Problem Set # 9

M392C: Morse Theory

1. Recall that to a finite dimensional real vector space V we attach a $\mathbb{Z}$-graded $\mathbb{Z}/2\mathbb{Z}$-torsor $\mathfrak{O}_V$ whose two elements are the two orientations of V and whose degree is $\dim V$. For a vector bundle $V \to S$ over a space S we obtain a double cover $\mathfrak{O}_V \to S$. Recall also the tensor product construction and the dual $\mathfrak{O}^\vee$ (with degree negative to that of $\mathfrak{O}$).

   (a) Suppose
   $$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$
   is a short exact sequence of vector spaces or vector bundles. Construct an isomorphism
   $$\mathfrak{O}_{V''} \otimes \mathfrak{O}_{V'} \longrightarrow \mathfrak{O}_V$$

   (b) Apply (a) several times to orient the transverse intersection $N_1 \cap N_2 \subset M$ of two submanifolds N_1, N_2 of a manifold M. Check your conventions against those in Guillemin and Pollack.

   (c) For a manifold M with boundary construct an isomorphism
   $$\mathfrak{O}_{\partial M} \longrightarrow \mathfrak{O}_M[1]\big|_{\partial M},$$
   where for a degree n $\mathbb{Z}/2\mathbb{Z}$-torsor $\mathfrak{O}$, we set $\mathfrak{O}[k]$ to be the same $\mathbb{Z}/2\mathbb{Z}$-torsor in degree $n-k$. Use the ONF=Outward Normal First=One Never Forgets convention (which follows from the Quotient Before Sub convention of part (a)).

   (d) Let $M = [0,1]$. Prove that the composite isomorphism of $\mathbb{Z}/2\mathbb{Z}$-torsors
   $$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathfrak{O}_{\{0\}} \longrightarrow \mathfrak{O}_M[1]\big|_{\{0\}} \longrightarrow \mathfrak{O}_M[1]\big|_{\{1\}} \longrightarrow \mathfrak{O}_{\{1\}} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$
   exchanges the two elements of $\mathbb{Z}/2\mathbb{Z}$. (Be sure to define each map carefully.)

2. Let $S^2 \subset \mathbb{E}^3_{x,y,z}$ be the sphere defined by $x^2 + y^2 + z^2 = 1$. Define $f = x^2 + 2y^2 + 3z^2$. Show it is a Morse function and analyze the critical points and negative gradient flow. Produce the Morse complex as in lecture. Do the same on the quotient $\mathbb{RP}^2 = S^2/\{\pm 1\}$ by the antipodal action. We recall the Morse complex below.

For this second problem you will need to recall the Morse complex of a function $f\colon M \to \mathbb{R}$ on a Riemannian manifold (M, g) assuming the Morse-Smale condition. Attached to a $\mathbb{Z}$-graded $\mathbb{Z}/2\mathbb{Z}$-torsor $\mathfrak{O}$ is a $\mathbb{Z}$-graded free abelian group $\mathbb{Z}_{\mathfrak{O}}$ of rank one in the same degree, defined as $\mathbb{Z}_{\mathfrak{O}} = \mathfrak{O} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}$ where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}$ by $n \mapsto -n$. As a $\mathbb{Z}$-graded abelian group the Morse complex is

$$C(f) = \bigoplus_{p \in \mathrm{Crit}(f)} \mathbb{Z}_{\mathfrak{O}_p},$$

where $\mathfrak{O}_p = \mathfrak{O}_{T_pM^-}$ is the orientation torsor of the sum of the negative eigenspaces of $S_p\colon T_pM \to T_pM$ defined by

$$(\mathrm{Hess}_p f)(\xi_1, \xi_2) = \langle \xi_1, S_p(\xi_2) \rangle, \qquad \xi_1, \xi_2 \in T_pM.$$

The differential ∂ is the direct sum of maps

$$\partial_{q,p}\colon \mathbb{Z}_{\mathfrak{O}_p} \longrightarrow \mathbb{Z}_{\mathfrak{O}_q}[1]$$

over pairs $p, q \in \mathrm{Crit}(f)$ with $\lambda(p) = \lambda(q) + 1$, where $\lambda(p) \in \mathbb{Z}$ is the index of the critical point p. The descending manifold $M^d(p)$ is diffeomorphic to $B^{\lambda(p)}$ and its tangent space at p is T_pM^-; the ascending manifold $M^a(p)$ is diffeomorphic to $B^{n-\lambda(p)}$ and its tangent space at p is T_pM^+, where $n = \dim M$. The Morse-Smale condition means implies that $M^d(p) \pitchfork M^a(q)$ and the intersection is a finite set of flow lines. To each we assign a map $\mathbb{Z}_{\mathfrak{O}_p} \to \mathbb{Z}_{\mathfrak{O}_q}[1]$, induced from a map $\mathfrak{O}_p \to \mathfrak{O}_q[1]$, and then $\partial_{q,p}$ is the sum of the maps over the flow lines. Choose a regular value c with $f(q) < c < f(p)$. The manifold $f^{-1}(c)$ is co-oriented by the negative gradient flow of f. The orientation torsor (double cover) of the normal bundle to $M^d(p)$ is $\mathfrak{O}_M \otimes \mathfrak{O}_p^\vee$. The orientation torsor of the normal bundle to $M^a(q)$ is $\mathfrak{O}_q$. Hence the orientation torsor of the normal bundle to the intersection $f^{-1}(c) \cap M^d(p) \cap M^a(q)$ is $\mathfrak{O}_M \otimes \mathfrak{O}_p^\vee \otimes \mathfrak{O}_q[-1]$, so the orientation torsor of the tangent bundle is $\mathfrak{O}_p \otimes \mathfrak{O}_q[1]$. Since the intersection is 0-dimensional we obtain a trivialization of this torsor (section of the double cover over the 0-dimensional intersection). We also obtain a section of the dual $\mathfrak{O}_p^\vee \otimes \mathfrak{O}_q[-1]$, which is precisely what we want.

Choose orientations for T_pM^- and T_qM^- then the section becomes a function ϵ to $\mathbb{Z}/2\mathbb{Z}$, and $(-1)^\epsilon$ is the sign in the usual description of the Morse complex. This definition works for unorientable manifolds, such as $\mathbb{RP}^2$.