THE (ALGEBRAIC) GEOMETRY OF PREPERIODIC POINTS IN $\mathbb{P}^N$, IN THREE LECTURES

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In these lecture notes, we examine a series of conjectures about the geometry of preperiodic points for endomorphisms of projective space $\mathbb{P}^N$, defined over the field $\mathbb{C}$ of complex numbers. Lecture 1 was focused on a conjecture formulated by Shouwu Zhang [Zh2], dubbed the "Dynamical Manin-Mumford Conjecture" (DMM) as an extension of the well-known Manin-Mumford Conjecture (which investigated the geometry of torsion points in abelian varieties and was proved in the early 1980s by Raynaud [Ra1, Ra2]). Zhang viewed certain endomorphisms of projective spaces and of abelian varieties on an equal footing, as polarizable endomorphisms. The DMM aims to classify the algebraic subvarieties of $\mathbb{P}^N$ containing a Zariski-dense set of preperiodic points, though few cases are known beyond the setting of abelian varieties. Lecture 2 was devoted to conjectures that treat algebraic families of maps on $\mathbb{P}^N$. The strongest of these conjectures was formulated in [DM2] and inspired by the recently-proved "Relative Manin-Mumford" theorem of Gao and Habegger for abelian varieties [GH]. Our dynamical version turns out to be related to the study of dynamical stability and to contain many previously-existing questions/conjectures/results about moduli spaces of maps on $\mathbb{P}^N$. Lecture 3 illustrated a special case of the conjecture from Lecture 2, related to the geometry of the Mandelbrot set, following [DM1]. These lecture notes are based on joint work with Myrto Mavraki.

1. Preperiodic points of a polarizable endomorphism

Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism, defined over the field $\mathbb{C}$ of complex numbers, of degree $d > 1$. That is, f is given in coordinates by

$$f = (f_0 : \cdots : f_N)$$

Date: February 3, 2025.

These notes were developed for the *États de la recherche SMF* program on analytic, algebraic, and arithmetic dynamics, CIRM, France, January 2025. Special thanks to the organizers Charles Favre, Thomas Gauthier, and Jasmin Raissy. Thanks to Serge Cantat for his formulation of Conjecture 2.7. A preliminary version of these lectures was delivered at Johns Hopkins University for the Algebra and Number Theory Day, November 2024; thanks to Yiannis Sekallaridis and Dori Bejleri for organizing that event. The research was supported by the National Science Foundation.

where each f_i is a homogeneous polynomial of degree d in $N+1$ variables, and the set $\{f_0, \ldots, f_N\}$ has no common zeroes in $\mathbb{P}^N(\mathbb{C})$. The following conjecture is about the algebraic geometry of periodic or preprediodic points of f.

We say that an endomorphism $f : Y \to Y$ of a complex projective algebraic variety Y is *polarizable* if there exists an ample line bundle L on Y so that $f^*L \simeq L^d$ for an integer $d > 1$. We say that d is the *degree* of f. Such an endomorphism will always extend to an endomorphism of a projective space of degree d, taking an embedding of Y into $\mathbb{P}^N$ defined by a power of L, as explained in [Fa1], so there is no loss of generality to assuming f is defined and regular on $\mathbb{P}^N$.

Conjecture 1.1. [Zh2, GTZ, GT] *Let Y be a smooth, projective complex algebraic variety, and suppose that $f : Y \to Y$ is polarizable endomorphism of degree $d > 1$, defined over $\mathbb{C}$. Let $\mathcal{P}$ be any infinite collection of preperiodic points for f, and let X be an irreducible component of the Zariski closure of $\mathcal{P}$. Then either X is preperiodic for f or X is preperiodic for an endomorphism that commutes with an iterate of f. More precisely, there exists a subvariety $X \subset X' \subset Y$ and a polarizable endomorphism $g : X' \to X'$ so that*

(1) X is preperiodic for g;
(2) $f^n(X') = X'$ for some integer $n \geq 1$; and
(3) $f^n \circ g = g \circ f^n$ on X'.

Remark 1.2. Except for special types of examples, we expect the conclusion in Conjecture 1.1 to be simply that X is preperiodic for f, and we can ignore the more complicated conclusion with g. That was introduced in [GT] to handle pesky counterexamples coming from elliptic curves and abelian varieties with large endomorphism rings; see [GTZ] and [Pa] for explicit constructions. We do not know if maps g satisfying (1), (2), and (3) will cover all possibilities, but Ghioca and Tucker show it covers all known examples.

1.1. **Abelian varieties.** Suppose that A is an abelian variety of dimension $g \geq 1$, and let L be an ample and symmetric line bundle on A; that is $[-1]^*L \simeq L$. Each nonzero integer n determines a multiplication-by-n endomorphism

$$[n] : A \to A$$

of topological degree n^{2g}. For $|n| \geq 2$, note that the torsion points of A coincide with the preperiodic points for the endomorphism $[n]$. The symmetry of L and the "theorem of the cube" guarantees that

$$[n]^*L \simeq L^{n^2},$$

so that $[n]$ is a polarizable endomorphism with polarization degree n^2 for all $|n| \geq 2$. In particular, for any $|n| \geq 2$, the endomorphism $[n]$ extends to a morphism f of a projective space, via an embedding $A \hookrightarrow \mathbb{P}^N$ [Fa1, Corollary 2.2].

Example 1.3. In §6.3 of [BoDa], Bonifant and Dabija provided explicit formulas for the "tangent process" on a smooth cubic curve in $\mathbb{P}^2$; this corresponds to multiplication by -2 on the associated elliptic curve. They work with the Hesse normal form of the elliptic curve:

$$C_\kappa = \{x_0^3 + x_1^3 + x_2^3 - 3\kappa x_0 x_1 x_2 = 0\}$$

for $\kappa^3 \neq 1$ in $\mathbb{C}$. (Details about this normal form are given in [BM].) Then

$$D(x_0 : x_1 : x_2) = (x_0(x_1^3 - x_2^3) : x_1(x_2^3 - x_0^3) : x_2(x_0^3 - x_1^3))$$

is a rational extension of $g = [-2]$ on the curve C_κ. Note that D preserves the entire pencil of curves C_κ, but it is not regular: it has indeterminacy points at $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$, as well as at the 9 points of the form $(1 : \omega : \omega')$ where ω and ω' are cubed roots of unity, though these indeterminacy points do not lie on the smooth C_κ's. For each κ, there is a 9-dimensional space of rational extensions of g from C_κ to $\mathbb{P}^2$ given by

$$\begin{aligned} D_{\kappa,A}(x_0 : x_1 : x_2) \;=\; & \Big(x_0(x_1^3 - x_2^3) + h_\kappa \sum_{j=0}^{2} a_{0j} x_j : \\ & x_1(x_2^3 - x_0^3) + h_\kappa \sum_{j=0}^{2} a_{1j} x_j : x_2(x_0^3 - x_1^3) + h_\kappa \sum_{j=0}^{2} a_{2j} x_j \Big) \end{aligned}$$

for a 3x3 matrix $A = (a_{ij})$ over $\mathbb{C}$, where h_κ is the defining polynomial of C_κ. A general choice of A defines a regular extension of g to $\mathbb{P}^2$. More about the dynamics of the *Desboves family* can be found in [BDM] and [BT].

