_{i,n} P_i \to X$ and $Q_n = \frac{1}{n} \sum_{i=1}^m b_{i,n} P_i \to Y$ with integers $a_{i,n}, b_{i,n}$, and compute that

$$\overline{D}_{P_n} \cdot \overline{D}_{Q_n} = \frac{1}{n^4} \overline{D}_{\sum_{i=1}^m a_{i,n} P_i} \cdot \overline{D}_{\sum_{i=1}^m b_{i,n} P_i} \geq \frac{c}{n^4} R_E \left( \sum_{i=1}^m a_{i,n} P_i, \sum_{i=1}^m b_{i,n} P_i \right) = c R_E(P_n, Q_n).$$

Letting $n \to \infty$ shows that $R_E(X,Y) = 0$, implying that $X, Y$ are linearly dependent over $\mathbb{R}$. Now assume (4), so that $F(\cdot, \cdot)$ is nondegenerate on the finite-dimensional $V = \Lambda \otimes \mathbb{R}$. Using the inner product $\langle \cdot, \cdot \rangle_E$ on $V$ and associated norm $\|\cdot\| = \hat{h}_E(\cdot)^{1/2}$, we have (by continuity and compactness) uniform positive upper and lower bounds on $\overline{D}_X \cdot \overline{D}_Y$ over all pairs $X, Y \in V$ satisfying $\langle X, Y \rangle_E = 0$ and $\|X\| = \|Y\| = 1$. On the other hand, $R_E(X,Y) = 1$ for all such pairs, and so there is a positive constant $c = c(V)$ so that

$$c R_E(X,Y) \leq \overline{D}_X \cdot \overline{D}_Y \leq c^{-1} R_E(X,Y)$$

all pairs $X, Y \in V$ satisfying $\langle X, Y \rangle_E = 0$ and $\|X\| = \|Y\| = 1$. By scaling the points, this extends to orthogonal pairs of any norm. For an arbitrary pair $X, Y \in V$, we write $Y = Y' + xX$ with $\langle Y', X \rangle_E = 0$ and $x \in \mathbb{R}$, and observe that $\overline{D}_X \cdot \overline{D}_Y = \overline{D}_X \cdot \overline{D}_{Y'}$ from Proposition 6.1. We also have $R_E(X,Y' + xX) = R_E(X,Y')$ and so (7.2) holds for all pairs $X, Y$ in $V$. 

Fix any pair \( X, Y \in \Lambda \otimes \mathbb{R} \), and express \( X \) and \( Y \) as \( \mathbb{R} \)-linear combinations of elements \( P_1, \ldots, P_m \in \Lambda \). Theorem 4.6 shows that \( D_X \) and \( D_Y \) are normalized, continuous, semipositive adelic metrizations on \( \mathbb{R} \)-divisors, and Proposition 5.1 shows that each has essential minimum equal to 0. Theorem 3.4 then implies that \( D_X \cdot D_Y = 0 \) if and only if the heights \( h_X \) and \( h_Y \) have a common small sequence in \( B(K) \).

Assume that (7) holds. We aim to prove the conclusion of Theorem 1.5 for this \( \Lambda \). Suppose that there is an infinite, non-repeating sequence \( t_n \in B(K) \) with \( h_{\Lambda_i}(t_n) \to 0 \) for all \( i = 1, \ldots, m \). From Lemma 5.3, we may choose point \( X_1 \in \Lambda_1 \) so that \( \liminf_{n \to \infty} h_{X_1}(t_n) = 0 \). Pass to a subsequence so that \( \lim_{n \to \infty} h_{X_1}(t_n) = 0 \). For each \( i = 2, \ldots, m \), we successively apply Lemma 5.3 to find \( X_i \in \Lambda_i \) for which \( \liminf_{n \to \infty} h_{X_i}(t_n) = 0 \) and then pass to a further subsequence so that \( \lim_{n \to \infty} h_{X_i}(t_n) = 0 \). In this way, we have an infinite, non-repeating sequence of points \( t_n \in B(K) \) so that \( \lim_{n \to \infty} h_{X_i}(t_n) = 0 \) for all \( i \). However, as \( \bigcap_{i=1}^m \Lambda_i = \{0\} \), at least two of the \( X_i \) must be independent. This contradicts (7).

