

Problem Set # 5

M392C: Morse Theory

1. Let P be a smooth manifold, not necessarily compact; $\pi: N \rightarrow P$ a finite rank real vector bundle; ξ a vertical vector field on N which vanishes on P . Prove that there exist neighborhoods $U \subset U'$ of the zero section and a smooth map $\varphi: [0, 1] \times U \rightarrow U'$ such that for each $n \in U$ the curve $t \mapsto \varphi_t(p)$ is an integral curve of ξ . Variation: Let the vector field ξ also depend on t .
2. Let V be a finite dimensional complex vector space and $\mathbb{P}(V)$ the set of all lines in V . For each codimension one subspace $W \subset V$ define

$$A_W = \{L \in \mathbb{P}(V) : L \not\subset W\}.$$

- (a) Give A_W the structure of an affine space over $\text{Hom}(V/W, W)$.
- (b) Given two codimension one subspaces W, W' consider the identity map on $A_W \cap A_{W'}$ as a map between open subsets of the affine spaces A_W and $A_{W'}$. Is this map affine? Is it affine polynomial? Is it holomorphic?
- (c) Explain how to use the foregoing to endow $\mathbb{P}(V)$ with the structure of a complex manifold.
- (d) Show that the incidence relation

$$\mathcal{L} := \{(L, \xi) \in \mathbb{P}(V) \times V : \xi \in L\}$$

defines a submanifold of $\mathbb{P}(V) \times V$. Prove carefully that the restriction of the projection onto $\mathbb{P}(V)$ is locally trivial, i.e., $\mathcal{L} \rightarrow \mathbb{P}(V)$ is a complex line bundle. Define and prove that it is holomorphic.

- (e) Let $h: \overline{V} \times V \rightarrow \mathbb{C}$ be an inner product. Recall how in lecture we use it to construct a connection on the principal \mathbb{T} -bundle $\pi: S(V) \rightarrow \mathbb{P}(V)$, where $S(V) \subset V$ is the unit sphere bundle. Write a formula for the connection form Θ .
- (f) Compute the curvature of the connection Θ .
- (g) Construct $\mathcal{L} \rightarrow \mathbb{P}(V)$ as an associated bundle of the principal \mathbb{T} -bundle π . As such it inherits a metric and covariant derivative ∇ . Prove that if s is a local holomorphic section of $\mathcal{L} \rightarrow \mathbb{P}(V)$, then ∇s is a $(1, 0)$ -form. Compute the curvature in terms of the section s .