Theorem 1.4 (Manin-Mumford Conjecture, proved by Raynaud in [Ra1, Ra2])**.** *Let A be an abelian variety over $\mathbb{C}$. Suppose that $X \subset A$ is an irreducible algebraic subvariety containing a Zariski-dense set of torsion points of A. Then X is a subgroup of A or a torsion translate of a subgroup.*

Proposition 1.5. *Theorem 1.4 is a special case of Conjecture 1.1.*

Proof. Let A be an abelian variety, and let $[2]$ be the multiplication-by-2 endomorphism on A. As explained above, $f = [2]$ is polarizable of degree $d = 4$. If $X \subset A$ is a complex algebraic subvariety of A containing a Zariski-dense set of torsion points, then it contains a Zariski-dense set of preperiodic points for the endomorphism f (because these are the same points). Conjecture 1.1 implies that X is either preperiodic for f or for an endomorphism that commutes with f.

Let us first assume that X is preperiodic for f itself. Passing to an iterated image of X, we may assume that X is periodic, so invariant under $[m]$ for some integer $m > 1$. The following argument is due to Bogomolov [Bo]. Let $k = \dim X$, so that the topological degree of $[m]|_X$ is equal to m^{2k}. As $[m]$ is a covering map on A, we know that X is stable under a finite group G of translations of size m^{2k}. Consider

$$B = \{a \in A : X + a = X\}.$$

This B is a subgroup of A, and it contains G. In particular,

$$\#\{b \in B : [m]b = 0\} \geq m^{2k}.$$

Now consider the iterates of the endomorphism $[m]$, namely the powers m^i for $i \geq 1$. The same reasoning implies that

$$\#\{b \in B : [m^i]b = 0\} \geq m^{2ik}$$

for all $i \geq 1$. We may conclude that B_0, the connected component of B containing $0 \in A$, must have dimension $\geq k$. Moreover, fixing any torision point $x_0 \in X$ and considering the map $B_0 \to X$ defined by $b \mapsto x_0 + b$, we see that X is itself a torsion translate of B_0.

Now suppose that we only know there is an invariant $X' \supset X$ for an iterate f^n so that X is preperiodic for an endomorphism $g : X' \to X'$ that commutes with f^n. The previous argument (applied to f^n and X') implies that X' is a sub-abelian variety of A (up to translation by a torsion point), so that g must also be a group endomorphism (again up to translation by a torsion point, as the commuting with f implies that some iterate of g preserves the 0). The argument above works verbatim for the isogeny g in place of $[m]$, where m^2 is replaced with the polarization degree of g. □

1.2. **Known cases.** As far as I am aware, all proofs of known cases of Conjecture 1.1 use arithmetic input (except for $N = 1$ where there is nothing to prove, and the generic result stated below as Theorem 1.7). The problem was first reduced to the setting where f and X are defined over $\overline{\mathbb{Q}}$, where additional tools are available. For example, there have been many proofs of Theorem 1.4 since the original by Raynaud (see [PZ, Ul, Zh1, Hr, PR]), but all rely somehow on number-theoretic properties of torsion points, in addition to the group structure of the abelian variety. Unfortunately, none of these proofs extend to the more general dynamical setting we consider in Conjecture 1.1.

Besides Theorem 1.4, little is known about Conjecture 1.1 for endomorphisms of $\mathbb{P}^N$ with $N > 1$. There is one class of maps where the conjecture has been proved in full, taking advantage of rigidity results from 1-dimensional complex dynamics:

Theorem 1.6. [GNY2, GNY1, MSW] *Suppose that $f : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N$ is a product map $f = (f_1, \ldots, f_N)$ defined over $\mathbb{C}$, with $N > 1$ and $\deg f_1 = \cdots = \deg f_N > 1$ for all i. Conjecture 1.1 holds for f.*

The degree condition on the factors f_i guarantees that f is polarizable. If we assume that no f_i is a *Lattès map* (meaning a quotient of a map on an elliptic curve), then the conclusion is stronger: a complex algebraic subvariety $X \subset (\mathbb{P}^1)^N$ contains a Zariski-dense set of preperiodic points for f if and only if it is preperiodic. See, for example, [Mi] for more about exceptional maps on $\mathbb{P}^1$ in general and Lattès maps in particular. The proof of Theorem 1.6 for $N = 2$, assuming the f_i are not Lattès, is contained in [GNY2]; for the non-Lattès exceptional maps (namely, power maps or $\pm$ Chebyshev polynomials), the authors rely on Laurent's proof of the analogue of Theorem 1.4 for the multiplicative group [La]. The final step of the proof to treat Lattès maps is spelled out in [GT, Section 2], using Theorem 1.4. The proof of Theorem 1.6 for higher N follows from [GNY1], when combined with the main results of [MSW]. When the map f of Theorem 1.6 is defined over $\overline{\mathbb{Q}}$, the proofs use the theory of canonical heights and Galois-equidistribution theorems introduced below, in §1.3.

Genuinely higher-dimensional dynamical cases of Conjecture 1.1 are hard to come by. Fakhruddin proved that Conjecture 1.1 holds *generically* [Fa2]. Let End_d^N denote the affine algebraic variety consisting of all endomorphisms of $\mathbb{P}^N$ of degree d, defined over $\mathbb{C}$. The *generic complex endomorphism* on $\mathbb{P}^N$ is a map defined over the function field $L = \mathbb{C}(\mathrm{End}_d^N)$ that corresponds to the generic point of End_d^N; see [Fa2, §3].

Theorem 1.7. [Fa2] *For the generic complex endomorphim f on $\mathbb{P}^N$, the $\mathrm{Gal}(\overline{L}/L)$-orbits of any infinite sequence of preperiodic points for f form a Zariski-dense subset of $\mathbb{P}^N(\overline{L})$.*

It follows from Theorem 1.7 that Conjecture 1.1 holds for a dense (analytically dense, that is) G_δ subset of End_d^N over $\mathbb{C}$, because no proper algebraic subvariety will contain infinitely many distinct preperiodic points for those maps. Interestingly, Fakhruddin's proof uses only properties of 1-dimensional systems, specifically the monodromy of preperiodic points for certain families of polynomials; he then considers appropriate products acting on $\mathbb{P}^N$.

