

Problem Set # 4

M392C: Morse Theory

1. Let X be a finite CW complex and F any field. Prove that the Euler number is

$$\text{Euler}(X) = \sum_{q=0}^{\infty} (-1)^q c_q = \sum_{q=0}^{\infty} (-1)^q \dim H_q(X; F),$$

where c_q is the number of q -cells. (Define the Euler number by the last expression with $F = \mathbb{Q}$.)

2. Let $C \subset E$ be a co-oriented curve in a Euclidean plane E . Choose a point $q \in E \setminus C$. Analyze the square distance function from q to C . For which q does this function have a degenerate critical point?
3. In lecture I analyzed the square distance function on an adjoint orbit of the connected compact Lie group SU_3 (to a point not on the orbit).
 - (a) Write a careful proof of the main theorem in that example; I gave all of the ingredients in lecture. (If f is the distance square function from a point Q , the theorem: (1) tells Q for which f is Morse, (2) identifies the critical point set $\text{Crit}(f)$, and (3) computes the index of each critical point.)
 - (b) Repeat the analysis for SU_2 . What are the orbits? Do you recognize the formula you compute for the second fundamental form?
 - (c) What is the general story for a connected compact Lie group G ?
4. Recall that the real projective space \mathbb{RP}^n is the smooth manifold of lines (one-dimensional subspaces) of \mathbb{R}^{n+1} .
 - (a) Define a Riemannian metric on \mathbb{RP}^n whose group of isometries acts transitively. What is that group of isometries?
 - (b) Does \mathbb{RP}^n occur as the adjoint orbit of a compact Lie group? Prove your answer.
 - (c) Find a perfect Morse function on \mathbb{RP}^n .