Assume now that (3) holds. Fix a pair of independent nonzero points \( X, Y \in \Lambda \otimes \mathbb{R} \), and suppose that there is an infinite nonrepeating sequence \( t_n \in B(K) \) so that

\[
\begin{align*}
    h_X(t_n) + h_Y(t_n) \to 0.
\end{align*}
\]

Then \( h_X(t_n) \to 0 \) and \( h_Y(t_n) \to 0 \). We write

\[
\begin{align*}
    X &= a_1 P_1 + \cdots + a_m P_m \\
    Y &= b_1 P_1 + \cdots + b_m P_m,
\end{align*}
\]

with \( a_i, b_j \in \mathbb{R} \) and independent \( P_i \in \Lambda \). We want to show that

\[
\begin{align*}
    \liminf_{n \to \infty} h_{\Lambda_i}(t_n) = 0
\end{align*}
\]

for all \( i = 1, \ldots, m \), contradicting (3). Fix \( i \in \{1, \ldots, m\} \). If \( b_i = 0 \), then \( Y \in \Lambda_i \otimes \mathbb{R} \) and (7.3) follows from Lemma 5.3. If on the other hand \( b_i \neq 0 \), then \( X - \frac{a_i}{b_i} Y \in \Lambda_i \otimes \mathbb{R} \) and by the parallelogram law we also have

\[
\begin{align*}
    h_{X - \frac{a_i}{b_i} Y}(t_n) \to 0.
\end{align*}
\]

As before, equation (7.3) follows by Lemma 5.3.

Assume that (7) holds. Fix \( X, Y \in \Lambda \otimes \mathbb{R} \), and suppose that \( \langle X_t, Y_t \rangle_t = 0 \) for all \( t \in B(K) \) for which the fiber \( E_t \) is smooth. By Proposition 5.1 there is an infinite
sequence \( t_n \in B(\mathbb{K}) \) with \( h_{X-Y}(t_n) \to 0 \). Since \( (X_{t_n}, Y_{t_n})_{t_n} = 0 \) we have
\[
 h_X(t_n) + h_Y(t_n) = h_X(t_n) - 2(X_{t_n}, Y_{t_n})_{t_n} + h_Y(t_n) = h_{X-Y}(t_n) \to 0.
\]
Thus by (7) we get that either \( X \) or \( Y \) is 0 or there are non-zero \( a, b \in \mathbb{R} \) such that \( aX = bY \).

In the latter case, our assumption that \( \langle X_{t_n}, Y_{t_n} \rangle_{t_n} = 0 \) for all \( t_n \) implies that both \( X \) and \( Y \) are 0. The assertion follows.

\[
(6) \implies (5)
\]
Assume that (6) holds. Fix \( X, Y \in \Lambda \otimes \mathbb{R} \), and suppose that \( h_X(t) = h_Y(t) \) for all \( t \in B(\mathbb{K}) \). If \( X = 0 \) or \( Y = 0 \) then our assumption that \( h_X(t) = h_Y(t) \) for all \( t \) implies that \( X = Y = 0 \) in \( \Lambda \otimes \mathbb{R} \). Thus we may assume that both \( X \) and \( Y \) are non-zero.

Since \( h_X(t) = h_Y(t) \) for all \( t \), Silverman’s specialization theorem [Si1, Theorem B] implies that \( \hat{h}_E(X) = \hat{h}_E(Y) \). From Theorem 3.4 we know that \( \overline{D}_X \cdot \overline{D}_Y = 0 \), and therefore, from Proposition 7.2, we have
\[
\langle X_{t_n}, Y_{t_n} \rangle_{t_n} = \langle X, Y \rangle_{E \hat{h}_E(Y)}\hat{h}_E(Y),
\]
for all \( t \). By our assumption (6) and since \( Y \neq 0 \), we have
\[
X = \frac{\langle X, Y \rangle_{E \hat{h}_E(Y)}}{\hat{h}_E(Y)} Y.
\]
Recalling that \( \hat{h}_E(X) = \hat{h}_E(Y) \), we get \( X = \pm Y \) in \( \Lambda \otimes \mathbb{R} \), as claimed.