There is one known case over $\mathbb{C}$ in dimension $N = 2$, due to Dujardin, Favre, and Ruggiero:

Theorem 1.8. [DFR] *Suppose that $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a regular polynomial endomorphism (meaning that it restricts to a polynomial endomorphism on an affine chart $\mathbb{A}^2 \subset \mathbb{P}^2$) defined over $\mathbb{C}$. Suppose that X is an algebraic curve in $\mathbb{P}^2$, defined over $\mathbb{C}$, so that at least one point of $X \cap (\mathbb{P}^2 \setminus \mathbb{A}^2)$ is* not *preperiodic to a superattracting periodic point. If X contains an infinite set of preperiodic points, then X is preperiodic.*

1.3. **Canonical heights and equidistribution.** Here we touch briefly on the canonical height theory for maps on $\mathbb{P}^N$. This provides a glimpse of the arithmetic that enters the existing dynamical proofs of cases of Conjecture 1.1, for maps defined over $\overline{\mathbb{Q}}$.

Fix a Weil height $h : \mathbb{P}^N(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$, and suppose that $f : \mathbb{P}^N \to \mathbb{P}^N$ is an endomorphism of degree $d > 1$, defined over $\overline{\mathbb{Q}}$. The *canonical height* for f was introduced by Call and

Silverman in [CS] and is defined by

$$\hat{h}_f(x) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(x))$$

for $x \in \mathbb{P}^N(\overline{\mathbb{Q}})$. The limit always exists, and it satisfies $\hat{h}_f(x) = 0$ if and only if x is preperiodic for f. One can also define heights of subvarieties defined over $\overline{\mathbb{Q}}$, and the "Zhang inequalities" (see [Zh3, Theorem 1.10]) allow us to reformulate Conjecture 1.1 as: *if $f : \mathbb{P}^N \to \mathbb{P}^N$ and subvariety $X \subset \mathbb{P}^N$ are defined over $\overline{\mathbb{Q}}$, and if $\hat{h}_f(X) = 0$, then X is preperiodic for f (or preperiodic for an endomorphism that commutes with f).* See [CL2] for background on computing heights of subvarieties.

In [Yu1], Yuan generalized existing equidistribution theorems, for example in [SUZ, Bi, Ru, Au, CL1, BR, FRL], and proved the following:

Theorem 1.9. [Yu1] *Suppose that $f : Y \to Y$ is a polarizable endomorphism defined over a number field K, and let $\hat{h}_f$ be the canonical height (defined by the polarization) on $Y(\overline{K})$. If $\{x_n\}$ is a generic sequence of points in Y for which $\hat{h}_f(x_n) \to 0$, then*

$$\frac{1}{|\operatorname{Gal}(\overline{K}/K) \cdot x_n|} \sum_{y \in \operatorname{Gal}(\overline{K}/K)\cdot x_n} \delta_y \longrightarrow \mu_f$$

weakly on $Y(\mathbb{C})$, where μ_f is the equilibrium measure for f.

A sequence $\{x_n\}$ in Y is *generic* if no infinite subsequence is contained in a proper subvariety of Y. The equilibrium measure μ_f will be defined in the next section for the case $Y = \mathbb{P}^N$. In fact, the theorem in [Yu1] demonstrates equidistribution for a more general class of height functions on Y, and also for sequences of small points at non-archimedean places of K on an appropriately-defined Berkovich analytification of Y, as was carried out in dimension one in [CL1, BR, FRL].

2. Preperiodic points in families

In this section, we present a conjecture from [DM2] and a partial selection of sub-conjectures, known cases, and consequences. The formulation of Conjecture 2.1 was inspired by the "Relative Manin-Mumford" theorem of Gao and Habegger [GH], which is stated below as Theorem 2.6.

2.1. **Invariant current.** Suppose that $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d > 1$ is defined over $\mathbb{C}$. Let ω_0 denote the Fubini-Study form normalized so that $\int \omega_0^N = 1$. Pulling back ω_0 by iterates of f yields a weak limit

$$T_f := \lim_{n\to\infty} \frac{1}{d^n} (f^n)^* \omega_0,$$

characterized as the unique positive (1,1)-current on $\mathbb{P}^N(\mathbb{C})$ in the cohomology class $[\omega_0]$ having continuous potential and satisfying $\frac{1}{d} f^* T_f = T_f$. See [Sib]. The top wedge-power $\mu_f = T_f^{\wedge N}$ is the unique measure of maximal entropy for f, and it is the limiting distribution of the repelling periodic points for f [BD1, BD2]. When f is defined over a number field K, the local canonical height at an archimedean place of K is a potential for T_f. Moreover, for any generic sequence of preperiodic points (whether periodic or not, whether repelling or not), their Galois orbits (over K) will be uniformly distributed with respect to μ_f, from Theorem 1.9.

2.2. **Holomorphic and algebraic families.** Suppose that Λ is a complex manifold; consider holomorphic maps

$$\Phi : \Lambda \times \mathbb{P}^N \to \Lambda \times \mathbb{P}^N$$

defined by $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$ for a family of endomorphisms $f_\lambda : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d > 1$. Now let

$$p : \Lambda \times \mathbb{P}^N \to \mathbb{P}^N$$

denote the projection. We set

$$\hat{T}_\Phi = \lim_{n\to\infty} \frac{1}{d^n} (\Phi^n)^* (p^* \omega_0)$$

in the sense of distributions. This current restricts to T_{f_λ} on each slice $\{\lambda\} \times \mathbb{P}^N$ for $\lambda \in \Lambda$. (Formally, one restricts the locally-defined continuous potential for $\hat{T}_\Phi$ to the slice.)

We say that Φ is an *algebraic family* of maps on $\mathbb{P}^N$ if Λ is a smooth, quasiprojective complex algebraic variety and Φ is a morphism.

Conjecture 2.1. [DM2] *Let $\Phi : \Lambda \times \mathbb{P}^N \to S \times \mathbb{P}^N$ be an algebraic family of morphisms of degree > 1, and let $\mathcal{X} \subset \Lambda \times \mathbf{P}^N$ be a complex, irreducible subvariety which is flat over S. The following are equivalent.*

(1) $\mathcal{X}$ contains a Zariski-dense set of Φ-preperiodic points.

(2) $\hat{T}_\Phi^{\wedge r(\Phi,\mathcal{X})} \wedge [\mathcal{X}] \neq 0$ for the relative special dimension $r(\Phi, \mathcal{X})$.

Some explanation is needed. If "flat" isn't part of your vocabulary, it is probably enough to assume that the dimension of $X_\lambda := \mathcal{X} \cap (\{\lambda\} \times \mathbb{P}^N)$ is constant in $\lambda \in \Lambda$.

