Problem Set # 6

M392C: Morse Theory

1. Proof or counterexample. Let $f\colon M \to \mathbb{R}$ be a proper Morse function on a smooth Riemannian manifold. Then the negative gradient flow exists for all time.

2. Proof or counterexample. Let M be a complete Riemannian manifold, $\gamma\colon (0,\infty) \to M$ a smooth curve, and suppose $|\dot\gamma(t)| \to 0$ as $t \to \infty$. Then $\lim_{t\to\infty} \gamma(t)$ exists.

3. If $f\colon M \to \mathbb{R}$ has a nondegenerate local minimum (or local maximum) at $p \in M$, then near p the negative gradient flow for a Riemannian metric on M is *smoothly* equivalent to the negative gradient flow near $0 \in T_pM$ of the Hessian. (For arbitrary index nondegenerate critical points the same is true *continuously*; this is the Hartman-Grobman theorem.) Work out a proof using the Moser method as follows.

   (a) Let $V = T_pM$, a vector space with an inner product. Use $-\operatorname{Hess}_p f$ to define a definite self-adjoint operator $L\colon V \to V$ and let ξ^0 be the associated linear vector field on V. Despite the fact that ξ^0 is unbounded, show that it generates a flow φ_t^0 which is defined for all t. Write an explicit formula for φ_t^0. Show that it is the negative gradient flow of an appropriate (which?) quadratic function.

   (b) Show the negative gradient of f in a neighborhood of $p \in M$, identified with a neighborhood of $0 \in V$ via a Morse coordinate system, is $\xi = \xi^0 + \eta$ for a vector field $\eta\colon U \to V$ in some open neighborhood U of $0 \in V$. Show that $\eta(0) = 0$.

   (c) Prove that in a possibly smaller neighborhood $U' \subset U$ there exists a unique vector field ζ such that $\zeta(0) = 0$ and $[\zeta, \xi^0] = \eta$. (Hint: Write the equation as $\frac{d}{dt}\Big|_{t=0} (\varphi_t^0)_*\zeta = -\eta$.)

   (d) Again possibly cutting down the neighborhood of the origin, show that ζ generates a flow ψ_t.

   (e) Prove that $(\psi_1)_*\xi^0 = \xi$. Conclude that the diffeomorphism ψ_1 intertwines the flows φ_t^0 and φ_t.