\[
(5) \implies (7)
\]
Suppose there exist nonzero \( X, Y \in \Lambda \otimes \mathbb{R} \) and an infinite, non-repeating sequence \( t_n \in B(\mathbb{K}) \) for which \( h_X(t_n) + h_Y(t_n) \to 0 \). By Theorem 4.6 we know that both \( h_X \) and \( h_Y \) are induced by normalized semipositive adelic metrizations on ample divisors \( D_X \) and \( D_Y \) on \( B \), of degrees \( h_E(X) \) and \( \hat{h}_E(Y) \), respectively. We may thus apply Theorem 3.4 to get
\[
h_X(t) = \frac{\hat{h}_E(X)}{\hat{h}_E(Y)} h_Y(t) = h_{xY}(t)
\]
for all \( t \in B(\mathbb{K}) \), where \( x = \sqrt{\frac{\hat{h}_E(X)}{\hat{h}_E(Y)}} \). Our assumption (5) then yields that \( X = \pm xY \) as claimed.

\[
(2) \iff (4)
\]
Fix any collection of points \( Q_1, \ldots, Q_\ell \) in \( \Lambda \), and let \( C \) be the irreducible curve in \( \mathcal{E}^\ell \) defined by a section \( (Q_1, \ldots, Q_\ell) \) over \( B \). To say that \( C \) is not contained in
flat subgroup scheme of positive codimension means that the points \( Q_1, \ldots, Q_\ell \) are linearly independent. To say that the curve \( C \) in \( \mathcal{E}^\ell \) defined by \( (Q_1, \ldots, Q_\ell) \) intersects the tube \( T(\mathcal{E}_m, \varepsilon) \) infinitely often for every \( \varepsilon > 0 \) means that there is an infinite non-repeating sequence of points \( t_n \in B(\overline{K}) \) and small points \( q_{i,n} \in E_{t_n}(\overline{K}) \) for each \( n \) so that the points \( \{Q_{1,t_n} - q_{1,n}, \ldots, Q_{\ell,t_n} - q_{\ell,n}\} \) satisfy two linear relations in \( E_{t_n} \). Therefore the equivalence of (2) and (4) is the statement of Proposition 7.3.

This completes the proof of the theorem.

8. **Equality of measures**

In this section we prove Theorem 1.12. We begin by describing the complex geometry of elements \( X \) of the real vector space \( E(k) \otimes \mathbb{R} \).

8.1. **Real points as holomorphic curves.** Given a non-isotrivial elliptic surface \( \mathcal{E} \to B \) defined over the number field \( K \), let \( S \subset B \) be the finitely-punctured Riemann surface over which all fibers \( E_t(\mathbb{C}) \) are smooth. Write \( \mathcal{E}_S \) for the open subset of \( \mathcal{E} \) over \( S \). Recall that each rational point \( P \in E(k) \) determines a holomorphic section of \( \mathcal{E} \to B \) defined by \( t \mapsto P_t \in E_t(\mathbb{C}) \) for \( t \in S(\mathbb{C}) \).

The Betti coordinates of \( P \in E(k) \) are defined as follows. Passing to the universal cover \( \tau : \tilde{S} \to S \), there is a holomorphic period function \( \tau : \tilde{S} \to \mathbb{H} \) taking values in the upper half plane, so that the fibers of \( \mathcal{E}_S \) satisfy

\[
E_{\pi(s)}(\mathbb{C}) \simeq \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau(s))
\]

for all \( s \in \tilde{S} \). Passing to the universal cover of \( E_{\pi(s)}(\mathbb{C}) \) for each fiber, we obtain a holomorphic line bundle over \( \tilde{S} \), trivialized by sending generator 1 of the lattice to 1 in \( \mathbb{C} \). For each \( P \in E(k) \), the corresponding section of \( \mathcal{E} \to B \) therefore lifts to a holomorphic function

\[
\xi_p : \tilde{S} \to \mathbb{C}
\]