An important role is played by what we call the *relative special dimension* $r(\Phi, \mathcal{X})$. First, a formal definition: The family Φ induces a morphism $\mathbf{\Phi} : \mathbf{P^N} \to \mathbf{P^N}$ defined over the function field $K = \mathbb{C}(\Lambda)$. A complex irreducible subvariety $\mathcal{Y} \subset \Lambda \times \mathbb{P}^N$, which is flat over a Zariski-open subset of Λ and Zariski-closed in $\Lambda \times \mathbb{P}^N$, is Φ-*special* if there exist a subvariety $\mathbf{Z} \subset \mathbf{P^N}$ over the algebraic closure $\overline{K}$ containing the generic fiber $\mathbf{Y}$, a polarizable endomorphism $\mathbf{\Psi} : \mathbf{Z} \to \mathbf{Z}$, and an integer $n \in \mathbb{N}$ so that the following hold:

- $\mathbf{\Phi}^n(\mathbf{Z}) = \mathbf{Z}$;

- $\mathbf{\Phi}^n \circ \mathbf{\Psi} = \mathbf{\Psi} \circ \mathbf{\Phi}^n$ on $\mathbf{Z}$; and
- $\mathbf{Y}$ is preperiodic for $\mathbf{\Psi}$.

Note that these are the same conditions as in the conclusion of Conjecture 1.1, just working over a function field, as $\mathcal{Y}$ might be preperiodic for Φ or it is preperiodic for something that commutes with (an iterate of) Φ, allowing for a possible base change. The *relative special dimension* of $\mathcal{X}$ is defined by

$$r(\Phi, \mathcal{X}) := \min\{\dim_\Lambda \mathcal{Y} \,:\, \mathcal{Y} \text{ is } \Phi\text{-special in } \Lambda \times \mathbb{P}^N \text{ and } \mathcal{X} \subset \mathcal{Y}\},$$

where

$$\dim_\Lambda \mathcal{Y} := \dim \mathcal{Y} - \dim \Lambda$$

is the dimension of a general fiber of the projection to Λ. Note that

$$0 \leq \dim_\Lambda \mathcal{X} \leq r(\Phi, \mathcal{X}) \leq N.$$

The relative special dimension is typically equal to N, and in that case the current $\hat{T}_\Phi^{\wedge r(\Phi,\mathcal{X})}$ restricts to the measure of maximal entropy on each fiber over Λ; therefore the current $\hat{T}_\Phi^{\wedge r(\Phi,\mathcal{X})} \wedge [\mathcal{X}]$ captures the intersection of $\mathcal{X}$ with the Julia sets as we move across Λ. (Here I am defining "Julia set" as the support of the measure of maximal entropy. Remember that repelling periodic points are dense in those Julia sets!) On the other hand, if for example $\mathcal{X}$ lies inside a family $\mathcal{Y}$ of Φ-invariant subvarieties, then we restrict our attention to $\Phi : \mathcal{Y} \to \mathcal{Y}$, and $\hat{T}_\Phi^{\wedge r(\Phi,\mathcal{X})}$ becomes the measure of maximal entropy on fibers of $\mathcal{Y}$ over Λ.

In particular, complex-dynamics arguments tell us that (2) implies (1) in Conjecture 2.1: we can find a dense set (in the usual analytic topology) of repelling points for the maps Φ_λ in the support of $\hat{T}_\Phi^{\wedge r(\Phi,\mathcal{X})} \wedge [\mathcal{X}]$. When $\mathcal{X}$ is not Φ-special, the argument builds on the methods of Dujardin [Du], Berteloot-Bianchi-Dupont [BBD], and Gauthier [Ga] to study supports of bifurcation currents and measures. For endomorphisms of abelian varieties, the implication (2) $\implies$ (1) was straightforward and observed in [ACZ]. See details in [DM2].

The challenge remains to prove that (1) implies (2), if it is true.

2.3. **A weaker form of the conjecture, but still powerful.** Many interesting statements – a few of which are described below – follow from a weaker form of Conjecture 2.1. Note that the positivity of the current in (2) implies that the dimension of $\mathcal{X}$ is at least as large as the power $r(\Phi, \mathcal{X})$.

Conjecture 2.2. [DM2] *Let $\Phi : \Lambda \times \mathbb{P}^N \to S \times \mathbb{P}^N$ be an algebraic family of morphisms of degree > 1, and let $\mathcal{X} \subset \Lambda \times \mathbf{P}^N$ be a complex, irreducible subvariety which is flat over S. Assume that $\mathcal{X}$ contains a Zariski-dense set of Φ-preperiodic points. Then the relative special dimension satisfies*

$$r(\Phi, \mathcal{X}) \leq \dim \mathcal{X}.$$

Simple examples show that the conclusion of Conjecture 2.2 is not equivalent to (1) or (2) in Conjecture 2.1: take an isotrivial family of maps on $\mathbb{P}^1$ over $\Lambda = \mathbb{C}$, for example $\Phi(\lambda, z) = (\lambda, z^2 - 1)$ in affine coordinates on $\mathbb{P}^1$, and let $\mathcal{X} = \Lambda \times \{5\}$. Then $r(\Phi, \mathcal{X}) = 1 = \dim \mathcal{X}$ but there are no preperiodic points in $\mathcal{X}$ and $\hat{T}_\Phi \wedge [\mathcal{X}] = 0$. More interesting examples appear in [GH, Gao] that show that the conclusion in Conjecture 2.2 is not enough to deduce conditions (1) or (2) in Conjecture 2.1 even in the absence of isotriviality.

2.4. **The case of $N = 1$.** Conjecture 2.1 is known to hold in dimension $N = 1$; in fact, it is logically equivalent to a theorem I proved in [De3], as we explain in [DM2, §3.2].

An irreducible flat subvariety $\mathcal{X}$ of $\Lambda \times \mathbf{P}^1$ is either all of $\Lambda \times \mathbf{P}^1$ or it projects finitely to Λ. In the former case, of course Preper(Φ) is Zariski dense in $\mathcal{X}$. We have $r(\Phi, \mathcal{X}) = 1$ and $\hat{T}_\Phi^{\wedge 1} \wedge [\mathcal{X}] = \hat{T}_\Phi \neq 0$.

In the latter case, after replacing the parameter space Λ with a branched cover, we may assume that $\mathcal{X}$ is the graph of a marked point $a : \Lambda \to \mathbf{P}^1$. If $r(\Phi, \mathcal{X}) = 0$, then $\mathcal{X}$ is itself Φ-special, meaning that the marked point a is persistently preperiodic for Φ. In this case, the preperiodic points are obviously dense in $\mathcal{X}$ and the current $\hat{T}_\Phi^0 \wedge [\mathcal{X}] = [\mathcal{X}]$ is clearly nonzero, so the equivalence of (1) and (2) in Conjecture 2.1 holds. If $r(\Phi, \mathcal{X}) = 1$, then the current of (2) is nonzero if and only if the point is unstable, in the sense of [De3]. In other words, the sequence of holomorphic maps $\{\lambda \mapsto \Phi_\lambda^n(a(\lambda))\}$ fails to formal a normal family on the parameter space Λ; see, for example, [De2, Theorem 9.1] for the equivalence of normality and the vanishing of $\hat{T}_\Phi \wedge [\mathcal{X}]$. As proved in [De3, Theorem 1.1], if the point a is not persistently preperiodic, then stability on all of Λ implies that the family Φ is isotrivial and the point a will never be preperiodic. On the other hand, instability implies, via Montel's theory of normal families, that the point a will be preperiodic for a Zariski-dense set of parameters; see, for example, [De3, Proposition 5.1]. So the equivalence in Conjecture 2.1 holds also for $r(\Phi, \mathcal{X}) = 1$.