The Betti map of \( P \) is the real-analytic map \( \beta_P : \tilde{S} \to \mathbb{R}^2 \) given by

\[
\beta_P(s) = (x(s), y(s)) \quad \text{such that} \quad \xi_P(s) = x(s) + y(s) \tau(s).
\]

The coordinates \( x \) and \( y \) themselves depend on the choices of \( \tau \) and \( \xi_P \), but as proved in [CDMZ], we have

\[
\omega_P = dx \wedge dy,
\]

independent of the choices, for the curvature distribution of \( \overline{D}_P \) at an archimedean place of \( K \).
For points \( Q \in E(\overline{k}) \) and integers \( n \) satisfying \( nQ = P \in E(k) \), the holomorphic functions
\[
\xi_Q = \frac{1}{n} \xi_P,
\]
for any choice of \( \xi_P \) (associated to a fixed choice of \( \tau \)), define holomorphic curves in \( E_S \) that are not necessarily sections over \( S \). Letting \( P \) range over all elements of \( E(k) \) and passing to limits as \( n \to \infty \), we obtain the following:

**Proposition 8.1.** Fix a choice of period function \( \tau : \tilde{S} \to \mathbb{H} \). Each \( X \in E(k) \otimes \mathbb{R} \) is represented by a holomorphic function \( \xi_X : \tilde{S} \to \mathbb{C} \) given by \( \xi_X = x_X + y_X \tau \) for which
\[
\omega_X = dx_X \wedge dy_X
\]
on \( S \). More precisely, if \( X = \sum_i x_i P_i \) for \( x_i \in \mathbb{R} \) and \( P_i \in E(k) \), then the Betti map of \( X \) can be taken to be
\[
\beta_X = \sum_i x_i \beta_{P_i}
\]
for choices of Betti maps \( \beta_{P_i} \). The map \( \beta_X \) is uniquely determined up to translation by a constant in \( \mathbb{R}^2 \).

Recall here that the curvature distribution for \( \overline{D}_X \) as in 4.10 at an archimedean place is given by
\[
\omega_X = \sum_i \left( x_i^2 - \sum_{j \neq i} x_i x_j \right) \omega_{P_i} + \sum_{i < j} x_i x_j \omega_{P_i + P_j},
\]
with \( P_i \in E(k) \).

**Remark 8.2.** The holomorphic functions \( \xi_X \) of Proposition 8.1 project to holomorphic curves in \( E_S \) over \( S \). For torsion points of \( E(k) \) representing the 0 of \( E(k) \otimes \mathbb{R} \), the holomorphic curves given by Proposition 8.1 are precisely the leaves of the Betti foliation, because we allow for arbitrary translation in \( \mathbb{R}^2 \). By definition, the leaves of the Betti foliation have constant Betti coordinates; see, e.g., [ACZ], [CDMZ], and [UU] for more information. For each nonzero \( X \in E(k) \otimes \mathbb{R} \), there is a corresponding foliation of \( E_S \). When an element \( X \) is represented by \( P \in E(k) \), the foliation is simply the corresponding Betti foliation for the elliptic surface with \( P \) chosen as the zero section.

**Proof of Proposition 8.1.** For each \( P \in E(k) \) we have Betti coordinates as defined above. For multiples \( \frac{1}{n} P \), there are natural choices of the functions \( \xi \), also as described above, defined up to translation by elements of \( \frac{1}{n}(\mathbb{Z} \oplus \mathbb{Z} \tau) \). Letting \( P_n \) be a sequence in \( E(k) \) such that \( Q_n := \frac{1}{n} P_n \) converges to \( X \) in \( E(k) \otimes \mathbb{R} \), the families of holomorphic functions \( \xi_{Q_n} \) converge, locally uniformly in \( \tilde{S} \). This defines a family of limit holomorphic functions \( \xi_X \). Because of the choices for \( \xi_{P_n} \) and \( \xi_{Q_n} \), we see that \( \xi_X \) is only defined up to translation by real multiples.
of lattice elements. Fix a choice of $\xi_X$ and consider the measure $\nu_X = dx_X \wedge dy_X$. This measure is clearly independent of the choices. Furthermore, it is the weak limit of measures $\omega_{Q_n}$ on $S$, by the formula (8.1) for $\omega_{Q_n}$ and local uniform convergence of $\xi_{Q_n}$ to $\xi_X$. We already know that $\omega_{Q_n} \to \omega_X$ for the curvature distributions (at a fixed archimedean place), from the definitions given in §4.3. It follows that $\nu_X = \omega_X$. □