The case of marked *critical* points in $\mathbb{P}^1$ was treated in [DF, Theorem 2.5] and [Mc, Lemma 2.1]; note the connection to J-stability in families!

2.5. **Dynamical stability.** Let $\Phi : \Lambda \times \mathbb{P}^N \to S \times \mathbb{P}^N$ be an algebraic family of morphisms of degree > 1, and suppose that $\mathcal{X} = \text{Crit}(\Phi)$ is the critical locus of Φ. Note that Crit(Φ) is a family of hypersurfaces of degree $(N + 1)(d - 1)$ in $\mathbb{P}^N$. The current

$$\hat{T}_\Phi^{\wedge N} \wedge [\text{Crit}(\Phi)]$$

projects to the *bifurcation current* on the parameter space Λ, as defined by Bassanelli and Berteloot in [BB], and we say the family is *stable* when this current vanishes. (For maps on $\mathbb{P}^1$, the current was introduced in [De1].) Note that $r(\Phi, \mathcal{X}) = N - 1$ if and only if $\mathcal{X} = \text{Crit}(\Phi)$ is Φ-special; recall this means that $\mathcal{X}$ is preperiodic for Φ (or for an endomorphism that commutes with Φ). Assuming $r(\Phi, \mathcal{X}) = N$, Conjecture 2.1 tells us that the family is unstable if and only if there is a Zariski-dense set of preperiodic points in $\mathcal{X}$. As we have already observed, it is known that condition (2) implies condition (1) in Conjecture 2.1; this implication for $\mathcal{X} = \mathrm{Crit}(\Phi)$ is contained in the work of Berteloot-Bianchi-Dupont in [BBD], where they prove the existence of Misiurewicz parameters in the bifurcation locus. It remains open, however, to show that the presence of many preperiodic points in the critical locus (whether repelling or not) forces instability, assuming that $\mathrm{Crit}(\Phi)$ is not Φ-special. (It is known for $N = 1$, as we observed in §2.4.) The article [BBD] contains several important characterizations of stability; the new work of Berteloot and Buff extends these characterizations to include the holomorphic motion of repelling periodic points in the Julia set [BB]. Conjecture 2.1 would provide a further characterization of stability, but only for algebraic families.

Moving beyond the critical locus, Gauthier and Vigny introduce and study the notion of stability for a more general flat family $\mathcal{X}$ of subvarieties of $\Lambda \times \mathbb{P}^N$ [GV]: the subvariety $\mathcal{X}$ is said to be *unstable for* Φ if

$$\hat{T}_\Phi^{\wedge(1+\dim_\Lambda \mathcal{X})} \wedge [\mathcal{X}] \neq 0, \tag{2.1}$$

where $\dim_\Lambda \mathcal{X}$ is the relative dimension of $\mathcal{X}$ over Λ. (See [DF, De3] for the case of $N = 1$.) Compare condition (2.1) to (2) of Conjecture 1.1 and condition (2.2) for non-degeneracy below.

2.6. **Equidistribution?** The book of Yuan and Zhang [YZ] explores intersection theory and heights in a very general context, allowing for a well-defined theory on quasiprojective (non-compact) varieties, and they prove an equidistribution result for the Galois orbits of points of small height, extending Theorem 1.9 of Yuan. One can ask if the preperiodic points in $\mathcal{X}$ will necessarily be equidistributed with respect to some natural measure in the context of Conjecture 2.1, assuming conditions (1) and (2) hold. The answer is: sometimes. The "non-degeneracy" hypothesis for the equidistribution result [YZ, Theorem 6.2.3] is equivalent to assuming that

$$\hat{T}_\Phi^{\wedge \dim \mathcal{X}} \wedge [\mathcal{X}] \neq 0, \tag{2.2}$$

in which case this power of the current defines a non-trivial measure on $\mathcal{X}(\mathbb{C})$. Condition (2.2) is neither stronger nor weaker than condition (2) of Conjecture 2.1, as the following three examples show:

Example 2.3 (Dense but degenerate, no equidistribution)**.** Take $\Lambda = \mathbb{C}$ and $\Phi_\lambda(z) = \lambda z + z^2$, with $\mathcal{X} = \Lambda \times \{0\} \subset \Lambda \times \mathbb{P}^1$. This $\mathcal{X}$ is fixed by Φ, so $r(\Phi, \mathcal{X}) = 0$ and $\hat{T}_\Phi^{\wedge 0} \wedge [\mathcal{X}] = [\mathcal{X}] \neq 0$, but $\hat{T}_\Phi^{\wedge 1} \wedge [\mathcal{X}] = 0$. Every point is fixed, so clearly we have a (Zariski-) dense set of preperiodic points in $\mathcal{X}$. However, taking any infinite sequence of algebraic parameters and their Galois orbits can lead to many different measure limits; one cannot expect a good equidistribution statement. Note that the current $\hat{T}_\Phi^{\wedge 1} \wedge [\mathcal{X}]$ detects the stability of $\mathcal{X}$ in the sense of (2.1).

Example 2.4 (Dense and non-degenerate, equidistribution)**.** Now instead take $\Lambda = \mathbb{C}$ and $\Phi_\lambda(z) = z^2 + \lambda$, with $\mathcal{X} = \Lambda \times \{0\} \subset \Lambda \times \mathbb{P}^1$. This $\mathcal{X}$ is not Φ-special, so $r(\Phi, \mathcal{X}) = 1$ and $\hat{T}_\Phi^{\wedge 1} \wedge [\mathcal{X}] \neq 0$ because it projects to the bifurcation measure (which is equal to the harmonic measure) on the boundary of the Mandelbrot set in Λ [De1, Example 6.1]. Since $\dim \mathcal{X} = 1$, it is non-degenerate in the sense of (2.2) and so we have Galois equidistribution of the preperiodic points in $\mathcal{X}$ from [YZ, Theorem 6.2.3]. Note that the preperiodic points in $\mathcal{X}$ correspond to the postcritically-finite parameters in Λ. (More will be said about this example in Lecture 3.) Note that this equidistribution result also follows from some of the Galois-equidistribution theorems on $\mathbb{P}^1$ [Au, CL1, BR, FRL] that I mentioned in §1.3; this case can also be deduced from the equidistribution theorem of Rumely [Ru] and is also stated explicitly in [BH, Theorem 8.15].