8.2. Proof of Theorem 1.12. Fix $X_1, X_2 \in E(k) \otimes \mathbb{R}$, and let $D_1$ and $D_2$ be the associated metrized $\mathbb{R}$-divisors on $B$, defined over the number field $K$. Fix an archimedean place of $K$, and let $\omega_1$ and $\omega_2$ be the curvature measures on $B(\mathbb{C})$ at this place. We assume that $\omega_1 = \omega_2$. As in §8.1, we fix a period function $\tau : \tilde{S} \to \mathbb{H}$, and we choose holomorphic functions $\xi_i = x_i + y_i\tau$, $i = 1, 2$, representing the points $X_1$ and $X_2$. From (8.1), we know that

\[(8.2) \quad dx_1 \wedge dy_1 = dx_2 \wedge dy_2\]
on $\tilde{S}$.

We break the proof into two steps. In the first, we exploit the holomorphic-antiholomorphic trick of [ACZ, §5], applied to a relation between holomorphic functions $\xi_1, \xi_2, \tau$ (and their derivatives) and the anti-holomorphic functions $\bar{\xi}_1, \bar{\xi}_2,$ and $\bar{\tau}$ (and their derivatives) coming from (8.2); the result is a relation on the holomorphic input alone. In the second step, we apply the transcendence result of [Be, Théorème 5] to this relation and deduce that the points $X_1$ and $X_2$ must be related in $E(k) \otimes \mathbb{R}$.

Step 1: Holomorphic-Antiholomorphic. We are grateful to Lars Kühne for teaching us this step.

Note that

\[d\xi_i = dx_i + y_i d\tau + \tau dy_i\]

so that

\[(d\xi_i - y_i d\tau) \wedge (d\bar{\xi}_i - y_i d\bar{\tau}) = (\bar{\tau} - \tau) dx_i \wedge dy_i.\]

Writing

\[y_i = \frac{\xi_i - \bar{\xi}_i}{\tau - \bar{\tau}}\]

we obtain a relation from (8.2) of the form

\[((\tau - \bar{\tau}) d\xi_1 - (\xi_1 - \bar{\xi}_1) d\tau) \wedge ((\tau - \bar{\tau}) d\bar{\xi}_1 - (\xi_1 - \bar{\xi}_1) d\bar{\tau}) = ((\tau - \bar{\tau}) d\xi_2 - (\xi_2 - \bar{\xi}_2) d\tau) \wedge ((\tau - \bar{\tau}) d\bar{\xi}_2 - (\xi_2 - \bar{\xi}_2) d\bar{\tau}).\]
Working in coordinates in the simply connected $\tilde{S}$, this gives

$$
(8.3) \quad (\xi_1 \xi_2' - \xi_2 \xi_1') (\tau - \bar{\tau})^2 - ((\xi_1 - \xi_1') \xi_1' - (\xi_2 - \xi_2') \xi_2') (\tau - \bar{\tau}) \bar{\tau}'
$$

$$
- ((\xi_1 - \xi_1') \xi_1' - (\xi_2 - \xi_2') \xi_2') (\tau - \bar{\tau}) \tau' + ((\xi_1 - \xi_1')^2 - (\xi_2 - \xi_2')^2) \tau' \bar{\tau}' = 0.
$$

Equation (8.3) can be expressed as

$$
\sum_{j=1}^{N} f_j(z) g_j(z) \equiv 0
$$

for holomorphic functions $f_j \in \mathbb{Z}[\xi_1, \xi_2, \xi_1', \xi_2']$ and antiholomorphic functions $g_j \in \mathbb{Z}[\xi_1, \xi_2, \xi_1', \xi_2', \tau, \bar{\tau}]$ in $z \in \tilde{S}$.