Example 2.5 (Non-degenerate but not dense, nothing to distribute)**.** As a third example, consider $\Lambda = \mathbb{C}$ and let

$$\Phi_\lambda(z, w) = (z^2 + \lambda, w^2 + \lambda + 10)$$

with $\mathcal{X} = \Lambda \times \{(0,0)\} \subset \Lambda \times \mathbb{P}^2$. See [DM2, Example 5.1]. For this example we have $r(\Phi, \mathcal{X}) = 2$, but $\dim \mathcal{X} = 1$. One can check that there are no preperiodic points for Φ in $\mathcal{X}$, and

$$\hat{T}_\Phi^{\wedge 2} \wedge [\mathcal{X}] = 0$$

for dimension reasons. But one can compute easily that

$$\hat{T}_\Phi^{\wedge 1} \wedge [\mathcal{X}] \neq 0$$

and so $\mathcal{X}$ is non-degenerate in the sense of Yuan-Zhang [YZ]. So equidistribution would hold, but the preperiodic points do not exist (and in fact, the height is bounded away from 0). In this example, $\mathcal{X}$ is also unstable in the sense of (2.1).

2.7. **Conjecture 1.1 is a special case.** When Λ is a single point, so that $\Phi : \mathbb{P}^N \to \mathbb{P}^N$ is one endomorphism over $\mathbb{C}$, Conjectures 2.1 and 2.2 reduce to Conjecture 1.1. Indeed, the current $T_\Phi^{\wedge r} \wedge [X]$ will obviously vanish if $r > \dim X$, but is nonzero for $r = \dim X$ for cohomological reasons. The implication that a Φ-special subvariety of $\mathbb{P}^N$ always contains a dense set of preperiodic points is well known; see [Fa1, BD1, DS].

2.8. **Families of abelian varieties.** The inspiration for Conjecture 2.1 came from the work of Gao and Habegger, especially their [GH, Theorem 1.3]. Suppose that $\mathcal{A} \to \Lambda$ is a family of abelian varieties of relative dimension g over quasiprojective Λ, defined over $\mathbb{C}$. Their "Relative Manin-Mumford" theorem is:

Theorem 2.6. [GH] *Suppose that the $\mathbb{Z}$-orbit of $\mathcal{X}$ is Zariski-dense in $\mathcal{A}$. Then $\mathcal{X}$ contains a Zariski-dense set of torsion points in $\mathcal{A}$ if and only if $\mathcal{X}$ has maximal Betti rank.*

Suppose that $\Phi = [2]$ is the multiplication-by-2 endomorphism on $\mathcal{A}$. The condition on the $\mathbb{Z}$-orbit in Theorem 2.6 guarantees that the relative special dimension $r(\Phi, \mathcal{X})$ is equal to g; the conclusion of maximal Betti rank means that $\hat{T}_\Phi^{\wedge g} \wedge [\mathcal{X}]$ is nonzero. A comparison of terminology is provided in [DM2, §2].

Theorem 2.6 implies – among other things – that uniform versions of the Manin-Mumford Conjecture hold. For example, Gao and Habbeger gave a new proof in [GH] of the uniform bound on torsion points of a curve of genus g in its Jacobian; see [DGH, Kü, Yu2].

2.9. **Invariant subvarieties.** Though a statement about preperiodic points, Conjecture 2.1 (and its weaker form, Conjecture 2.2) implies statements about preperiodic subvarieties of positive dimension and their prevalence in families. As an example, consider the following conjecture. Let M_d^N denote the moduli space of endomorphisms on $\mathbb{P}^N$ of degree d defined over $\mathbb{C}$; see [Si] for background on the construction of M_d^N.

Conjecture 2.7. [DM2] *Fix integers $N > 1$, $d > 1$. For each integer $e \geq 1$, there exists a proper Zariski-closed subvariety V_e of M_d^N so that any $f : \mathbb{P}^N \to \mathbb{P}^N$ having a positive-dimensional preperiodic subvariety of degree e has conjugacy class $[f]$ lying in V_e.*

Question 2.8. *Is the union $\bigcup_{e \geq 1} V_e$ contained in a proper subvariety of M_d^N? That is, should we expect that a* general *choice of $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d \geq 2$ has no proper invariant subvarieties of positive dimension?*

An important special case of Conjecture 2.7 was proved by Gauthier, Taflin, and Vigny: for any dimension $N \geq 2$ and degree $d \geq 2$, the set of maps for which the *critical locus* is preperiodic is contained in a proper algebraic subvariety of M_d^N [GTV].

It is shown in [DM2, §4] that Conjecture 2.2 implies Conjecture 2.7. We consider families $\mathcal{X} \subset \Lambda \times \mathbb{P}^N$ and maps Φ for which the natural map from Λ to M_d^N is dominant. The argument uses Fakhruddin's result that the monodromy action on periodic points of a given period is transitive over Λ [Fa2] (similar to what goes into his proof of Theorem 1.7). We look at fiber powers of $\mathcal{X}$ over Λ for the action of fiber product $\Phi^{\times_\Lambda m}$ on $\Lambda \times (\mathbb{P}^N)^m$, for each positive integer m. The Fakhruddin result is used to show that the relative special dimensions $r(\Phi^{\times_\Lambda m}, \mathcal{X}^{\times_\Lambda m})$ are maximal (equal to mN), which contradicts the dimension estimate of Conjecture 2.2 when m is large.

More refined statements than Conjecture 2.7 can be made, for example involving the growth of the relative special dimension $r(\Phi^{\times_\Lambda m}, \mathcal{X}^{\times_\Lambda m})$ with m when Λ does not project dominantly to M_d^N.

2.10. **DAO.** Many of you have heard me speak (many times over the years) on a conjecture about the geometry of postcritically-finite maps in the moduli space M_d^1 of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of a given degree $d \geq 2$. Ji and Xie called this conjecture "DAO" (a pun-intended acronym for the Dynamical André-Oort conjecture) in their paper [JX]. Roughly, the statement is that a subvariety of M_d^1 should contain a Zariski-dense set of postcritically-finite maps if and only if the subvariety is defined by critical orbit relations; see [BD]. After a series of works by a number of different authors, Favre and Gauthier completed the case of algebraic curves in spaces of polynomial maps in [FG]. Ji and Xie completed the more general setting of algebraic curves in M_d^1 [JX]. Higher-dimensional subvarieties are still unresolved, as is the more general question treating arbitrary marked points, even for curves in M_d^1.

The following is a mild generalization of DAO (treating arbitrary marked points, not just critical points), and it can be viewed as a special case of Conjecture 2.2, the weaker form of Conjecture 2.1.