For each $j$, define holomorphic function

$$
\hat{g}_j(w) := g_j(\bar{w}).
$$

Then

$$
(8.4) \quad F(z, w) := \sum_{j=1}^{N} f_j(z) \hat{g}_j(w)
$$

is holomorphic on $\tilde{S} \times \tilde{S}$ and vanishes identically on the real-analytic subvariety $\{w = \bar{z}\}$, where it coincides with (8.3). It follows that $F$ must vanish identically on $\tilde{S} \times \tilde{S}$; see [ACZ, Lemma 5.2]. In particular, if we fix any $w_0 \in \tilde{S}$, we have $F(z, w_0) \equiv 0$ on $\tilde{S}$, and we obtain a polynomial relation in the holomorphic functions $\xi_1, \xi_2, \xi_1', \xi_2', \tau, \bar{\tau}'$ that holds on all of $\tilde{S}$.

**Step 2: Algebraic Independence.** Suppose that $P_1, \ldots, P_m \in E(k)$ define a basis for $E(k) \otimes \mathbb{R}$, so that

$$
X_i = \sum_{j=1}^{m} a_{i,j} P_j
$$

for $a_{i,j} \in \mathbb{R}$, $i = 1, 2$. From Proposition 8.1, we know that we can write

$$
\xi_i = \sum_{j} a_{i,j} \xi_{P_j}
$$

for choices of lifts $\xi_{P_j}$ of each point $P_j$. From Step 1, for each $w_0 \in \tilde{S}$, the function (8.4) satisfies $F(\cdot, w_0) \equiv 0$ on $\tilde{S}$, giving a polynomial relation on the holomorphic functions

$$
\xi_{P_1}, \ldots, \xi_{P_m}, \xi'_{P_1}, \ldots, \xi'_{P_m}, \tau, \tau'.
$$

But the functions $\xi_{P_j}$ come from algebraic points $P_j \in E(k)$ and so satisfy the hypothesis of Théorème 5 in [Be]. Consequently, if the relation is nontrivial, the functions $\xi_{P_j}$ and therefore the points $P_j$ must themselves satisfy a linear relation. As they were assumed to be a basis for $E(k) \otimes \mathbb{R}$, we conclude that the relation must have been trivial. In other words,
for any choice of \( w_0 \), the coefficients of \( F(z, w_0) \) – as polynomials in \( \xi_{P_1}, \ldots, \xi_{P_m}, \xi'_{P_1}, \ldots, \xi'_{P_m} \) – must vanish.

Examining the relation (8.3), we can determine these coefficients explicitly. The “constant” term, having no dependence on the \( \xi_{P_j} \) or \( \xi'_{P_j} \), gives

\[
C_1(w_0) \tau' + C_2(w_0) \tau \tau' = 0
\]
as a function of \( z \in \tilde{S} \), with coefficients \( C_1, C_2 \) that are antiholomorphic functions of \( w_0 \) on \( \tilde{S} \). For fixed \( w_0 \), if \( C_1 \) or \( C_2 \) is nonzero, this would imply that \( \tau \) is constant, which is absurd because the elliptic surface \( \mathcal{E} \to B \) is nonisotrivial. This implies that \( C_2(w_0) = 0 \) for all \( w_0 \).

But, again looking at the formula from (8.3), we have

\[
C_2(w_0) = \xi_1'(w_0)\xi_1(w_0) - \xi_2'(w_0)\xi_2(w_0) = 0
\]
for all \( w_0 \). Taking complex conjugates, we have

\[
0 \equiv \xi_1'\xi_1 - \xi_2'\xi_2 = \sum_{j,\ell=1}^{m} (a_{1,j}a_{1,\ell} - a_{2,j}a_{2,\ell})\xi_{P_j}\xi_{P_\ell}.
\]

In other words, we find another relation on the holomorphic functions \( \xi_{P_j} \) and \( \xi_{P_\ell} \) which must therefore be trivial [Be, Théorème 5]. We conclude that either

\[
a_{1,j} = a_{2,j}
\]
for all \( j \) or that

\[
a_{1,j} = -a_{2,j}
\]
for all \( j \). In other words,

\[
X_1 = \pm X_2.
\]
This completes the proof of Theorem 1.12.

9. Proofs of the main theorems

In this section, we prove our main theorems.

9.1. Proof of Theorem 1.1. Recall that the Néron-Tate height \( \hat{h}_E \) on \( E(k) \) extends to a positive definite quadratic form on \( E(k) \otimes \mathbb{R} \). It follows (by Cauchy-Schwarz) that the Néron-Tate regulator

\[
R_E(X,Y) := \hat{h}_E(X)\hat{h}_E(Y) - \langle X,Y \rangle_E^2 \geq 0
\]
extends to a biquadratic form on \( E(k) \otimes \mathbb{R} \) satisfying \( R_E(X,Y) = 0 \) if and only if \( X \) and \( Y \) are linearly dependent over \( \mathbb{R} \). As

\[
F(X,Y) := \overline{D_X} \cdot \overline{D_Y}
\]
is also biquadratic on $E(k) \otimes \mathbb{R}$ and satisfies $F(X,Y) \geq 0$ (see Proposition 6.1), the upper bound on $\overline{D}_X \cdot \overline{D}_Y$ in Theorem 1.1 follows.

From Theorem 7.1, we know that Theorem 1.1 holds for $\mathcal{E} \to B$ if and only if $\overline{D}_X \cdot \overline{D}_Y \neq 0$ for all pairs of linearly independent $X, Y \in E(k) \otimes \mathbb{R}$. So assume we have nonzero elements $X, Y \in E(k) \otimes \mathbb{R}$ satisfying $\overline{D}_X \cdot \overline{D}_Y = 0$. By scaling $X$ and $Y$, we may assume that $\hat{h}_E(X) = \hat{h}_E(Y) = 1$. We proved in Theorem 4.6 that $\overline{D}_X$ and $\overline{D}_Y$ are normalized, semipositive, continuous adelic metrizations on $\mathbb{R}$-divisors on $B$, each on divisors of degree 1. Theorem 3.2 then implies that the $\overline{D}_X$ and $\overline{D}_Y$ are isomorphic, so the curvature forms for $\overline{D}_X$ and for $\overline{D}_Y$ on $B^\text{an}$ must coincide at all places $v$ of the number field $K$. Fixing a single archimedean place, we deduce from Theorem 1.12 that $X = \pm Y$. This completes the proof.

9.2. **Proof of Theorem 1.3.** Suppose that $C$ is an algebraic curve in $\mathcal{E}^m$ that dominates the base curve $B$. Passing to a finite branched cover $B' \to B$, we may view $C$ as a section $C'$ of the $m$-th fibered power of the pull-back elliptic surface $\mathcal{E}' \to B'$. As Theorem 1.1 holds for $\mathcal{E}' \to B'$, we apply Theorem 7.1 to conclude that the intersection of $C'$ with the tube $T((\mathcal{E}')^m,\{2\},\epsilon)$ is contained in a finite union of flat subgroup schemes of positive dimension, for all sufficiently small $\epsilon > 0$. Projecting back to $\mathcal{E}^m \to B$, we can make the same conclusion about the intersection of $C$ with $T(\mathcal{E}^m,\{2\},\epsilon)$. This completes the proof.

9.3. **Proof of Theorem 1.5.** The theorem follows immediately from Theorem 1.1 and Theorem 7.1.

**References**


ELLiptic Surfaces and ArithmetiC EquidistriBution for \( \mathbb{R} \) -Divisors on Curves


*Email address: demarco@math.harvard.edu*

*Email address: mavraki@math.harvard.edu*