Conjecture 2.9. [BD, GHT, De3] *Suppose that $\Phi : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ is an algebraic family of maps on $\mathbf{P}^1$ of degree $d \geq 2$. Assume that $m = \dim \Lambda > 0$ and that the image of the induced map $\Lambda \to \mathrm{M}_d^1$ also has dimension m. Let $a_0, \ldots, a_m : \Lambda \to \mathbf{P}^1$ be $m+1$ marked points, and assume that there is a Zariski-dense set of parameters $\lambda \in \Lambda$ such that $a_0(\lambda), \ldots, a_m(\lambda)$ are simultaneously Φ_λ-preperiodic. Then the marked points $a_0, \ldots, a_m$ are dynamically related along Λ.*

By definition, marked points $(a_0, \ldots, a_m)$ are *dynamically related along* Λ if there exists a proper subvariety $\mathcal{Y} \subset \Lambda \times (\mathbf{P}^1)^{m+1}$ projecting dominantly to Λ which is preperiodic for the fiber-product $\Phi^{\times_\Lambda (m+1)}$ and which contains the graph of $(a_0, \ldots, a_m)$ over Λ.

To see Conjecture 2.9 as a consequence of Conjecture 2.2, note that $\Phi^{\times_\Lambda (m+1)}$ defines a family of polarized endomorphisms of $(\mathbf{P}^1)^{m+1}$ and so extends to a family of endomorphisms of projective space $\mathbf{P}^M$ for some dimension $M \geq m+1$. In particular, the collection $(a_0, \ldots, a_m)$ is dynamically related if and only if the graph Γ of this $(m+1)$-tuple lies in a $\Phi^{\times_\Lambda (m+1)}$-special subvariety of relative dimension $\leq m$ over Λ. (For powers of nonisotrivial maps on $\mathbb{P}^1$, the special subvarieties in $\Lambda \times (\mathbf{P}^1)^{m+1}$ will be preperiodic.) The graph Γ contains a Zariski dense set of preperiodic points if and only if there is a Zariski dense set of $\lambda \in \Lambda$ at which $a_0(\lambda), \ldots, a_N(\lambda)$ are simultaneously preperiodic. So, assuming that Conjecture 2.2 holds for $\mathcal{X} = \Gamma$, we have $r(\Phi^{\times_\Lambda (m+1)}, \Gamma) \leq m = \dim \Gamma$. In other words, the graph Γ must lie in a $\Phi^{\times_\Lambda (m+1)}$-special subvariety of relative dimension $\leq m$. In other words, the marked points are dynamically related. See [DM2, §3.3].

3. Geometry of the Mandelbrot set

In this third lecture, we focus on a fun fact that was predicted by Conjecture 2.1, in a setting where we already know a lot but can still discover surprising statements. The material here follows [DM1]. Our hope is that Conjecture 2.1, whether true or false, can lead to interesting questions and deeper understanding in new settings.

3.1. **Lines through the Mandelbrot set.** Let $f_c(z) = z^2 + c$, for $c \in \mathbb{C}$. The *Mandelbrot set* is the set $\mathcal{M}$ of all parameters c for which the orbit of 0 is bounded. A polynomial f_c is PCF (short for postcritically finite) if the orbit of 0 is finite. The PCF parameters are the centers of the hyperbolic components in $\mathcal{M}$ (when 0 is periodic) and the Misiurewicz parameters (when 0 is strictly preperiodic). Recall that the latter are dense in the boundary $\partial\mathcal{M}$.

Theorem 3.1. [DM1] *There are only finitely many real lines in $\mathbb{C}$ that contain more than two PCF parameters.*

It follows from the main result of Ghioca, Krieger, Nguyen, and Ye in [GKNY] that the real axis is the only (irreducible) real algebraic curve in $\mathbb{C}$ containing infinitely many PCF parameters. So Theorem 3.1 can be reformulated as: *other than the PCF parameters on the real line, there are only finitely many collinear triples of PCF points.*

Here is a bold conjecture:

Conjecture 3.2. *The only real lines in $\mathbb{C}$ containing more than two PCF parameters are the real and imaginary axes. More precisely, every collinear triple of PCF points is either real or equal to $\{i, 0, -i\}$.*

The recent numerical results of Mihalache and Vigneron [MV] can be used to confirm that there are no unexpected collinear triples of PCF points with critical orbit length < 10, and we hope to check the conjecture to orbit length 30 or more within the next few weeks [DH].

3.2. **From real to complex.** Suppose $P(x, y) \in \mathbb{R}[x, y]$ is a polynomial with degree $d \geq 1$. Writing $x = \frac{1}{2}(c + \bar{c})$ and $y = \frac{1}{2i}(c - \bar{c})$, we obtain a polynomial in c and $\bar{c}$ of degree d with complex coefficients. In this way, any real algebraic curve in $\mathbb{R}^2$ passing through a collection $\{c_1, \ldots, c_m\}$ of PCF parameters in $\mathbb{C}$ gives rise to a complex algebraic curve in $\mathbb{C}^2$ passing through special points $\{(c_1, \bar{c}_1), \ldots, (c_m, \bar{c}_m)\}$. (Recall that the set of PCF parameters is symmetric under complex conjugation.) In particular, if we begin with the line in $\mathbb{R}^2$ defined by

$$\big\{(x, y) \in \mathbb{R}^2 : a\,x + b\,y = r\big\}$$

with $a, b, r \in \mathbb{R}$, then this line contains $c \in \mathbb{C}$ if and only if the complex line

$$\big\{(x, y) \in \mathbb{C}^2 : (a - ib)\,x + (a + ib)\,y = 2\,r\big\}$$

contains the point $(c, \bar{c})$ in $\mathbb{C}^2$. For example, taking $b = 1$ and $a = r = 0$ shows that the real axis in $\mathbb{C}$ corresponds to the diagonal line $x = y$ in $\mathbb{C}^2$. Note that the vertical and horizontal lines in $\mathbb{C}^2$ cannot arise by this construction.

3.3. **Products of quadratic polynomials.** Fix a positive integer m, and let $\mathbb{C}^m$ parameterize the space of m-tuples of quadratic polynomials $(f_{c_1}, \dots, f_{c_m})$. If each f_{c_i} is PCF, we say that $c \in \mathbb{C}^m$ is a PCF m-tuple. In [DM1], we extended the main result of [GKNY] to prove:

Theorem 3.3. [DM1] *Let $m \geq 2$. Suppose that Λ is a Zariski-closed irreducible complex algebraic subvariety in $\mathbb{C}^m$. There is a Zariski dense set of PCF m-tuples in Λ if and only Λ is an intersection of hyperplanes in $\mathbb{C}^m$ of the form*

$$\{c_i = x\} \text{ or } \{c_j = c_k\}$$

in coordinates $(c_1, \dots, c_m) \in \mathbb{C}^m$, for PCF parameters $x \in \mathbb{C}$.

The subvarieties of the form given in Theorem 3.3 are called *PCF-special*.

The proof of Theorem 3.3 is similar to the proof of [GKNY] for the case of $m = 2$. But there were a few significant advances since that paper was published that allowed us to complete the argument in arbitrary dimension (and shorten the proof and strengthen the statement for $m = 2$ [DM1, Theorem 2.1, Remark 2.3]). I would like to sketch the argument here, so you see how the arithmetic tools from §1.3 are typically used for these problems, and how they can be combined with one-dimensional complex-dynamics arguments.

Sketch proof of Theorem 3.3. Suppose that $\Lambda \subset \mathbb{C}^m$ has dimension k, and consider the projections from Λ to copies of $\mathbb{C}^k$ coming from choices of k of the m coordinates. Unless $m - k$ of the coordinates are constant (in which case we already know that Λ is PCF-special), our Λ projects dominantly to at least two unequal copies of $\mathbb{C}^k$. Consider two functions defined on $\Lambda(\overline{\mathbb{Q}})$ given by the sum of the critical heights $\hat{h}_{f_{c_i}}(0)$ for two unequal choices of k coordinates. It turns out that these define "non-degenerate" height functions on Λ in the sense of Yuan-Zhang [YZ]. If we assume the existence of a generic "critically small" sequence (or, in particular, a Zariski-dense set of PCF m-tuples) in Λ, then the equidistribution theorem of [YZ] implies that their Galois orbits are uniformly distributed with respect to two measures of the form

$$\mu_{i_1, \dots, i_k} = \pi_{i_1}^* \mu_{\mathcal{M}} \wedge \cdots \wedge \pi_{i_k}^* \mu_{\mathcal{M}}$$

from the two sets of k-tuples of coordinates, which must then be equal. Here π_i denotes the projection from Λ to the i-th coordinate, and $\mu_{\mathcal{M}}$ is the bifurcation measure on the boundary of the Mandelbrot set. In other words, Λ induces an algebraic correspondence over k-fold powers of the Mandelbrot set that preserves these measures. Luo's theorem on the inhomogeneity of Mandelbrot set [Lu] then implies that Λ must be the identity correspondence. (See also the inhomogeneity statements about $\mathcal{M}$ in [GKN, GKNY].) ☐

3.4. **Completing the proof of Theorem 3.1.** Now consider the space of all non-horizontal and non-vertical lines in $\mathbb{C}^2$, viewed as a line bundle $\mathcal{L} \to V$, where V is the two-dimensional quasiprojective variety that parameterizes these lines. We will work with

$$\Lambda = \mathcal{L}_V^3 := \mathcal{L} \times_V \mathcal{L} \times_V \mathcal{L}. \tag{3.1}$$

An element of Λ is a triple (P, Q, R) of points on a line $L \in V$. There is a natural map

$$\rho : \Lambda \to \mathbb{C}^6 \tag{3.2}$$

by recording the coordinates of the points P, Q, R. The geometry of $\mathcal{L} \to V$ implies that ρ is generically finite to its image; see [DM1, Proposition 3.1]. Note that Λ has dimension 5.

Now suppose that $\{L_n\}$ is an infinite sequence of points in V, each corresponding to a line in $\mathbb{C}^2$ that contains at least three points with both coordinates PCF. These triples of points (P_n, Q_n, R_n) on L_n define an infinite subset of Λ. Let $Z \subset \rho(\Lambda) \subset \mathbb{C}^6$ be any positive-dimensional irreducible component of the Zariski-closure of these collinear triples, viewed as PCF 6-tuples. We apply Theorem 3.3 to determine the structure of Z. A careful case-by-case analysis (which is admittedly a bit painful, but is not hard and only uses basic geometry of lines in $\mathbb{C}^2$) shows which PCF-special subvarieties Z can possibly lie in $\rho(\Lambda)$ [DM1, Section 4]. For example, $\rho(\Lambda)$ itself cannot have the structure of a PCF-special subvariety because there is no single coordinate kept constant and there are no two coordinates that must coincide throughout Λ. We conclude that the only possibility for Z comes from conditions of the form $P = Q$ or $P = R$ or $Q = R$. In other words, the collinear triples on L_n cannot consist of three *distinct* points.

3.5. **And how is all this related to Conjecture 2.1?** Let's see how Theorem 3.3 implies cases of the conjecture.

Fix any integer $m \geq 2$, and suppose that Λ is an irreducible complex algebraic subvariety in $\mathbb{C}^m$. The space Λ parameterizes an algebraic family of maps

$$\Phi : \Lambda \times \mathbb{P}^m \to \Lambda \times \mathbb{P}^m$$

that are products of quadratic polynomials $(f_{c_1}, \ldots, f_{c_m})$ in an affine chart. We consider the subvariety

$$\mathcal{X} = \Lambda \times \{(0 : \cdots : 0 : 1)\} \subset \Lambda \times \mathbb{P}^m$$

of relative dimension 0 over Λ. Note that a preperiodic point for Φ in $\mathcal{X}$ means that the associated parameter $\lambda \in \Lambda$ is a PCF m-tuple. Therefore, $\mathcal{X}$ contain a Zariski-dense set of preperiodic points if and only if Λ is PCF-special if only if Λ has the simple form of Theorem 3.3. (The case of $m = 1$ was described in Example 2.4.)

Suppose now that Λ is PCF-special. To prove Conjecture 2.1 in this case, we need to show that the current $\hat{T}_\Phi^{\wedge r(\Phi,\mathcal{X})} \wedge [\mathcal{X}]$ is nonzero.

Let $r = \dim \Lambda$. Let $\pi : \mathcal{X} \to \Lambda$ be the projection isomorphism. Since $\mathcal{X}$ follows the critical point in each coordinate, we observe that

$$\hat{T}_\Phi|_\mathcal{X} = \pi^*(T_1 + \cdots + T_m)$$

as a $(1,1)$-current on $\mathcal{X}$, where $T_i = \pi_i^* \mu_\mathcal{M}$ is the pullback of the bifurcation current on the boundary of $\mathcal{M}$ for the projection $\pi_i : \mathbb{C}^m \to \mathbb{C}$ to the i-th factor; all pullbacks are defined in the sense of currents. From the structure of Λ, given in Theorem 3.3, we see that each coordinate c_i which is constant PCF throughout Λ reduces the relative special dimension of $\mathcal{X}$ in $\Lambda \times \mathbb{P}^m$ by one, because the orbit of $\mathcal{X}$ lies in an invariant hypersurface $\left(\bigcup_n \{f_{c_i}^n(0)\}\right) \times \mathbb{C}^{m-1}$ in an affine chart. Moreover, each pair satisfying $c_j = c_k$ throughout Λ again reduces the relative special dimension because $\mathcal{X}$ will lie in the diagonal $\{z_i = z_j\}$ in that coordinate chart. This shows that

$$r(\Phi, \mathcal{X}) = r.$$

Moreover, we will have exactly r independent coordinates in the linear subvariety Λ of $\mathbb{C}^m$, implying that $(T_1 + \cdots + T_m)^{\wedge r}$ is nonzero on Λ. This proves condition (2) in Conjecture 2.1 for this $\mathcal{X}$ over Λ.

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